About Yor's problem.

3. Non-standard σ-fields: proofs

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Deducing Theorem 15 from Lemma 20

27. If $\rho', \rho'' \in \mathrm{CM}(\mathcal{X})$ and $\varepsilon > 0$ are such that $\forall x, y \in \mathcal{X} \quad \rho''(x,y) \le \rho'(x,y) + \varepsilon$, then evidently $\forall \mu, \nu \in \mathcal{P}(\mathcal{X}) \quad \rho''_{\mathrm{KR}}(\mu,\nu) \le \rho'_{\mathrm{KR}}(\mu,\nu) + \varepsilon$. Hence, in the situation of Item 13,

$$\rho''_N \le \rho'_N + \varepsilon \quad \Longrightarrow \quad \forall k \quad \rho''_k \le \rho'_k + \varepsilon. \tag{28}$$

29. It is enough to prove Theorem 15 for any finite-dimensional $\rho_N \in \mathrm{CM}(\mathcal{X}_N)$, that is, ρ_N of the form (6). Indeed, the general case may be approximated with finite-dimensional one uniformly in k, see (6) and (28).

30. It is enough to prove Theorem 15 for ρ_N of the form

$$\rho_N(x'_N, x''_N) = |f(x'_N) - f(x''_N)|$$

with a bounded measurable $f : \mathcal{X}_N \to \mathbb{R}^n$. Indeed, such a metric majorizes a metric from (6):

$$\max_{m=1,\dots,n} |f_m(x'_N) - f_m(x''_N)| \le \left(\sum_m |f_m(x'_N) - f_m(x''_N)|^2 \right)^{1/2} = |f(x'_N) - f(x''_N)|,$$

where $f_1, \dots, f_n$ are coordinate components of f. And each $\tilde\nu$ (see (11)) is evidently isotonic: $\rho' \le \rho'' \quad \Longrightarrow \quad \tilde\nu\rho' \le \tilde\nu\rho''$.

31. Due to Lemma 20, it is enough to prove that

$$\mathbb{E}\,\big(f(X_N) - \mathbb{E}\,(f(X_N) \,|\, Y_{M+1}, \dots, Y_N)\,\big)^2 \;\to\; 0 \quad \text{when } M \to -\infty.$$

But it follows immediately from the fact that $\sigma(X_N) \subset \sigma\{Y_k : k \le N\}$. So, Theorem 15 is deduced from Lemma 20.

Proof of Lemma 20

32. We know (see Item 11), that a map $\nu : \mathcal{X}_0 \to \mathcal{P}(\mathcal{X}_1)$ induces $\tilde\nu : \mathrm{SM}(\mathcal{X}_1) \to \mathrm{SM}(\mathcal{X}_0)$. Now, let α be a one-step parametrization of ν. (It means that $\alpha : [0,1] \times \mathcal{X}_0 \to \mathcal{X}_1$,

$$\forall x_0 \in \mathcal{X}_0 \quad \int_{\mathcal{X}_1} f(x_1)\,\nu(x_0)(dx_1) = \int_0^1 f(\alpha(y, x_0))\,dy$$

for any bounded measurable $f : \mathcal{X}_1 \to \mathbb{R}$.) Define $\tilde{\alpha} : \mathrm{SM}(\mathcal{X}_1) \to \mathrm{SM}(\mathcal{X}_0)$ as follows:

$$(33) \qquad \rho_0 = \tilde{\alpha}\rho_1 \quad \Longleftrightarrow \quad \forall x_0', x_0'' \in \mathcal{X}_0 \quad \rho_0(x_0', x_0'') = \int_0^1 \rho_1(\alpha(y, x_0'), \alpha(y, x_0''))\, dy.$$

Then

$$(34) \qquad \tilde{\nu}\rho_1 \le \tilde{\alpha}\rho_1$$

for any $\rho_1 \in \mathrm{SM}(\mathcal{X}_1)$. It follows immediately from (8): we may take $\xi(t) = \alpha(t, x_0')$ and $\psi(t) = \alpha(t, x_0'')$. (In fact, $(\tilde{\nu}\rho_1)(x_0', x_0'') = \inf_\alpha (\tilde{\alpha}\rho_1)(x_0', x_0'')$).

35. Consider a finite (in time) Markov chain $\{X_M, X_{M+1}, \dots, X_N\}$, with corresponding $\{\mu_M, \dots, \mu_N\}$ and $\{\nu_{M+1}, \dots, \nu_N\}$, as in Lemma 20, and its parametrization $\{\alpha_{M+1}, \dots, \alpha_N\}$. Take a bounded measurable $f : \mathcal{X}_N \to \mathbb{R}^n$ and the corresponding $\rho_N(x_N', x_N'') = |f(x_N') - f(x_N'')|$, as in Lemma 20. But, unlike to Lemma 20, define ρ_k by

$$(36) \qquad \rho_{k-1} = \tilde{\alpha}_k \rho_k \quad \text{for } k = M+1, \dots, N.$$

It is enough to prove Lemma 20 for these ρ_k, because they majorize metrics defined by (14), see (34).

The final variable X_N may be considered a function of the initial one X_M and the parameters:

$$(37) \qquad X_N = \alpha_M^N(Y_N, \dots, Y_{M+1}, X_M),$$

$\{Y_k\}_k$ being uniform independent. For this end, define functions α_k^N recursively:

$$\alpha_{k-1}^N(y_N, \dots, y_{k+1}, y_k, x_{k-1}) = \alpha_k^N(y_N, \dots, y_{k+1}, \alpha_k(y_k, x_{k-1})).$$

We claim that

$$(38) \qquad \rho_M(x_M', x_M'') = \\ = \int_0^1 \dots \int_0^1 \rho_N\left(\alpha_M^N(y_N, \dots, y_{M+1}, x_M'), \alpha_M^N(y_N, \dots, y_{M+1}, x_M'')\right) dy_N \dots dy_{M+1}.$$

Indeed, the equality

$$\rho_k(x_k', x_k'') = \int_0^1 \dots \int_0^1 \rho_N\left(\alpha_k^N(y_N, \dots, y_{k+1}, x_k'), \alpha_k^N(y_N, \dots, y_{k+1}, x_k'')\right) dy_N \dots dy_{k+1}$$

may be proved inductively for $k = N-1, N-2, \dots, M$ by using (36) and (31).

From (38) we obtain

$$\rho_M^2(x'_M, x''_M) \leq$$
$$\leq \int_0^1 \dots \int_0^1 \left| f\left(\alpha_M^N(y_N, \dots, y_{M+1}, x'_M)\right) - f\left(\alpha_M^N(y_N, \dots, y_{M+1}, x''_M)\right) \right|^2 dy_N \dots dy_{M+1}$$

and hence

$$\iint \rho_M^2(x'_M, x''_M)\, \mu_M(dx'_M)\mu_M(dx''_M) \leq$$
$$\leq \int_0^1 \dots \int_0^1 dy_N \dots dy_{M+1} \iint \mu_M(dx'_M)\mu_M(dx''_M)$$
$$\left| f\left(\alpha_M^N(y_N, \dots, y_{M+1}, x'_M) - \alpha_M^N(y_N, \dots, y_{M+1}, x''_M)\right) \right|^2.$$

Applying the general identity

$$\iint |\varphi(x) - \varphi(y)|^2 \mu(dx)\mu(dy) = 2\left(\int |\varphi(x)|^2 \mu(dx) - \left| \int \varphi(x)\, dx \right|^2 \right),$$

we obtain

$$\iint \rho_M^2(x'_M, x''_M)\, \mu_M(dx'_M)\mu_M(dx''_M) \leq$$
$$\int_0^1 \dots \int_0^1 dy_N \dots dy_{M+1}$$
$$2\left(\int \left| f\left(\alpha_M^N(y_N, \dots, y_{M+1}, x_M)\right) \right|^2 \mu_M(dx_M) - \left| \int f\left(\alpha_M^N(y_N, \dots, y_{M+1}, x_M)\right) \mu_M(dx_M) \right|^2 \right);$$

and this is another form of (22). So, Lemma 20 is proved.

Deducing Theorem 17 from Lemma 23

39. For each $n = 0, 1, \dots$ apply Lemma 23 for $N = 2^n N_0$ and combine the results into a Markov chain $\{X_k\}_{k \in \{\dots,-2,-1,0\}}$, $X_k \in \mathcal{X}_k$, $2 \leq |\mathcal{X}_k| < \infty$, with following properties.

(39a) each X_{-2n} (for $n = 0, 1, 2, \dots$) is uniformly distributed within $\mathcal{X}_{-2n}$;

(39b) X_{-2n} and X_{-2n-2} are independent;

(39c) for any metric ρ_0 on $\mathcal{X}_0$, the corresponding sequence of metrics ρ_k (defined following (14)) satisfies the condition

$$\min_{x'_{-2n-2} \neq x''_{-2n-2}} \rho_{-2n-2}(x'_{-2n-2}, x''_{-2n-2}) \geq \exp\left(-(2^n N_0)^{-\alpha}\right) \min_{x'_{-2n} \neq x''_{-2n}} \rho_{-2n}(x'_{-2n}, x''_{-2n}).$$

40. It follows from (39c), that for some $\varepsilon > 0$

$$\forall n \quad \min_{x'_{-2n} \neq x''_{-2n}} \rho_{-2n}(x'_{-2n}, x''_{-2n}) \geq \varepsilon,$$

because

$$\prod_{n=0}^{\infty} \exp\left(-(2^n N_0)^{-\alpha}\right) > 0.$$

Together with (39a) it implies

$$\iint \rho_{-2n}(x,y)\, \mu_{-2n}(dx)\mu_{-2n}(dy) \ge \frac{|\mathcal{X}_{-2n}|-1}{|\mathcal{X}_{-2n}|}\varepsilon \ge \frac{\varepsilon}{2}.$$

41. The sequence of numbers $\iint \rho_k(x,y)\, \mu_k(dx)\mu_k(dy)$ is increasing (in k). It is a general fact; we will prove it for any one-step Markov transition $X_0 \to X_1$. Taking independent ξ, ψ in (8), we obtain

$$\rho_{\text{KR}}(\mu,\nu) \le \iint \rho(x,y)\, \mu(dx)\mu(dy).$$

Hence,

$$\begin{aligned}
&\iint \rho_0(x_0', x_0'')\, \mu_0(dx_0')\mu_0(dx_0'') = \\
&\quad = \iint \rho_{1,\text{KR}}(\nu(x_0'), \nu(x_0''))\, \mu_0(dx_0')\mu_0(dx_0'') \le \\
&\quad \le \iint \mu_0(dx_0')\mu_0(dx_0'') \iint \nu(x_0')(dx_1')\nu(x_0'')(dx_1'')\rho_1(x_1', x_1'') = \\
&\quad = \iint \left(\int \mu_0(dx_0')\nu(x_0')\right)(dx_1') \cdot \left(\int \mu_0(dx_0'')\nu(x_0'')\right)(dx_1'')\, \rho_1(x_1', x_1'') \\
&\quad = \iint \mu_1(dx_1')\mu_1(dx_1'')\, \rho_1(x_1', x_1'').
\end{aligned}$$

42. From (40) and (41) we see that

$$\iint \rho_k(x,y)\, \mu_k(dx)\mu_k(dy) \ge \frac{\varepsilon}{2} \quad \text{for all } k,$$

so, (18) is satisfied.

43. It remains to prove that $\{X_k\}_k$ is tail-trivial. This fact follows from (39b). Indeed, (39b) together with Markov property shows that $\sigma\{X_k : k \le -2n-2\}$ and $\sigma\{X_k : k \ge -2n\}$ are independent. Hence, $\sigma_{-\infty}$ is independent of $\sigma\{X_k : k \ge -2n\}$ for any n. Now Theorem 17 is deduced from Lemma 23.

Proof of Lemma 23

44. Take some prime integer p and consider the five-dimensional linear space $\mathbb{Z}_p^5$ over the finite field $\mathbb{Z}_p$. Let $\mathcal{X}_0$ be the set of all two-dimensional linear subspaces ("planes" in

what follows) of $\mathbb{Z}_p^5$, $\mathcal{X}_1$ —the set of all one-dimensional affine subspaces (that is, translated linear subspaces; "lines" in what follows) of $\mathbb{Z}_p^5$, and $\mathcal{X}_2 = \mathbb{Z}_p^5$ —the set of all points of $\mathbb{Z}_p^5$.

For given plane $x_0 \in \mathcal{X}_0$, the Markov process may jump from x_0 to any line $x_1 \in \mathcal{X}_1$ that is parallel to the plane x_0, that is, x_1 is a translation of a one-dimensional linear subspace lying in the plane x_0 (the case $x_1 \subset x_0$ is permitted, too). The process jumps to any such x_1 equiprobably.

For a given line $x_1 \in \mathcal{X}_1$, the Markov process may jump from x_1 to any point $x_2 \in \mathcal{X}_2$ lying on the line x_1, equiprobably.

45. Each $x_0 \in \mathcal{X}_0$ leads, after two Markov jumps, to the uniform distribution on $\mathcal{X}_2$. Indeed, after the first jump the distribution becomes translation invariant, and remains translation invariant after the second jump. So, X_2 is uniformly distributed and independent of X_0. The distribution of X_0 may be choosen arbitrarily; we choose it as the uniform one.

46. We have $|\mathcal{X}_0| \ge p^6$. Indeed, each pair of points is contained in a plane (at least one). Each plane contains p^2 points and hence p^4 pairs. And the whole number of pairs in $\mathbb{Z}_p^5$ is p^{10}.

47. Take the metric ρ_2 on $\mathcal{X}_2$ such that $\rho_2(x_2', x_2'') = 1$ for all $x_2' \ne x_2''$. The corresponding Kantorovich-Rubinstein metric coincides with the norm metric:

$$\rho_{2,\mathrm{KR}}(\mu, \nu) = \frac{1}{2} ||\mu - \nu||.$$

Hence,

$$\rho_1(x_1', x_1'') = \rho_{2,\mathrm{KR}}(\nu_2(x_1'), \nu_2(x_1'')) = \frac{1}{2} ||\nu_2(x_1') - \nu_2(x_1'')|| = \frac{|x_1'| - |x_1' \cap x_1''|}{|x_1'|};$$

but $|x_1'| = p$ (the number of points on a line), and $|x_1' \cap x_1''| \le 1$ (the number of common points of two lines). So,

$$\rho_1(x_1', x_1'') \ge 1 - \frac{1}{p} \quad \text{for any } x_1' \ne x_1''.$$

48. The same argumentation is applicable for $X_0 \to X_1$ transition, giving the factor $\ge 1 - \frac{1}{p+1}$. Indeed, the number of lines containing 0 and contained in a given plane is equal to $p+1$. Only one of these lines is contained in another given plane. And the condition that the lines contain 0 does not affect the ratio.

49. It remains to choose $\alpha > 0$ and $N_0 < \infty$ such that for any $N \ge N_0$ there exists a prime p such that $p^6 \ge 2N$, $p^5 \le N$, and

$$\left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{p+1}\right) \ge \exp\left(-N^{-\alpha}\right).$$

Surely it is possible. So, Lemma 23 is proved.