About Yor's problem.

6. Supplement: from discrete to continuous

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501. Let $\mathcal{X}$ be a Borel space. By multiplying it to the Borel space $[0,1]$ we obtain a new Borel space $\mathcal{X} \times [0,1]$.

502. $\mathrm{SM}(\mathcal{X}) \subset \mathrm{SM}(\mathcal{X} \times [0,1])$. More precisely, $\mathrm{SM}(\mathcal{X})$ may be considered embedded into $\mathrm{SM}(\mathcal{X} \times [0,1])$ by identifying a pseudometric $\rho \in \mathrm{SM}(\mathcal{X})$ with the pseudometric

$$\hat\rho((x_1,t_1),(x_2,t_2)) = \rho(x_1,x_2);$$

here $x_1, x_2 \in \mathcal{X}$ and $t_1, t_2 \in [0,1]$.

503. **Lemma.** Let $\mathcal{X}$ be a Borel space, $\rho \in \mathrm{SM}(\mathcal{X})$, and $\hat\rho \in \mathrm{SM}(\mathcal{X}\times[0,1])$ corresponds to ρ as above. Then

$$\hat\rho_{\mathrm{KR}}(\mu \times \mathrm{mes},\ \nu \times \mathrm{mes}) = \rho_{\mathrm{KR}}(\mu,\nu)$$

for any $\mu, \nu \in \mathcal{P}(\mathcal{X})$. (Of course, $\mu \times \mathrm{mes}$ denotes the product of the measure μ on $\mathcal{X}$ and the Lebesgue measure mes on $[0,1]$).

Proof. Use the definition of Kantorovich-Rubinstein metric, given in Item 401 (via Lipschitz functions). Clearly, a Lipschitz function $\hat f$ on $(\mathcal{X} \times [0,1], \hat\rho)$ does not depend on $t \in [0,1]$ and may be identified with a Lipschitz function f on $(\mathcal{X}, \rho)$. Now,

$$\int \hat f \, d(\mu \times \mathrm{mes}) = \int f \, d\mu$$

and so

$$\sup_{\hat f} \left| \int \hat f \, d(\mu \times \mathrm{mes}) \ - \int \hat f \, d(\nu \times \mathrm{mes}) \right| = \sup_{f} \left| \int f \, d\mu \ - \int f \, d\nu \right|,$$

as was to be proved.

504. Let $\{X_k\}_k$ be a random sequence, $X_k \in \mathcal{X}_k$. Form a two-component random sequence $\{\hat X_k\}_k$, $\hat X_k \in \hat{\mathcal{X}}_k = \mathcal{X}_k \times [0,1]$, as follows:

$$\hat X_k = (X_k, X'_k)$$

with X'_k independent, uniform on $[0,1]$, and $\{X'_k\}_k$ independent of $\{X_k\}_k$.

505. What is the "markovization" of such a sequence $\{\hat X_k\}_k$? We have

$$\hat X^k_{-\infty} = (X^k_{-\infty},\ X'^k_{-\infty}),$$

and the conditional distribution of $\hat{X}_k$ for given $\hat{X}_{-\infty}^{k-1}$ is the product measure: the conditional distribution of X_k for given $X_{-\infty}^{k-1}$, multiplied by mes. So, the conditional distribution $\hat{\nu}_k(\hat{x}_{-\infty}^{k-1})$ of $\hat{X}_{-\infty}^k$ for given $\hat{X}_{-\infty}^{k-1} = \hat{x}_{-\infty}^{k-1}$ is essentially $\nu_k(x_{-\infty}^{k-1}) \times \mathrm{mes}$. More exactly, the second term is the infinite product of δ-measures multiplied by mes, but it does not change the following argument.

506. Consider a chain of metrics $\{\rho_k\}_k$ for $\{X_k\}_k$, and form $\hat{\rho}_k$ as in Item 502, that is,

$$\hat{\rho}_k(\,(x_{-\infty}^k, x_{-\infty}'^k),\,(y_{-\infty}^k, y_{-\infty}'^k)\,) = \rho_k(x_{-\infty}^k, y_{-\infty}^k).$$

Then $\{\hat{\rho}_k\}_k$ is a chain of metrics for $\{\hat{X}_k\}_k$. This fact follows from Item 505 and Lemma 503 (slightly modified). And note that the numbers $\overline{\rho}_k$, defined in (210), are insensitive to the distinction between ρ_k and $\hat{\rho}_k$.

507. **Corollary.** Let $\{\hat{X}_k\}_k$ be the two-component random sequence constructed from a random sequence $\{X_k\}_k$ as in Item 504. If $\{X_k\}_k$ admits a chain of metrics with non-zero $\overline{\rho}_{-\infty}$, then $\{\hat{X}_k\}_k$ admits such a chain, too.

508. The distribution of $\{\hat{X}_k\}_k$ is

$$\hat{P} = P \times \mathrm{Mes},$$

where $P \in \mathcal{P}(\mathcal{X}_{-\infty}^0)$ is the distribution of $\{X_k\}_k$, and Mes is the distribution of $\{X_k'\}_k$, that is, Mes is the Lebesgue product measure on the infinite-dimensional cube.

509. If P is equivalent to another distribution P_0, then clearly $P \times \mathrm{Mes}$ is equivalent to $P_0 \times \mathrm{Mes}$. And if P_0 is a product,

$$P_0 = \prod_k \mu_k,$$

then $P_0 \times \mathrm{Mes}$ is a product, too:

$$P \times \mathrm{Mes} \sim P_0 \times \mathrm{Mes} = \prod_k (\mu_k \times \mathrm{mes}).$$

510. But, from general point of view, each measure space $(\mathcal{X}_k \times [0,1],\ \mu_k \times \mathrm{mes})$ is anyway isomorphic to the standard space $([0,1], \mathrm{mes})$. Indeed, all nonatomic measures are isomorphic.

511. **Corollary.** Let $\{\hat{X}_k\}_k$ be the two-component random sequence constructed from a random sequence $\{X_k\}_k$ as in Item 504. If the distribution P of $\{X_k\}_k$ is equivalent to a product measure, then the distribution $\hat{P}$ of $\{\hat{X}_k\}_k$ is equivalent to the corresponding product of nonatomic measures.

512. **Proof of Theorem 101.** Take $\{X_k\}_k$ as in Theorem 103, form $\{\hat{X}_k\}_k$ as in Item 504, and consider $\{f_k(\hat{X}_k)\}_k$, where each $f_k : \hat{\mathcal{X}}_k \to [0,1]$ is some isomorphism between $(\{0,1\} \times [0,1], \mu \times \text{mes})$ and $([0,1], \text{mes})$; here μ is the uniform distribution on the two-element set $\{0,1\}$, so we may choose f_k simply as

$$f_k(x,t) = \frac{x+t}{2} \quad \text{for } x = 0 \text{ or } 1, \text{ and } 0 < t < 1.$$

Now Corollary 511 shows that the distribution of $\{f_k(\hat{X}_k)\}_k$ is equivalent to the Lebesgue product measure. It follows that the sequence is tail-trivial. And Corollary 507 together with Theorem 15 shows that the chain of σ-fields

$$\sigma\{\dots, f_{k-1}(\hat{X}_{k-1}), f_k(\hat{X}_k)\} = \sigma\{\dots, \hat{X}_{k-1}, \hat{X}_k\}$$

is non-standard. So, Theorem 101 is proved.