About Yor's problem.

7. Another proof of a standardness theorem

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Consider a Markov chain $\{X_k\}_k$, $X_k \in \mathcal{X}_k$. As before (see Item 2), we may describe it by means of $\mu_k \in \mathcal{P}(\mathcal{X}_k)$ and $\nu_k : \mathcal{X}_{k-1} \to \mathcal{P}(\mathcal{X}_k)$; that is, μ_k is the distribution of X_k, and $\nu_k(x_{k-1})$ is the conditional distribution of X_k for given $X_{k-1} = x_{k-1}$.

Take the following infimum of measures:*

$$\nu_k^{\min} = \operatorname*{ess\,inf}_{x_{k-1}} \nu_k(x_{k-1});$$

that is, $\nu_k^{\min}$ is the greatest measure satisfying the inequality

$$\nu_k^{\min} \le \nu_k(x_{k-1})$$

for μ_{k-1}-almost all $x_{k-1} \in \mathcal{X}_{k-1}$. Of course, $\nu_k^{\min}$ is not a probability measure; consider its total mass:

$$m_k = \nu_k^{\min}(\mathcal{X}_k) = \int_{\mathcal{X}_k} \nu_k^{\min}(dx_k);$$

clearly $m_k \in [0,1]$.

Theorem. If $\sum_k m_k = \infty$, then $\{X_k\}_k$ is tail-trivial, and admits a generating parametrization.

Proof. We have to construct the needed function

$$\alpha_k : [0,1] \times \mathcal{X}_{k-1} \to \mathcal{X}_k$$

for each k. First, choose a function

$$\beta_k : [0, m_k] \to \mathcal{X}_k$$

such that

$$\beta_k(\mathrm{mes}) = \nu_k^{\min}.$$

Second, let

$$\alpha_k(y, x_{k-1}) = \beta_k(y) \quad \text{for } y \le m_k.$$

Third, construct $\alpha_k(y, x_{k-1})$ for $y > m_k$ such that

$$\alpha_k\left(\mathrm{mes}|_{[m_k,1]},\, x_{k-1} \right) = \nu_k(x_{k-1}) - \nu_k^{\min};$$

* It is well-known that the space of measures is a lattice, and any set of measures, bounded from above (or from below) has its supremum (or, correspondingly, its infimum).

then clearly

$$\alpha_k(\mathrm{mes}, x_{k-1}) = \nu_k(x_{k-1})$$

and $\{\alpha_k\}_k$ is a parametrization, indeed. We have to prove that for almost any sequence $\{y_k\}_k$ there exists only one sequence $\{x_k\}_k$ satisfying

$$\forall k \quad x_k = \alpha_k(y_k, x_{k-1}).$$

But the condition $\sum m_k = \infty$ ensures that $y_k \le m_k$ for an infinite set of k. Now,

$$y_k \le m_k \quad \Longrightarrow \quad x_k = \beta_k(y_k),\ x_{k+1} = \alpha_{k+1}(y_{k+1}, x_k),\ \dots$$

and all the x_{k+i} are uniquely determined. We see that

$$\sigma(X_{-\infty}^k) \subset \sigma(Y_{-\infty}^k),$$

so the parametrization is generating. Now, $\sigma(X_{-\infty}^{k-1})$, $\sigma(Y_k^0)$ are independent; hence $\sigma_{-\infty}(X)$, $\sigma(Y_k^0)$ are independent; hence $\sigma_{-\infty}(X)$, $\sigma(Y_{-\infty}^0)$ are independent; hence $\sigma_{-\infty}(X)$, $\sigma(X_{-\infty}^0)$ are independent; and hence $\sigma_{-\infty}(X)$ is trivial. So, the theorem is proved.

Corollary. If $\sum m_k = \infty$ and $\nu_k(x_{k-1})$ is non-atomic for μ_{k-1}-almost all x_{k-1}, then the corresponding chain of σ-fields is standard.

Proof. Due to the above Theorem, the chain is tail-trivial and admits a generating parametrization. And it is conditionally non-atomic. Due to the theorem of the section "Parametrizations," it admits an exact parametrization. Hence, it is standard.

Note. Suppose existence of the density

$$\rho_k(x_k, x_{k-1}) = \frac{\nu_k(x_{k-1})(dx_k)}{\mu_k(dx_k)}.$$

Then

$$m_k = \int_{\mathcal{X}_k} \rho_k^{\min}(x_k)\, \mu_k(dx_k),$$

where

$$\rho_k^{\min}(x_k) = \operatorname*{ess\,inf}_{x_{k-1}} \rho_k(x_k, x_{k-1}).$$