About Yor's problem.

1. Parametrizations

Boris Tsirelson (Tel Aviv University)

Consider a Markov distribution P, that is, the distribution of a Markov chain $\{X_k\}_k$, $X_k \in \mathcal{X}_k$ ($\mathcal{X}_k$ is a Borel space), k runs over $\mathbb{Z}$, or over a segment (finite or half-infinite) of $\mathbb{Z}$.

It is easy to see that the two following definitions are essentially equivalent.

(A) A parametrization of P is a sequence of measurable maps

$$\alpha_k : [0,1] \times \mathcal{X}_{k-1} \to \mathcal{X}_k$$

such that

$$\mathbb{E}(\, f(X_k) \mid X_{k-1}\,) = \int_0^1 f(\alpha_k(y, X_{k-1}))\, dy$$

for any k and any bounded measurable $f : \mathcal{X}_k \to \mathbb{R}$.

(B) A parametrization of P is a two-component random sequence $\{(X_k, Y_k)\}_k$ (on a probability space) such that

the distribution of $\{X_k\}_k$ coincides with P,

Y_k are independent and uniform on $[0,1]$,

$\forall k \quad \sigma(X_k) \subset \sigma(Y_k, X_{k-1})$,

$\forall n \quad \sigma\{Y_k : k > n\}$ is independent of $\sigma\{X_k, Y_k : k \le n\}$.

The connection between (A) and (B) is given by

$$X_k = \alpha_k(Y_k, X_{k-1}). \qquad (*)$$

For any $\{\alpha_k\}_k$ as in (A) we may build $\{(X_k, Y_k)\}_k$ as in (B), obeying (*). And conversely, for any $\{(X_k, Y_k)\}_k$ as in (B) we may build $\{\alpha_k\}_k$ as in (A), obeying (*).

A parametrization will be called *non-redundant*, if

$$\forall k \quad \sigma(Y_k) \subset \sigma(X_k, X_{k-1}).$$

An equivalent condition:

$$\sigma(X_{k-1}, Y_k) = \sigma(X_{k-1}, X_k).$$

For the case when k may tend to $-\infty$ (that is, k runs over $\mathbb{Z}$ or $(-\infty, n] \cap \mathbb{Z}$):

A parametrization will be called *generating*, if

$$\forall n \quad \sigma\{X_k : k \le n\} \subset \sigma\{Y_k : k \le n\} \vee \sigma_{-\infty}(X),$$

where $\sigma_{-\infty}(X)$ is the tail σ-algebra,

$$\sigma_{-\infty}(X) = \bigcap_n \sigma\{X_k : k \le n\}.$$

A parametrization will be called *exact,* if it is both generating and non-redundant. In this case

$$\forall n \quad \sigma\{X_k : k \le n\} = \sigma\{Y_k : k \le n\} \vee \sigma_{-\infty}(X).$$

And if in addition X is *tail-trivial,* that is, $\sigma_{-\infty}(X) = \{\emptyset, \Omega\}$, then

$$\forall n \quad \sigma\{X_k : k \le n\} = \sigma\{Y_k : k \le n\}.$$

A Markov distribution P is called *conditionally non-atomic,* iff for all k and almost all $x \in \mathcal{X}_{k-1}$ the conditional distribution of X_k for given $X_{k-1} = x$ is non-atomic.

Theorem. Let a Markov distribution P is conditionally non-atomic and tail-trivial. If P admits a generating parametrization, then it admits an exact parametrization.

Each one of α_k may be called a *one-step parametrization.* So, a parametrization is nothing else but a collection of one-step parametrizations. The "non-redundant" property is local: it is imposed on each one step component. On the contrary, the "generating" property is global.

Consider a one-step Markov measure P_1; that is, now k runs over $\{0, 1\}$ only, so we have one Markov step $X_0 \to X_1$. The space A of all (one-step) parametrizations α becomes a metrizable topological space, being equipped with the following topology (i is an index rather than exponent):

$$\alpha^i \to \alpha \quad \text{when } i \to \infty,$$

iff

$$\mathbb{E}\, |\, f(\alpha^i(Y_1, X_0) - f(\alpha(Y_1, X_0))\, | \to 0 \quad \text{when } i \to \infty$$

for any bounded measurable $f : \mathcal{X}_1 \to \mathbb{R}$. Or, what is equivalent, iff

$$\mathbb{E}\, |\, f(X_0, \alpha^i(Y_1, X_0) - f(X_0, \alpha(Y_1, X_0))\, | \to 0 \quad \text{when } i \to \infty$$

for any bounded measurable $f : \mathcal{X}_0 \times \mathcal{X}_1 \to \mathbb{R}$.

Lemma 1. If a one-step Markov measure is conditionally non-atomic, then the set of all non-redundant parametrizations is dense in the space of all parametrizations.

Now we return to the many-step case.

Lemma 2. If a parametrization $\{\alpha_k\}_k$ is generating, and each α_k is the limit of a sequence of α_k^i when $i \to \infty$ (each α_k^i being a one-step parametrization of the $X_{k-1} \to X_k$ transition), then there are $i_1, i_2, \dots$ such that the parametrization

$$\{\alpha_k^{i_k}\}_k$$

is generating, too.