About Yor's problem.

2. Non-standard σ-fields: formulations

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1. A Borel space $\mathcal{X}$ determines the set $\mathcal{P}(\mathcal{X})$ of all probability measures on $\mathcal{X}$. The set $\mathcal{P}(\mathcal{X})$ is a Borel space, too.

2. A convenient way to determine the distribution of a Markov chain $\{X_k\}_{k\in\mathbb{Z}}$, $X_k \in \mathcal{X}_k$, is to give for each k a measure $\mu_k \in \mathcal{P}(\mathcal{X}_k)$ and a measurable map $\nu_k : \mathcal{X}_{k-1} \to \mathcal{P}(\mathcal{X}_k)$ such that

$$\mu_k = \int \nu_k(x)\, \mu_{k-1}(dx). \tag{3}$$

4. For a Borel space $\mathcal{X}$ introduce the set $\mathrm{SM}(\mathcal{X})$ of all measurable pseudometrics $\rho(\cdot,\cdot)$ on $\mathcal{X}$, turning $\mathcal{X}$ into a separable metric space after identifying points having zero distance.

A general form of such a metric is

$$\rho(x,y) = \sup_n |f_n(x) - f_n(y)|, \tag{5}$$

where $f_1, f_2, \dots$ are measurable functions on $\mathcal{X}$ such that the supremum is finite everywhere.

Distinguish the set $\mathrm{CM}(\mathcal{X}) \subset \mathrm{SM}(\mathcal{X})$ of pseudometrics giving a precompact space (that is, compact after completion). Such a metric may be described by (5) under condition that each f_n is bounded, and the finite-dimensional pseudometric

$$\rho_n(x,y) = \sup_{m=1,\dots,n} |f_m(x) - f_m(y)| \tag{6}$$

converges when $n \to \infty$ to $\rho(x,y)$ uniformly in x, y.

7. Any pseudometric $\rho \in \mathrm{SM}(\mathcal{X})$ determines the corresponding Kantorovich-Rubinstein pseudometric $\rho_{\mathrm{KR}} \in \mathrm{SM}(\mathcal{P}(\mathcal{X}))$ as follows:

$$\rho_{\mathrm{KR}}(\mu,\nu) = \inf \left\{ \int_0^1 \rho(\xi(t),\psi(t))\, dt : \xi,\psi : [0,1] \to \mathcal{X},\ \xi(\mathrm{mes}) = \mu,\ \psi(\mathrm{mes}) = \nu \right\}; \tag{8}$$

here $\mu,\nu \in \mathcal{P}(\mathcal{X})$, mes is the Lebesgue measure on $[0,1]$, and $\xi(\mathrm{mes}) = \mu$ means that ξ is a measure-preserving map from $([0,1],\mathrm{mes})$ to $(\mathcal{X},\mu)$.

9. If $\rho \in \mathrm{CM}(\mathcal{X})$, then $\rho_{\mathrm{KR}} \in \mathrm{CM}(\mathcal{P}(\mathcal{X}))$. Further, if $\rho \in \mathrm{CM}(\mathcal{X})$ is a metric (that is, it separates points), then ρ_{KR} provide $\mathcal{P}(\mathcal{X})$ with the weak topology:

$$\rho_{\mathrm{KR}}(\mu_n,\mu) \underset{n}{\to} 0 \quad \Longleftrightarrow \quad \forall f \in C(\mathcal{X},\rho) \quad \int f\, d\mu_n \underset{n}{\to} \int f\, d\mu. \tag{10}$$

11. A measurable map $\nu : \mathcal{X}_0 \to \mathcal{P}(\mathcal{X}_1)$ induces its associate map $\tilde{\nu} : \mathrm{SM}(\mathcal{X}_1) \to \mathrm{SM}(\mathcal{X}_0)$ as follows:

$$\rho_0 = \tilde{\nu}\rho_1 \quad \Longleftrightarrow \quad \forall x, y \in \mathcal{X}_0 \quad \rho_0(x,y) = \rho_{1,\mathrm{KR}}(\nu(x), \nu(y)). \tag{12}$$

Also, $\tilde{\nu} : \mathrm{CM}(\mathcal{X}_1) \to \mathrm{CM}(\mathcal{X}_0)$.

13. When a Markov chain is given for $k \in (-\infty, N] \cap \mathbb{Z}$ by means of $\{\mu_k\}_k$, $\{\nu_k\}_k$ as in (2), then any $\rho_N \in \mathrm{SM}(\mathcal{X}_N)$ determines a sequence $\{\rho_k\}_k$, $\rho_k \in \mathrm{SM}(\mathcal{X}_k)$, as follows:

$$\rho_{k-1} = \tilde{\nu}_k \rho_k \quad \text{for } k \le N. \tag{14}$$

15. **Theorem.** If a Markov chain admits a generating parametrization and is tail-trivial, then

$$\iint \rho_k(x,y)\, \mu_k(dx)\mu_k(dy) \to 0 \quad \text{when } k \to -\infty \tag{16}$$

for any $\rho_N \in \mathrm{CM}(\mathcal{X}_n)$.

17. **Theorem.** There exists a Markov chain with finite sets $\mathcal{X}_k$, tail-trivial, and satisfying

$$\liminf_{k \to -\infty} \iint \rho_k(x,y)\, \mu_k(dx)\mu_k(dy) > 0 \tag{18}$$

for any metric ρ_N on $\mathcal{X}_N$.

19. **Corollary.** There exists a non-standard tail-trivial, conditionally non-atomic chain of σ-fields.

Proof. The Markov chain of Theorem 17 may be easily converted into a conditionally non-atomic one by taking the two-component Markov process (X_k, X'_k) with $\{X_k\}_k$ as in Theorem 17, and X'_k i.i.d. and independent of $\{X_k\}_k$. Then, taking ρ_N non-depending on the additional component, we keep ρ_k essentially the same as in (18). So, (18) remains true, and according to Theorem 15 the chain admits no generating parametrization. Hence, the corresponding chain of σ-fields is non-standard.

The following lemma is the key to Theorem 15.

20. **Lemma.** Let a Markov chain is given for $k \in [M, N] \cap \mathbb{Z}$ by means of $\{\mu_k\}_k$, $\{\nu_k\}_k$ as in (2). Define $\rho_N \in \mathrm{CM}(\mathcal{X}_N)$ as

$$\rho_N(x,y) = |f(x) - f(y)| \tag{21}$$

with a measurable bounded $f : \mathcal{X}_N \to \mathbb{R}$. Then for any parametrization

$$\iint \rho_M^2(x,y)\, \mu_M(x)\mu_M(y) \le 2\mathbb{E}\,\big(f(X_N) - \mathbb{E}\,(f(X_N) \,|\, Y_{M+1}, \dots, Y_N)\,\big)^2. \tag{22}$$

The same remains true, if f takes its values in an Euclidean or Hilbert space instead of $\mathbb{R}$.

The following lemma is the key to Theorem 17.

23. **Lemma.** There exist $\alpha > 0$ and $N_0 < \infty$ such that for any $N \ge N_0$ there exists a Markov chain for $k \in \{0, 1, 2\}$, with finite sets $\mathcal{X}_k$, satisfying the following conditions.

(a) $|\mathcal{X}_0| \ge 2N$; $|\mathcal{X}_2| \le N$
($|\mathcal{X}|$ denoting the number of elements in $\mathcal{X}$).

(b) X_0 and X_2 are independent and uniformly distributed (within $\mathcal{X}_0$ and $\mathcal{X}_2$, correspondingly).

(c) If a metric ρ_2 on $\mathcal{X}_2$ is defined by

$$\rho_2(x, y) = 1 \quad \text{for } x \ne y,$$

then the corresponding metric ρ_0 on $\mathcal{X}_0$ satisfy

$$\rho_0(x, y) \ge \exp(-N^{-\alpha}) \quad \text{for } x \ne y.$$

24. Let me note, in addition, the following "probability-free" fact, which seems to be the "cause" of the existence of non-standard chain of σ-fields.

Any separable metric space $(\mathcal{X}, \rho)$ may be considered embedded into the space $(\mathcal{P}(\mathcal{X}), \rho_{\text{KR}})$, the Kantorovich-Rubinstein extension of the given space. And the new space may be extended, too. By continuing this way, we obtain a huge metric space—the union of the sequence of successive extensions:

$$\mathcal{X}_\infty = \bigcup_n \mathcal{X}_n, \quad \mathcal{X}_{n+1} = \mathcal{P}(\mathcal{X}_n), \quad \mathcal{X}_0 = \mathcal{X}. \tag{25}$$

26. **Proposition.** If a metric space $\mathcal{X}$ contains more than one point, then $\mathcal{X}_\infty$ is not precompact.