About Yor's problem.

8. A strengthening of the standardness theorem

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Consider a Markov chain $\{X_k\}_k$, $X_k \in \mathcal{X}_k$. As before (see "Another proof of a standardness theorem"), we may describe it by means of $\mu_k \in \mathbb{P}(\mathcal{X}_k)$ and $\nu_k : \mathcal{X}_{k-1} \to \mathbb{P}(\mathcal{X}_k)$; that is, μ_k is the distribution of X_k, and $\nu_k(x_{k-1})$ is the conditional distribution of X_k for given $X_{k-1} = x_{k-1}$.

For $x'_{k-1}, x''_{k-1} \in \mathcal{X}_{k-1}$ consider the infimum of three measures

$$\nu(x'_{k-1}) \wedge \nu(x''_{k-1}) \wedge \mu_k,$$

take its total mass, and its essential infimum in x'_{k-1}, x''_{k-1}; more exactly, define m_k as the maximal number satisfying the following condition:

$$(\nu(x'_{k-1}) \wedge \nu(x''_{k-1}) \wedge \mu_k)(\mathcal{X}_k) \ge m_k \tag{1}$$

for all pairs (x'_{k-1}, x''_{k-1}) in a product set of full measure.

Note 1. m_k introduced above is clearly no less than m_k introduced in "Another proof of a standardness theorem."

Note 2. If there exists the density

$$\rho_k(x_k, x_{k-1}) = \frac{\nu_k(x_{k-1})(dx_k)}{\mu_k(dx_k)},$$

then (1) means that

$$\int (\rho_k(x'_{k-1}, x_k) \wedge \rho_k(x''_{k-1}, x_k) \wedge 1)\, \mu_k(dx_k) \ge m_k.$$

Theorem. If $\sum_k m_k = \infty$, then $\{X_k\}_k$ is tail-trivial, and admits a generating parametrization.

Proof. We have to construct the needed function

$$\alpha_k : [0,1] \times \mathcal{X}_{k-1} \to \mathcal{X}_k$$

for each k. But $[0,1]$ may be replaced with another probability space; we prefer to construct functions

$$\alpha_k : (\mathcal{X}_k \times [0,1]) \times \mathcal{X}_{k-1} \to \mathcal{X}_k,$$

the first space $\mathcal{X}_k \times [0,1]$ being equipped with the measure $\mu_k \times \mathrm{mes}$. Introduce the density*

$$p_k(x_{k-1}, x_k) = \frac{(\nu_k(x_{k-1}) \wedge \mu_k)(dx_k)}{\mu_k(dx_k)}. \tag{2}$$

First, define

$$\alpha_k(x_k, t, x_{k-1}) = x_k \quad \text{when } t \le p_k(x_{k-1}, x_k); \tag{3}$$

it "parametrizes" the measure $\nu_k(x_{k-1}) \wedge \mu_k$. Second, define $\alpha_k(x_k, t, x_{k-1})$ for $t > p_k(x_{k-1}, x_k)$ in any way providing that α_k is a parametrization. We have to prove that

$$\sigma(X_{-\infty}^k) \subset \sigma(Y_{-\infty}^k),$$

because the rest of the proof is the same as in "Another proof of a standardness theorem."

We shall construct functions $f_r^s : \mathcal{Y}_r^s \to \mathcal{X}_s$ such that

$$\mathbb{P}\{X_s = f_r^s(Y_r, \dots, Y_s)\} \ge 1 - \prod_{k=r+1}^{s} (1 - m_k); \tag{4}$$

when $r \to -\infty$, it gives us $X_s \in \sigma(Y_{-\infty}^s)$. Of course, $\mathcal{Y}_r^s = \mathcal{Y}_r \times \mathcal{Y}_{r+1} \times \ldots \times \mathcal{Y}_s$ and $\mathcal{Y}_k = \mathcal{X}_k \times [0,1]$.

Define the functions by recursion in s:

$$\begin{aligned} f_r^r(x_k, t) &= x_k \quad \text{for any } t \in [0,1]; \\ f_r^{s+1}(y_r^{s+1}) &= \alpha_{s+1}(y_{s+1}, f_r^s(y_r^s)). \end{aligned}$$

Consider events

$$A_r^s = \{\omega : X_s \ne f_r^s(Y_r^s)\}.$$

We have

$$A_r^{s+1} \subset A_r^s;$$

indeed, $X_{s+1} = \alpha_{s+1}(Y_{s+1}, X_s)$ by definition of a parametrization, and $f_r^{s+1}(y_r^{s+1}) = \alpha_{s+1}(y_{s+1}, f_r^s(y_r^s))$ by definition of f, so $X_s = f_r^s(Y_r^s) \implies X_{s+1} = f_r^{s+1}(Y_r^{s+1})$. It remains to prove the inequality

$$\mathbb{P}\,(\, A_r^{s+1} \,|\, \sigma(X_{-\infty}^s, Y_{-\infty}^s)\,) \le 1 - m_{s+1}, \tag{5}$$

because it implies $\mathbb{P}(A_r^{s+1} | A_r^s) \le 1 - m_{s+1}$, hence $\mathbb{P}(A_r^{s+1}) \le (1 - m_{s+1})\mathbb{P}(A_r^s)$, and hence (4).

* Clearly, $p_k(x_{k-1}, x_k) = \rho_k(x_{k-1}, x_k) \wedge 1$, provided that ρ_k exists. Even if it does not, we may take the absolutely continuous part.

Let us prove (5). We have

$$A_r^{s+1} = \{\omega : X_{s+1} \neq f_r^{s+1}(Y_r^{s+1})\} = \{\omega : \alpha_{s+1}(Y_{s+1}, X_s) \neq \alpha_{s+1}(Y_{s+1}, f_r^s(Y_r^s))\};$$

here X_s and $f_r^s(Y_r^s)$ are measurable with respect to the given σ-field $\sigma(X_{-\infty}^s, Y_{-\infty}^s)$, whereas Y_{s+1} is independent of the σ-field (by definition of parametrization). So, it is enough to prove that

$$\mathbb{P}\{\,\alpha_{s+1}(Y_{s+1}, x_s') \neq \alpha_{s+1}(Y_{s+1}, x_s'')\,\} \leq 1 - m_{s+1}$$

for any $x_s', x_s'' \in \mathcal{X}_s$. But

$$\begin{gathered}\mathbb{P}\{\,\alpha_{s+1}(Y_{s+1}, x_s') \neq \alpha_{s+1}(Y_{s+1}, x_s'')\,\} = \\ = (\mu_{s+1} \times \mathrm{mes})\,\{\,(x_{s+1}, t) : \alpha_{s+1}(x_{s+1}, t, x_s') \neq \alpha_{s+1}(x_{s+1}, t, x_s'')\,\} = \\ = \int \mu_{s+1}(dx_{s+1})\,\mathrm{mes}\,\{\,t \in [0,1] : \alpha_{s+1}(x_{s+1}, t, x_s') \neq \alpha_{s+1}(x_{s+1}, t, x_s'')\,\} \leq \\ \leq 1 - \int \mu_{s+1}(dx_{s+1}) \cdot (p_{s+1}(x_s', x_{s+1}) \wedge p_{s+1}(x_s'', x_{s+1}) \wedge 1),\end{gathered}$$

because, according to (3),

$$t \leq p_{s+1}(x_s', x_{s+1}) \wedge p_{s+1}(x_s'', x_{s+1}) \quad \Longrightarrow \quad \alpha_{s+1}(x_{s+1}, t, x_s') = x_{s+1} = \alpha_{s+1}(x_{s+1}, t, x_s'').$$

Now,

$$\begin{gathered}\int \mu_{s+1}(dx_{s+1}) \cdot (p_{s+1}(x_s', x_{s+1}) \wedge p_{s+1}(x_s'', x_{s+1}) \wedge 1) = \\ = (\nu_{s+1}(x_s') \wedge \nu_{s+1}(x_s'') \wedge \mu_{s+1})\,(\mathcal{X}_{s+1}) \geq m_{s+1}\end{gathered}$$

due to (1), and the theorem is proved.