Foundations of Physics, Vol. 21, No. 8, 1991

A Quantum Time Machine

Lev Vaidman[1]

Received August 2, 1990

A novel description of quantum systems is employed for constructing a "time machine" capable of shifting in time the wave function of a quantum system. This device uses gravitational time dilations and a peculiar quantum interference effect due to preselection and postselection. In most trials this time machine fails to operate but when it does succeed it accomplishes tasks which no other machine can.

1. INTRODUCTION

Recently a novel approach in quantum theory was developed.[1] In this approach a quantum system at a given time is described by two vectors in a Hilbert space instead of one, the usual state vector, evolving from the time of the latest complete measurement in the past, and another one evolving backward in time from the time of the earliest complete measurement in the future. Using this approach, several surprising features of quantum systems between two measurements were uncovered. These features can be explained in the standard, single state vector approach as well. Such explanations involve, however, peculiar mathematical identities, which seem to be rather paradoxical. That explains why these novel features have not been discovered before. In this paper we discuss one of these peculiar phenomena: a quantum time machine.[2]

To avoid possible misinterpretations due to the name "time machine," let us explain from the outset what our machine can do and how it differs from the familiar concept of "time machine." Our device is not for time travel. All that it can accomplish is to change the rate of time flow for a

[1] Physics Department, University of South Carolina, Columbia, South Carolina 29208, and School of Physics and Astronomy, Tel-Aviv University, Ramat Aviv 69978, Israel.

0015-9018/91/0800-0947$06.50/0

closed quantum system. Classically, one can slow down the time flow of a system relative to an external observer, for example, by fast travel. Our quantum time machine is able to change the rate of time flow of a system for a given period by an arbitrary, even *negative*, factor. Therefore, our machine, contrary to any classical device, is capable of moving the system to its "past." In that case, at the moment the machine completes its operation the system is in a state in which it was some time *before the beginning* of the operation of the time machine. Our machine can also move the system to the future, i.e., at the end of the operation of the time machine the system is in a state corresponding to some later time of the undisturbed evolution.

A central role in the operation of our time machine is played by a peculiar mathematical identity which we discuss in Sec. 2. A specific superposition of time evolutions for short periods of time δt_n yields a time evolution for a large period of time Δt:

$$\sum_{n=0}^{N} c_n U(\delta t_n) \,|\Psi\rangle \cong U(\Delta t)\,|\Psi\rangle \tag{1}$$

This approximate equality holds (with the same δt_n and c_n) for a large class of states $|\Psi\rangle$ of the quantum system, and in some cases even for all states of the system.

In order to obtain different time evolutions of the system we use the gravitational time dilation effect which is discussed in Sec. 3. In Sec. 4 we describe the design and the operation of our time machine, i.e., the procedure for the preparation of the state given by the left-hand side of Eq. (1). The success of the operation of our time machine depends on obtaining a specific outcome in the postselection quantum measurement. The probability of the successful postselection measurement is analyzed in Sec. 5. Section 6 concludes the paper by discussing the limitations and the advantages of our time machine.

2. A PECULIAR MATHEMATICAL IDENTITY

The mathematical identity states that a linear combination with coefficients c_n of the values of a function at arguments shifted by a_n is approximately equal to the values of the same function shifted by an amount α which is very different from all the a_n's:

$$\sum_{n=0}^{N} c_n f(t-a_n) \cong f(t-\alpha) \tag{2}$$

What is peculiar about this equality is that the same values of a_n and c_n are appropriate not just for one specific function, but for a wide class of functions. This approximate equality can be made arbitrarily precise by increasing the number of terms in the sum.

This type of mathematical identity was discovered through the consideration of the "weak" measurement of a quantum variable A performed at a time between two measurements. The approach, which associates two vectors in the Hilbert space of states with the quantum system, implies that the result of the weak measurement has to be the "weak value" of A.[1] This weak value might be very different from the eigenvalues of A. The left-hand side of Eq. (2) represents the wave function of the measuring device which measures the weak value of A, where a_n are eigenvalues of A. The right-hand side of Eq. (2) represents the wave function of the measuring device corresponding to the outcome α—the weak value of A.

In this work we consider a particular example:

$$a_n = \frac{n}{N} \tag{3a}$$

$$c_n = \frac{N!}{(N-n)!\, n!}\, \alpha^n (1-\alpha)^{N-n} \tag{3b}$$

where $n = 0, 1, \ldots, N$. Note, that the coefficients c_n are terms in the binomial expansion of $[\alpha + (1-\alpha)]^N$ and, in particular, $\sum_{n=0}^{N} c_n = 1$.

For Eq. (2) to be correct, the Fourier transform of the function $f(t)$ has to decrease fast enough for large w (w is the Fourier conjugate to t). A sufficient condition for our particular choice is that for large w's:

$$|\tilde{f}(w)| < e^{-b|w|}, \qquad \text{where} \quad b > |\alpha(\alpha-1)| \tag{4}$$

To prove this, we notice that Eq. (2), with the choice of a_n and c_n given by (3a) and (3b) respectively, is the Fourier transform of

$$\sum_{n=0}^{N} c_n e^{iwa_n} \tilde{f}(w) = [1 + \alpha(e^{iw/N} - 1)]^N \tilde{f}(w) \cong e^{i\alpha w} \tilde{f}(w) \tag{5}$$

For large N we can expand:

$$\ln[1 + \alpha(e^{iw/N} - 1)]^N \cong iw\alpha + \frac{\alpha(\alpha-1)}{2} \frac{w^2}{N} + \cdots$$

Thus, for $|w| < N^{1/2-\varepsilon}$ we have $[1 + \alpha(e^{iw/N} - 1)]^N \cong e^{i\alpha w}$. Therefore, if the contribution to the Fourier transform of the left-hand side from $|w| > N^{1/2-\varepsilon}$ can be neglected, we obtain Eq. (2). Requirement (4) ensures

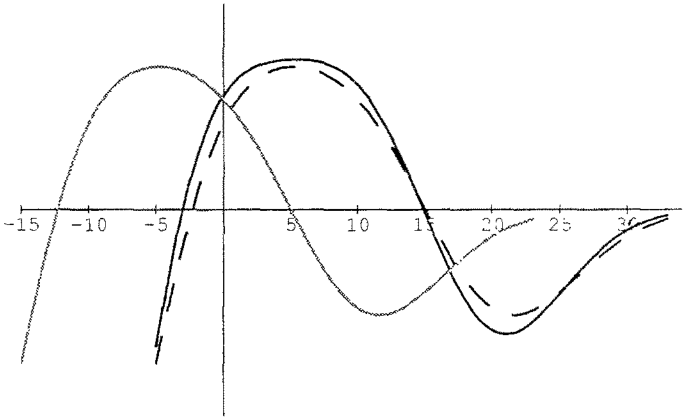

Fig. 1. Demonstration of an approximate equality given by Eq. (2). The sum of a function shifted by the 14 values a_n between 0 and 1 and multiplied by the coefficient c_n (a_n and c_n are given by Eqs. (3a) and (3b) with $N=13$, $\alpha=10$) yields approximately the same function shifted by the value 10: $\sum_{n=0}^{13} \binom{13}{n}(-10/9)^n f(t-n/13) \cong f(t-10)$. The gray line shows $f(t)$; the dashed line shows $f(t-10)$, the RHS of Eq. (2); and the solid line shows the sum, the LHS of Eq. (2).

that this contribution can be neglected. To see this, we estimate the absolute value of $[1+\alpha(e^{iw/N}-1)]^N$ as follows:

$$|[1+\alpha(e^{iw/N}-1)]^N| = |1+2\alpha(\alpha-1)\sin^2(w/2N)|^{N/2} < e^{|\alpha(\alpha-1)w|}$$

Thus, if $\tilde{f}(w)$ fulfills the requirement (4) then, indeed, the contribution to the Fourier transform of the left-hand side of Eq. (5) due to the integration over w, $|w| > N^{1/2-\varepsilon}$, can be neglected. Therefore, we have proved that the approximate equality (2) with parameters a_n and c_n given by Eq. (3) is correct for all functions which fulfill the requirement (4).

Figure 1 shows an example of identity (2). Even for a relatively small number of terms in the sum (14 in our example), the method works remarkably well. The shifts from 0 to 1 yield the shift by 10. The distortion of the shifted function is not very large. By increasing the number of terms in the sum the distortion of the shifted function can be made arbitrarily small.

3. CLASSICAL TIME MACHINES

A well-known example of a time machine is a rocket which takes a system to a fast journey. If the rocket is moving with velocity V and the duration of the journey (in the laboratory frame) is T, then we obtain the time shift (relative to the situation without the fast journey):

$$\delta t = T\,(1 - \sqrt{1 - V^2/c^2}) \tag{6}$$

For typical laboratory velocities this effect is rather small, but it has been observed experimentally in precision measurements in satellites. In such a "time machine," however, the system necessarily experiences external force, and we consider this a conceptual disadvantage.

In our time machine we use, instead of the time dilation of special relativity, the gravitational time dilation. The relation between the proper time of the system placed in a gravitational potential ϕ and the time of the external observer ($\phi = 0$) is given by $d\tau = dt\sqrt{1 + 2\phi/c^2}$. We produce the gravitational potential by surrounding our system with a spherical shell of mass M and radius R. The gravitational potential inside the shell is $\phi = -GM/R$. Therefore, the time shift due to the massive shell surrounding our system, i.e., the difference between the time period T of the external observer at a large distance from the shell and the period of the time evolution of the system (the proper time), is

$$\delta t = T\,(1 - \sqrt{1 - 2GM/c^2R}) \tag{7}$$

This effect, for any man-made massive shell, is too small to be observed by today's instruments. However, the conceptual advantage of this method is that we do not "touch" our system. Even the gravitational field due to the massive spherical shell vanishes inside the shell.

The classical time machine can only *slow down* the time evolution of a system. For any reasonable mass and radius of the shell the change of rate of the time flow is extremely small. In the next section we shall describe our quantum time machine which amplifies the effect of the classical gravitational time machine (for a spherical shell of the same mass), and makes it possible to speed up the time flow for an evolution of a system, as well as to change its direction.

4. QUANTUM GRAVITATIONAL TIME MACHINE

In our machine we use the gravitational time dilation and a quantum interference phenomenon which, due to the peculiar mathematical property discussed in Sec. 2, amplifies the time translation. We produce the superposition of states shifted in time by the small values δt_n (due to spherical shells of different radii) given by the left-hand side of Eq. (1). Thus, we obtain a time shift by a possibly large, positive or negative, time interval Δt.

The wave function of a quantum system $\Psi(q, t)$, considered as a function of time, usually has a Fourier transform which decreases rapidly for large frequencies. Therefore, the sum of the wave function shifted by the small periods of time $\delta t_n = \delta t\, a_n$, with a_n given by (3a), and multiplied by the coefficients c_n [of Eq. (3b)] is approximately equal to the wave function shifted by the large time $\Delta t = \delta t\, \alpha$. Since the equality (2) is correct with the same coefficients for all functions with rapidly decreasing Fourier transforms, we obtain for each q, and therefore for the whole wave function,

$$\sum_{n=0}^{N} c_n \Psi(q, t - \delta t_n) \cong \Psi(q, t - \Delta t) \tag{8}$$

Thus, a device which changes the state of the system from $\Psi(q, t)$ to the state given by the left-hand side of Eq. (8) generates a time shift of Δt. Let us now present a design for such a device and explain how it operates.

Our machine consists of the following parts: a massive spherical shell, a mechanical device—"the mover"—with a quantum operating system, and a measuring device which can prepare and verify states of this quantum operating system.

The massive shell of mass M surrounds our system and its radius R can have any of the values $R_0, R_1, \ldots, R_N$. Initially, $R = R_0$.

The mover changes the radius of the spherical shell at time $t = 0$, waits for an (external) time T, and then moves it back to its original state, i.e., to the radius R_0.

The quantum operating system (QOS) of the mover controls the radius to which the shell is moved for the period of time T. The Hamiltonian of the QOS has $N+1$ nondegenerate eigenstates $|n\rangle$, $n = 0, 1, \ldots, N$. If the state of the QOS is $|n\rangle$, then the mover changes the radius of the shell to the value R_n.

The measuring device preselects and postselects the state of the QOS. It prepares the QOS before the time $t = 0$ in the initial state:

$$|\psi_{\rm in}\rangle_{\rm QOS} = \mathcal{N} \sum_{n=0}^{N} c_n |n\rangle \tag{9}$$

with the same c_n as given above in Eq. (3b) and with a normalization factor

$$\mathcal{N} = \left(\sum_{n=0}^{N} |c_n|^2 \right)^{-1/2} \tag{9a}$$

After the mover completes its operation, i.e., after the time $t = T$, we perform another measurement on the QOS. One of the nondegenerate eigenstates of this measurement is the specific "final state":

$$|\psi_f\rangle_{\text{QOS}} = \frac{1}{\sqrt{N+1}} \sum_{n=0}^{N} |n\rangle \tag{10}$$

Our machine works only if the postselection measurement yields the state (10). Unfortunately, this is a very rare event. We shall discuss the probability of obtaining the appropriate outcome in the next section.

Assume that the postselection measurement is successful, i.e., that we do obtain the final state (10). We will next show that in this case, assuming an appropriate choice of the radii R_n, our "time machine" shifts the wave function of the system by the time interval Δt. The time shift is defined relative to the situation in which the machine has not operated, i.e., the radius of the shell was not changed from the initial value R_0. In order to obtain the desired time shift $\Delta t = \delta t\, \alpha$ we chose the radii R_n such that

$$\delta t_n \equiv \frac{n\, \delta t}{N} = T(\sqrt{1 - 2GM/c^2 R_0} - \sqrt{1 - 2GM/c^2 R_n}) \tag{11}$$

The maximal time shift in the different terms of the superposition [left-hand side of Eq. (8)] is $\delta t_N = \delta t$. The parameter α is the measure of a "quantum amplification" relative to the maximal (classical) time shift δt. If the radius R_0 of the shell is large enough that the time dilation due to the shell in its initial configuration can be neglected, Eq. (11) simplifies to

$$\delta t_n = T(1 - \sqrt{1 - 2GM/c^2 R_n}) \tag{11a}$$

Let us assume then that we have arranged the radii according to Eq. (11a), and we have prepared the quantum operating system of the mover in the state (9). Then, just prior to the operation of the time machine the overall state is the direct product of the corresponding states of the system, the shell, and the mover:

$$\mathcal{N}\, |\Psi(q, 0)\rangle\, |R_0\rangle \sum_{n=0}^{N} c_n\, |n\rangle \tag{12}$$

where $|R_0\rangle$ signifies that the shell, together with the mechanical part of the mover, is at the radius R_0. Although these are clearly macroscopic bodies, we assume that we can treat them quantum-mechanically. We also make an idealized assumption that these bodies do not interact with the environment, i.e., no element of the environment becomes correlated to the radius of the shell.

Once the mover has operated, changing the radius of the spherical shell, the overall state becomes

$$\mathcal{N}\,|\Psi(q,0)\rangle \sum_{n=0}^{N} c_n\,|R_n\rangle\,|n\rangle \tag{13}$$

For different radii R_n, we have different gravitational potentials inside the shell and, therefore, different relations between the flow of the proper time of the system and the flow of the external time. Thus, after the external time T has elapsed, just before the mover takes the radii R_n back to the value R_0, the overall state is

$$\mathcal{N} \sum_{n=0}^{N} c_n\,|\Psi(q, T-\delta t_n)\rangle\,|R_n\rangle\,|n\rangle \tag{14}$$

Note that now the system, the shell, and the QOS are correlated: the system is not in a pure quantum state. After the mover completes its operation, the overall state becomes

$$\mathcal{N} \sum_{n=0}^{N} c_n\,|\Psi(q, T-\delta t_n)\rangle\,|R_0\rangle\,|n\rangle \tag{15}$$

There is still a correlation between the system and the QOS.

The postselection measurement performed on the QOS puts the QOS and, consequently, also our quantum system, in a pure state. After the successful postselection measurement, the overall state is

$$\left(\sum_{n=0}^{N} c_n\,|\Psi(q, T-\delta t_n)\rangle\right)|R_0\rangle\left(\frac{1}{\sqrt{N+1}}\sum_{n=0}^{N}|n\rangle\right) \tag{16}$$

We showed that the wave function of the quantum system $\Psi(q,t)$ is changed by the operation of the time machine into $\sum_{n=0}^{N} c_n\,\Psi(q, t-\delta t_n)$. Up to the precision of the approximate equality (8) (which can be arbitrarily improved by increasing the number of terms N in the sum), this wave function is indeed $\Psi(q, t-\Delta t)$! Note that for $\Delta t > T$ the state of the system at the moment the time machine has completed its operation is the state in which the system was *before* the beginning of the operation of the time machine.

5. THE PROBABILITY OF THE SUCCESS OF THE QUANTUM TIME MACHINE

The main conceptual weakness of our time machine is that usually it does not work. Successful postselection measurements corresponding to large time shifts are extremely rare. Let us estimate the probability of the successful postselection measurement in our example. The probability is given by the square of the norm of the vector obtained by projecting the state (15) on the subspace defined by state (10) of the QOS:

$$\text{Prob} = \left\| \frac{\mathcal{N}}{\sqrt{N+1}} \sum_{n=0}^{N} c_n \, |\Psi(q, T - \delta t_n)\rangle \, |R_0\rangle \right\|^2 \tag{17}$$

To obtain a time shift without significant distortion, the wave functions shifted by different times δt_n have to be such that the scalar products between them can be approximated by 1. Taking then the explicit form of c_n from (3b), we evaluate the probability (17), obtaining:

$$\text{Prob} \cong \frac{\mathcal{N}^2}{N} \tag{18}$$

The normalization factor $\mathcal{N}$ given by Eq. (9a) decreases very rapidly for large N. Even if we use a more efficient choice of the initial and the final states of the QOS (see Ref. 1) the probability, for the amplification $\alpha > 1$, decreases with N as $1/(2\alpha - 1)^N$.

The small probability of the successful operation of our time machine is, in fact, unavoidable. At the time just before the postselection measurement, the system is in a mixture of states correlated to the orthogonal states of the QOS [see Eq. (15)]. The probability of finding the system at that time in the state $|\Psi(q, T - \Delta t)\rangle$, for Δt which differs significantly from the time periods δt_n, is usually extremely small. This is the probability to find the system, by a measurement performed "now," in the state in which it was supposed to be at some other time. For any real situation this probability is tiny but not equal precisely to zero, since all systems with bounded energies have wave functions with nonvanishing tails. The successful operation of our time machine is a particular way of "finding" the state of the quantum system shifted by the period of time $\Delta t = \delta t \, \alpha$. Therefore, the probability for success cannot be larger than the probability of finding the shifted wave function by direct measurement.

One can wonder what has been achieved by all this rather complicated procedure if we can obtain the wave function of the system shifted by the time period Δt simply by performing a quantum verification measurement at the time T of the state $|\Psi(T - \Delta t)\rangle$. There is a very small chance for the success of this verification measurement, but using our procedure the chance is even smaller. What our machine can do, and we are not aware of any other method which can achieve this, is to shift the wave function in time *without knowing* the wave function. If we obtain the desired result of the postselection measurement (the postselection measurement performed on *the measuring device*), we know that the wave function of the system, whatever it is, is shifted by the time Δt. Not only is knowledge of the wave function of the system inessential for our method, but even the very nature of the physical system whose wave function is shifted by our time machine need not be known. The only requirement is that the energy distribution of the system decreases rapidly enough. Let us discuss this requirement more quantitatively.

In order to be able to perform the time shift of the state of the system without significant distortion, the Fourier transform of $\Psi(q, t)$ (as the function of t) should decrease rapidly enough for large frequencies. The wave function of the system in the energy representation is

$$|\Psi(t)\rangle = \int \exp(-iEt)\, f(E)\, |\Psi_E\rangle\, dE \tag{19}$$

where $|\Psi_E\rangle$ is the energy eigenstate. For a discrete spectrum the integration should be replaced by the sum. In the case of degenerate eigenstates the integral should include summation (and/or integration) on the degeneracy. The state [left-hand side of Eq. (8)] is, in this energy representation,

$$\int \sum_{n=0}^{N} c_n \exp(iE\,\delta t_n) \exp(-iEt)\, f(E)\, |\Psi_E\rangle\, dE \tag{20}$$

Equation (8) becomes, in the energy representation,

$$\sum_{n=0}^{N} c_n \exp(iE\,\delta t_n)\, f(E) \cong \exp(iE\,\Delta t)\, f(E) \tag{21}$$

Note that this is a version of the identity given in Eq. (5).

For every given E we can obtain any desirable precision of the approximate equality

$$\sum_{n=0}^{N} c_n \exp(iE\,\delta t_n) \cong \exp(iE\,\Delta t) \tag{22}$$

by increasing the number of terms in the sum N. (This number, however, grows very fast with E.) Therefore, the time machine works for systems with bounded energies and for systems with rapidly decreasing energy distribution.

The operation of our time machine can be considered as *a superposition of time evolutions*[2] for different periods of time δt_n. This name is especially appropriate if the Hamiltonian of the system is bounded, since in this case the approximate equality (1) is correct for *all* states $|\Psi\rangle$.

If the expectation value of the energy (constituting a very partial knowledge about the state of the system) can be estimated, then we can improve dramatically the probability of the success of our procedure. Given the expectation value of the energy $\langle E\rangle$, we can modify the coefficients c_n in such a way that the minimal number of terms in the superposition required for shiting the state in time can be significantly reduced. This, in turn, enormously increases the probability of successful postselection measurement. Indeed, let us take in our procedure, instead of the coefficients c_n given by Eq. (3b), the new coefficients c_n':

$$c_n' \equiv c_n \exp[i\langle E\rangle(\Delta t - \delta t_n)] \tag{23}$$

The requirement for time translation without significant distortion then becomes

$$\sum_{n=0}^{N} c_n \exp[i(E - \langle E\rangle)\, \delta t_n]\, f(E) \cong \exp[i(E - \langle E\rangle)\, \Delta t]\, f(E) \tag{24}$$

The number of terms in the sum, necessary for the approximate equality (24) to be true for energies E for which $f(E)$ is substantial, depends, therefore, on the energy dispersion ΔE. Thus, the level of difficulty of the time shift without distortion depends on the magnitude of the energy dispersion ΔE and not on the expectation value of energy $\langle E\rangle$.

6. TIME TRANSLATION TO THE PAST AND TO THE FUTURE

Let us spell out again what our machine does. Assume that the time evolution of the state of the system is given by $|\Psi(t)\rangle$. By this we mean that this is the evolution *before* the operation of the time machine and this is also the evolution later, provided we *do not* operate the time machine. The state $|\Psi(t)\rangle$ describes the actual past states of the system and the counterfactual future states of the system, i.e., the states which will be in the case we do not disturb the evolution of the system by the operation of our time machine. Define "now," $t=0$, to be the time at which we begin the operation of the time machine. The time interval of the operation of the time machine is T. Moving the system to the *past* means moving it to the state in which the system actually was at some time $t<0$. Moving the system to the future means moving it to the state in which it would have wound up after undisturbed evolution at some future time $t>T$. Evidently, the classical time machine does neither of these, since all it can achieve is that at time T the system is in the state corresponding to the time t, $0<t<T$.

When we speed up or slow down the rate of the time evolution, the system passes through all states of its undisturbed evolution only once. More bizarre is the situation when we reverse the direction of the time flow, thus ending up, after completing the operation of the time machine, in the state in which the system was before $t=0$. In this case the system passes *three* times through some states during its evolution.

For our time machine to operate properly, it is essential that the system is isolated from the external world. In the case of the time translation to the past state, the system has to be isolated not only during the time of operation of the time machine, but also during the whole period of intended time translation. If the system is to be moved to the state in which it was at the time t, $t<0$, then it has to be isolated from the time t until the end of the operation of the time machine. This seems to be a limitation of our time machine. It leads, however, to an interesting possibility. We can send a system to its *counterfactual past*, i.e., to the past in which it was supposed to be if it were isolated (or if it were in any environment chosen by us). Consider an excited atom which we isolate in vacuum at time $t=0$ inside our time machine. And assume that our time machine made a successful time translation to a negative time t, such that $|t|$ is larger than the lifetime of the excited atomic state. Since the atom, now, is not in the environment it was in the past, we do not move the atom to its actual state in the past. Instead, we move the atom to the state of its counterfactual past. By this we mean the state of the isolated atom which under its normal evolution in the vacuum during the time period $|t|$ winds up in the excited state. In fact, this is the state of the atom together with an incoming radiation field. The radiation field is exactly such that it will be absorbed by the atom. Although our procedure is very complicated and only very rarely successful, still, it is probably the easiest way to prepare the precise incoming electromagnetic wave which excites a single atom with probability one.

ACKNOWLEDGMENTS

Many thanks are due to Yakir Aharonov for valuable discussions. This work was supported in part by the National Science Foundation under grant No. PHY-9008048.

REFERENCES

1. Y. Aharonov and L. Vaidman, *Phys. Rev. A* **41**, 11 (1990).
2. Y. Aharonov, J. Anandan, S. Popescu, and L. Vaidman, *Phys. Rev. Lett.* **64**, 2965 (1990).