A QUANTITATIVE CRITERION OF THE APPLICABILITY OF THE CLASSICAL DESCRIPTION WITHIN THE QUANTUM THEORY

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1. INTRODUCTION

The Copenhagen interpretation stipulates for a joint application of the two description modes, the classical and the quantum ones, to the physical world, which apparently is "connected" in the sense that it cannot be naturally (from the nature of things) divided into two realms, well-separated with some gap, one being covered with the quantum description, the other - with the classical one. The Copenhagen interpretation specifies an interface between the application domains (not "applicability domains"!) of the two description modes; being sharp, this interface is conventional and movable. It can be shifted within the limits of a "neutral domain", where both description modes are applicable and conform; this is nothing but the correspondence principle for the quantum and classical theories. The "neutral domain" is supposed to be roomy enough to contain all physical phenomena immediately perceivable by human observers and also all those which can be successfully described by classical physics and treated as a part of the classically described macroscopic environment.

The physical contents of the correspondence principle is obviously connected with the roominess of the "neutral area". Is it really so vast as indicated above? That was declared rather than proved. Two main approaches are known to the problem of the quantum-classical correspondence: "the quasi-classical limit for $h \to 0$" and the Everett's approach.

The first approach seems thorough, being based on Hamiltonian mechanics and on the short-wave asymptotics for the Schrödinger equation. However, Hamiltonian dynamics can turn (rather quickly, during the so-called Ehrenfest time) a microscopically localized wave function into a macroscopically delocalized one. Indeed, fast divergence of trajectories is ordinary for non-well-integrable systems of classical mechanics.

The second (Everett's) approach obtains the correspondence principle from a certain scenario of the quantum correlation propagation. However, the mentioned scenario is over--simplified. The correlation propagation is described by highly abstract models for interactions; their correspondence to more realistic models remains problematic, as well as quantitative criteria.

We propose a new approach to the problem of the quantum-classical correspondence, combining features of the two mentioned approaches and by virtue of that free from their drawbacks pointed above. Our approach uses Hamiltonian mechanics and the short-wave asymptotics for the Schrödinger equation, and also the quantum correlation propagation. The former saves us from over-simplifying the interactions description (our description is realistic in some cases, but, of course, not in all cases), the latter - from the delocalization of wave functions.

We have arrived at this approach when developing investigations of Bell-type inequalities for quasi-classical systems, which can be described by localized wave functions moving along the classical trajectories; in this case the quantum theory provides small corrections to classical predictions. It was natural to suppose that there are quantum corrections to the Bell inequalities too. Just so was this problem treated in our previous paper[1]. It was shown there that the maximum value of the possible violation of the Bell-CHSH inequality for a quasi-classical system depends essentially on such factors as the accuracy of measurements and the thermal fluctuations intensity; these factors introduce discrepancies. It was shown that in the absence of discrepancies the Bell-CHSH inequality may be violated for quasi-classical systems to the same extent as for spin systems. In the opposite case, when the quantum uncertainties are small in comparison with the discrepancies, we have obtained an upper bound vor violations of the Bell-CHSH inequality showing that these violations are small. However, no lower bound was given. Further investigations presented here led us to an unexpected result. It turns out that, if the discrepancies exceed the quantum uncertainties, then a classical description with hidden variables is possible and therefore the Bell inequalities hold exactly, without any corrections. (All hidden variables are assumed local throughout this paper).

Our approach inherits from investigations of Bell inequalities the idea that the informal notion of the possibility of a classical description may be formalized as the possibility of a description with hidden variables. And hidden variables are treated here somewhat non-traditionally; instead of a continuous-in-time description, we obtain a "dotted-in-time" one. The continuous evolution of the quantum state of a quasi-classical system, weakly interacting with a heat bath, is accompanied

(in the sense explained throughout this paper) by a discrete--in-time evolution of the classical state. The time pitch is nothing but the characteristic time for the process of establishing quantum correlations between the system and the bath. And it is determined by the thermal fluctuations intensity. This time can be estimated easily by means of the method presented here and involving fluctuation-dissipation relations. Several examples are given. For macroscopical systems belonging to the above-mentioned "neutral domain" the time pitch turns out to be small, and the sequence of classical states turns out to approximate the classical trajectory.

The mathematical examination of the presented approach is due to B.S.Tsirelson. The general outlines were formed in the process of authors' collaboration.

2. QUANTUM MEASUREMENTS OF LIMITED ACCURACY AND BELL INEQUALITIES

Apparently there is no accepted rigorous definition of the quantum measurement accuracy. We accept that a measurement of a coordinate with an accuracy Δq is equivalent (of course, in the outcomes' distribution, not in affecting the object) to the exact coordinate measurement followed by adding $\zeta\Delta q$ to the outcome; here ζ is a normally distributed random variable with zero mean and unity variance. A measurement of a momentum with an accuracy Δp is understood similarly. We want to realize, how the possibility of the Bell inequality violation is related to $\Delta q\Delta p$. This problem was considered in our paper[1]; here we obtain essentially more complete results by virtue of use of the "measurement compatibility" notion. Measurements of limited accuracy are, of course, not ideal ones, therefore it is better to avail of the general treatment of a quantum measurement (see for example[2]) as a transformation of quantum states into classical ones:

$$M : L_1(H) \longmapsto L_1(X);$$

H is the Hilbert space, describing the quantum system; $L_1(H)$ denotes the Banach space of all Hermitian trace-class operators in H; X is a measurable space describing possible measurement outcomes; $L_1(X)$ denotes the Banach space of all measures on X having densities with respect to some prescribed "background" measure (or, what is the same, the space of all functions integrable with respect to this "background" measure). The map M, describing a measurement, must be linear, positive and normalization preserving; that is, if $W\in L_1(H)$, $f\in L_1(X)$ and $f=M(W)$, then $Tr(W)=\int f$ and $W\geq 0 \Rightarrow f\geq 0$. We set aside some possible mathematical refinements related to the infinite dimension, because our conclusions are in fact insensitive

to them. The conjugate transformation $M^*: L_\infty(X) \rightarrowtail L_\infty(H)$ maps the commutative algebra $L_\infty(X)$ of classical observables into the non-commutative C^*-algebra $L_\infty(H)$ of quantum observables. For the measurement outcome the probability of hitting a subset Y of X equals $\int_Y \rho$, where $\rho=M(W)$, W is the density matrix; this probability can be written as $\langle M^*(\mathbb{1}_Y)\rangle_W = \mathrm{Tr}(W\cdot M^*(\mathbb{1}_Y))$, where $\mathbb{1}_Y$ is the indicator of the event Y (the function that equals 1 on Y and 0 out of Y). The possibility of a violation of the Bell-CHSH inequality (one of the family of Bell inequalities) for quantum objects means in this terms that there exist two Hilbert spaces H_1, H_2, four measurements $M_{nm}: L_1(H_n) \rightarrowtail L_1(X_{nm})$ $(1\le n\le 2, 1\le m\le 2)$, four classical observables $f_{nm}\in L_\infty(X_{nm})$ taking values ± 1 only, and a density matrix on the tensor product $W\in L_1(H_1\otimes H_2)$ such that

$$| \langle A_1\otimes B_1 + A_1\otimes B_2 + A_2\otimes B_1 - A_2\otimes B_2 \rangle_W | > 2,$$

where $A_k=M^*_{1k}(f_{1k})\in L_\infty(H_1)$, $B_l=M^*_{2l}(f_{2l})\in L_\infty(H_2)$. The existence of such W is equivalent to the relation

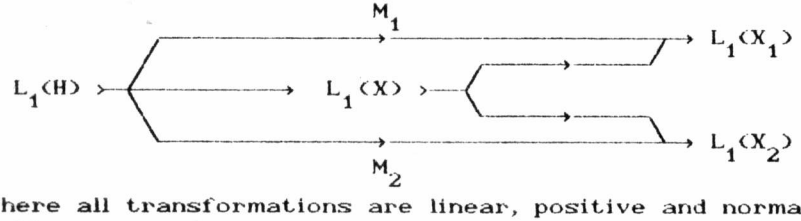
$$\| A_1\otimes B_1 + A_1\otimes B_2 + A_2\otimes B_1 - A_2\otimes B_2 \| > 2,$$

which can be written as

$$\| \Sigma_{kl}\ \tau_{kl} M^*_{1k}(f_{1k})\otimes M^*_{2l}(f_{2l}) \| > 2;$$

here and in what follows $\tau_{11}=\tau_{12}=\tau_{21}=+1$, $\tau_{22}=-1$. In the case of arbitrary measurements M_{nm} the above norm can be equal to $2\sqrt{2}$, but cannot exceed this boundary (see [1]).

2.1. Definition. Two measuremets $M_k: L_1(H) \rightarrowtail L_1(X_k)$, k=1,2, are *compatible*, if there is a commutative diagram



where all transformations are linear, positive and normalization preserving.●

This is equivalent to a definition in [2] (see Ch. 2, § 6). The Bell inequalities can result from the measurements compatibility.

2.2. Theorem. Let H_1, H_2 be Hilbert spaces and $M_k: L_1(H_1) \rightarrowtail L_1(X_k)$ are compatible measurements $(k=1,2)$. Then

for every $f_k \in L_\infty(X_k)$ such that $\|f_k\| \le 1$ and every $B_l \in L_\infty(H_2)$ such that $\|B_l\| \le 1$ $(l=1,2)$

$$\|\Sigma_{kl}\ \tau_{kl} M_k^*(f_k) \otimes B_l\| \le 2.$$

Proof. By virtue of the compatibility, the two measurements M_l can be replaced by one measurement $M : L_1(H_1) \to L_1(X)$, two functions f_k being transferred to X (see the diagram in 2.1). It is sufficient to prove that $\| \Sigma\ \tau_{kl} M^*(f_k) \otimes B_l \| \le 2$ for $f_k \in L_\infty(X)$, $\|f_k\| \le 1$. For any $x \in X$ $\| \Sigma\ \tau_{kl} f_k(x) B_l \| =$
$= \| f_1(x)B_1 + f_1(x)B_2 + f_2(x)B_1 - f_2(x)B_2 \| \le |f_1(x)+f_2(x)| \cdot \|B_1\| +$
$+ |f_1(x)-f_2(x)| \cdot \|B_2\| \le 2$, hence $\| \Sigma\ \tau_{kl} f_k \otimes B_l \| \le 2$. It only remains to prove that the transformation $M^* \otimes \mathbb{1} : L_\infty(X) \otimes L_\infty(H_2) \to$
$\to L_\infty(H_1) \otimes L_\infty(H_2) = L_\infty(H_1 \otimes H_2)$ is positive (then it cannot increase the norm, because it preserves the identity operator). And this follows from the Stinespring's[3] Theorem 4, according to which the positive transformation M^* with a commutative domain is completely positive.●

The proved theorem shows that $\|\Sigma \tau_{kl} M_{1k}^*(f_{1k}) \otimes M_{2l}^*(f_{2l})\| \le 2$, that is, the Bell-CHSH inequality holds provided at least one of two measurement pairs $\{M_{11}, M_{12}\}$, $\{M_{21}, M_{22}\}$ satisfies the compatibility condition. Comparing the above proof with the classical proof of the Bell inequality we see that a point x of X is nothing but a hidden variable.

Armed with the mathematical tools presented above, we return to measurements of limited accuracy. The measurement of a coordinate with an accuracy Δq is the transformation $L_1(H) \to L_1(-\infty,+\infty)$ which transforms W into ρ according to the equation

$$\rho(q) = \frac{1}{\sqrt{2\pi} \cdot \Delta q} \left\langle \exp\left[-\frac{1}{2}\left(\frac{Q-q}{\Delta q}\right)^2\right] \right\rangle_W ;$$

here Q is the coordinate operator. And the same holds for a momentum. It is known[2] that these measurements are compatible provided $\Delta q \Delta p \ge \hbar/2$. We see that the violation of Bell inequalities in a spinless coordinate-momentum Gedankenexperiment, discussed in [1], Sect. 3, requires the measurement accuracy high enough. In this case inadequate accuracy cannot be compensated with any reiterations of the experiment.

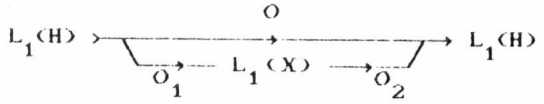
3. OPERATIONS DESTROYING QUANTUM CORRELATIONS

The quantum theory asserts that Bell inequalities can be violated in an experiment with two particles or, more generally, two quantum objects, jointly prepared with one source and

then separately measured with two instruments. And it is of principle that the quantum object transmission from the source to the instrument cannot be replaced with any classically describable signal transmission whatsoever (since otherwise the classical proof of Bell inequalities would have been applicable). On the other hand, in quasi-classical situations quantum states can be approximated by classical ones. Can this reason be used to obtain Bell inequalities?

Any quantum object transmission can be approximated to a certain extent by a classical signal transmission according to the following plan. At first, one subjects the quantum object to a quantum measurement, getting some aggregate of numbers (or, more generally, some classical state). Then one prepares a quantum state (of the same or another but identical object), using the above numbers as classical parameters affecting the prepared state. As it were, the classical system absorbs and re-emits the quantum object. An operation which transforms one quantum state into another according to such plan will be called here a classically factorizable operation. The rigorous definition follows (H, X, $L_1(H)$, $L_1(X)$ are as in Sect. 2).

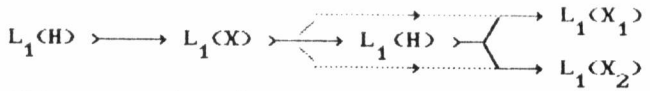
3.1. Definition. A linear transformation $O : L_1(H) \rightarrowtail L_1(H)$ is called a *classically factorizable operation*, if there is a commutative diagram



where transformations O_1, O_2 are linear, positive and normalization preserving. ●

The class of all classically factorizable operations over H will be denoted by Opf(H). It is a part of the class Op(H) of all operations over H (see [4] for a definition); this follows easily from Stinespring's[3] Theorems 3, 4, showing that any positive transformation is completely positive provided at least one of the two algebras (the domain and the range) is commutative. It is clear that O_1 describes a measurement (or absorbtion), O_2 - a preparation (or re-emitting).

If $O \in Opf(H)$, then measurements of the form M_1O, M_2O are compatible for arbitrary measurements $M_k : L_1(H) \rightarrowtail L_1(X_k)$; this is evident by virtue of the diagram



(In fact, this property of an operation is equivalent to its

classical factorizability.) This statement coupled with Theorem 2.2 leads to the following result.

3.2. Theorem. Let H_1, H_2 be Hilbert spaces, and $O \in Opf(H_1)$. Then for any Hermitian operators A_1, A_2 in H_1 and B_1, B_2 in H_2, having norm ≤ 1,

$$\| O^*(A_1)\otimes B_1 + O^*(A_1)\otimes B_2 + O^*(A_2)\otimes B_1 - O^*(A_2)\otimes B_2 \| \leq 2. \bullet$$

So any violation of Bell inequalities becomes impossible if at least one of the two quantum objects is subjected to a classically factorizable operation. And it is not at all necessary that the object be actually absorbed and re-emitted by a classical system. We will see that in many cases an equivalent effect is caused by thermal fluctuations in a medium between the source and the instrument.

Theorem 3.2 was formulated essentially as a property of the transformation $O^*\otimes\mathbb{1}$ in $L_\infty(H_1\otimes H_2)$; it transforms observables of the form $\sum \tau_{kl} A_k \otimes B_l$ into observables with the norm ≤ 2 (τ_{kl} was introduced in Sect. 2). It is natural to ask, what is remarkable in the transformation $O\otimes\mathbb{1}$ in $L_1(H_1\otimes H_2)$. It turns out that it transforms any density matrix into a mixture of non-correlated (that is, having the form $W_1\otimes W_2$) density matrices. And this means destroying quantum correlations between two correlated objects when at most one of them is subjected to a classically factorizable operation. Only classical correlations remain, which are describable by means of a classical parameter affecting both objects. Of course, this rules out the possibility of any violation of the Bell inequalities.

The class $Op(H)$ of all operations is vast, as well as its subclass $Opf(H)$. In the rest of this paper we confine ourself to "Gaussian" operations defined below. A special part played by such operations will be argued in Sect. 4. The irreducible representation of the finite-dimensional canonical commutation relations is supposed given in the Hilbert space H. The Weyl correspondence is well-known[5] between operators in H and functions on the phase space (which is finite-dimensional symplectic linear space); the function corresponding to an operator is called its Weyl symbol. The Weyl symbol of a density matrix is known as its Wigner distribution too. A density matrix is called[2] Gaussian if its Wigner distribution is equal to the density of a Gaussian measure. And the last obeys a known condition[2] connected with the Heisenberg uncertainty relation. It is convenient to formulate this condition in terms of two bilinear forms on the space L of all linear real-valued functions on the phase space, namely, the Poisson bracket $\{f, g\}$ considered as a skew real-valued bilinear form on L, and

the covariation form $Cov_\gamma(f,g) = \int fg d\gamma - (\int f d\gamma)\cdot(\int g d\gamma)$ or rather the corresponding norm $\|f\|_\gamma = \sqrt{Cov_\gamma(f,f)}$. So a Gaussian measure γ on the phase space is a Wigner distribution if and only if

$$h\cdot\{f,\ g\} \le 2\|f\|_\gamma\|g\|_\gamma \qquad (1)$$

for all $f,g\in L$.

3.3. Definition. An operation $O\in Op(H)$ is called *Gaussian* if there is a Gaussian measure γ on the phase space such that for any $W\in L_1(H)$ the Wigner distributions w for W and w_1 for $W_1=O(W)$ are connected through the convolution relation $w_1 = \gamma * w$, that is, $w_1(x) = \int w(x+y)\gamma(dy)$. ●

It is easy to see that one and only one Gaussian operation O_γ exists for any Gaussian measure γ on the phase space. See also [1], Sect. 3.

3.4. Theorem. A Gaussian operation O_γ is classically factorizable provided

$$h\cdot\{f,\ g\} \le \|f\|_\gamma\|g\|_\gamma \qquad (2)$$

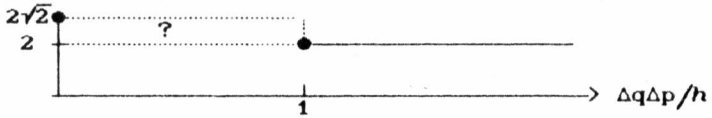
for all $f,g\in L$.

Proof. Consider the Gaussian measure γ_0 such that $\gamma_0*\gamma_0=\gamma$. Then $\|f\|_{\gamma_0} = 2^{-1/2}\|f\|_\gamma$ for all $f\in L$. Accordingly to the criterion formulated above γ_0 is the Wigner distribution for some Gaussian density matrix W_0. We define the transformation O_1: $L_1(H) \rightarrowtail L_1(X)$, where X is the phase space, through the relation $Ws(O_1(W)) = \gamma_0 * Ws(W)$ for any $W\in L_1(H)$; here Ws denotes the Weyl symbol. Then $O_1(W)(x) = Tr(W_0(x)\cdot W)$, where $W_0(x)$ is the Gaussian density matrix whose Wigner distribution is γ_0 translated to x. Hence O_1 is positive. Further, we define the transformation O_2: $L_1(X) \rightarrowtail L_1(H)$ through the relation $Ws(O_2(f)) = \gamma_0 * f$ for any $f\in L_1(X)$. Then $O_2(f) = \int W_0(x)f(x)dx$, therefore O_2 is positive too. And

$Ws(O_2(O_1(W))) = \gamma_0*\gamma_0*Ws(W) = \gamma*Ws(W)$, hence $O_2O_1 = O_\gamma$. ●

The proved theorem yields a sufficient condition for a Gaussian operation to be classically factorizable; we do not know whether this condition is necessary. In the case of one degree of freedom, that is, two-dimensional phase space (q,p), any Gaussian operation is unitary equivalent (through a linear canonical transformation) to one of operations $O_{\Delta q,\Delta p}$ which are O_γ for γ such that $\|q\|_\gamma=\Delta q$, $\|p\|_\gamma=\Delta p$ and $Cov_\gamma(q,p)=0$. And

such operations are mutually unitary equivalent for any fixed value of $\Delta q\Delta p$. They are classically factorizable provided $\Delta q\Delta p \geq h$ due to Theorem 3.4. We do not know whether the critical value of the parameter $\Delta q\Delta p/h$ separating the presence and the absence of the classical factorizability is equal to 1. But we know that such critical value exists in (0,1]. Its positiveness can be obtained from properties of $sup \; \| \Sigma \tau_{kl} O^*_{\Delta q,\Delta p}(A_k)\otimes B_l \|$ considered as a function of $\Delta q\Delta p/h$, the supremum being taken over all A_k, B_l having norm ≤ 1. Our information about this function is scanty; but we can prove that it is decreasing, continuous on the right, constant and equal to 2 on $[1, +\infty)$; at the origin however it is equal to $2\sqrt{2}$ and continuous.



This behavior looks like a phase transition, the more so that the argument of this function turns out to be depending on the temperature in situations considered further.

4. QUASI-CLASSICAL SYSTEMS AS SPECIAL OPEN SYSTEMS

One is speaking about a classical Hamiltonian mechanical system when one neglects all deviations in the behavior of a physical system from the classical dynamics. When the deviations are to be taken into account as small corrections, then it is said about a quasi-classical system. Usually one bears in mind, however, only the deviations resulting from the quantum nature of the system. But other deviations prevail quite often, those occuring through the fact that the considered system is not well isolated, and first of all through the thermal fluctuations. In this section quasi-classical systems are treated as weakly non-isolated quantum systems of a special form. The first approximation to their dynamics is naturally classical Hamiltonian mechanics. The second approximation is a Gaussian operation introduced in Sect. 3 rather than conservative quantum mechanics or classical diffusion.

4.1. Elimination of Tensor Product from Chronological Descrition of Interaction

An interaction of two quantum systems is described by a unitary operator U on the tensor product $H_1\otimes H_2$. Assuming a non-correlated initial state $W_1\otimes W_2$ with a prescribed W_2, the quantum state change for the first system is described with the operation $O\in Op(H)$ defined by the equality $\langle M \rangle_{O(W_1)} = \langle U^*MU\rangle_{W_1\otimes W_2}$ for all $M\in L_\infty(H_1)$. (We do not subject the second system to any measurements; accordingly O is non-selective,

that is, normalization preserving). Further, $U = T\exp(-ih^{-1}\int dt H_{int}(t))$, where $H_{int}(t)$ is the interaction Hamiltonian, T is the chronological ordering, and the interaction representation is used. We accept the following form of interaction:

$$H_{int}(t) = \sum_\nu A^{(1)}_\nu(t)\otimes A^{(2)}_\nu(t). \qquad (3)$$

This is not too restrictive because operators of this form can approximate any operator in $H_1\otimes H_2$. And this form is convenient to formulate needed conditions for an interaction in terms of $A^{(1)}$, $A^{(2)}$.

A way to eliminate tensor products is suggested by the following formula:

$$\langle U \rangle_{W_1\otimes W_2} = \langle T\mathcal{F}(A_1) \rangle_{W_1}, \qquad (4)$$

where $\mathcal{F}$ is the functional defined as

$$\mathcal{F}(a) = \langle T\exp(-ih^{-1}\int dt \cdot \sum a_\nu(t)A^{(2)}_\nu(t))\rangle_{W_2}$$

for any path $a = \{a_\nu(t)\}$. From here on indices ν and sums over ν will be omitted (but meant), and $A_k(t)$ will be written rather than $A^{(k)}(t)$. Let us prove (4).

$$\mathcal{F}(a) = \left\langle \sum_{n=0}^{\infty} \left(-\frac{i}{\hbar}\right)^n \int\!\!\cdots\!\!\int_{t_n>\ldots>t_1} dt_n\ldots dt_1 a(t_n)\ldots a(t_1)A_2(t_n)\ldots A_2(t_1) \right\rangle_{W_2} =$$

$$= \sum_{n=0}^{\infty} \int\!\!\cdots\!\!\int_{t_n>\ldots>t_1} dt_n\ldots dt_1 f_n(t_n,\ldots,t_1)a(t_n)\ldots a(t_1),$$

where $f_n(t_n,\ldots,t_1) = \left(-\frac{i}{\hbar}\right)^n \langle A_2(t_n)\ldots A_2(t_1)\rangle_{W_2}$; so,

$$\langle T\mathcal{F}(A_1)\rangle_{W_1} = \left\langle \sum_{n=0}^{\infty} \int\!\!\cdots\!\!\int_{t_n>\ldots>t_1} dt_n\ldots dt_1 f_n(t_n,\ldots,t_1)A_1(t_n)\ldots A_1(t_1) \right\rangle_{W_1} =$$

$$= \sum_{n=0}^{\infty} \left(-\frac{i}{\hbar}\right)^n \int\!\!\cdots\!\!\int_{t_n>\ldots>t_1} dt_n\ldots dt_1 \langle A_2(t_n)\ldots A_2(t_1)\rangle_{W_2}\langle A_1(t_n)\ldots A_1(t_1)\rangle_{W_1} =$$

$$= \left\langle \sum_{n=0}^{\infty} \left(-\frac{i}{\hbar}\right)^n \int\!\!\cdots\!\!\int_{t_n>\ldots>t_1} dt_n\ldots dt_1 A_1(t_n)\ldots A_1(t_1)\otimes A_2(t_n)\ldots A_2(t_1) \right\rangle_{W_1\otimes W_2} =$$

$$= \left\langle T\exp\left[-\frac{i}{\hbar}\int dt A_1(t)\otimes A_2(t)\right] \right\rangle_{W_1\otimes W_2} = \langle U \rangle_{W_1\otimes W_2},$$

and (4) is proved. The following formula can be proved similarly:

$$\langle U^* M U \rangle_{W_1\otimes W_2} = \langle \hat{T}(M\mathcal{F}(A_1,A_1)) \rangle_{W_1}, \qquad (5)$$

where $\mathcal{F}$ is the two-place (with two path arguments) functional defined as $\mathcal{F}(a,b) =$

$$= \langle (T\exp(-\frac{i}{\hbar}\int dt b(t)A_2(t)))^* \cdot (T\exp(-\frac{i}{\hbar}\int dt a(t)A_2(t))) \rangle_{W_2}, \qquad (6)$$

and $\hat{T}$ denotes the ordering according to the following rules: (a) operators A(t) substituted in $\mathcal{F}$ in place of the first argument are ordered chronologically; (b) the inverse order for

the second argument; (c) the operators mentioned in (a) acting before M (that is, are written on the right of M) and those mentioned in (b) acting after M (on the left of M). Note that the right-hand side of (6) can be considered as the scalar product of two vectors, one depending on a, the other - on b (in fact, any state defines a scalar product on the algebra of observables), hence there is an expansion $\mathcal{F}(a,b) = \sum \mathcal{F}_k(a)\overline{\mathcal{F}_k(b)}$ with some one-place functionals $\mathcal{F}_1, \mathcal{F}_2, \ldots$ Then (5) gives

$$\langle U^* M U \rangle_{W_1 \otimes W_2} = \sum_k \langle (T\mathcal{F}_k(A_1))^* \cdot M \cdot (T\mathcal{F}_k(A_1)) \rangle_{W_1}; \qquad (7)$$

unlike to (5), only the usual chronological ordering is used in (7). The formula (7) can be considered as a specific version of the formula (3.7) in [4]. Of course, the functionals $\mathcal{F}_k$ are not uniquely defined by $\mathcal{F}$, but their arbitrariness does not affect the right-hand side of (7). At last, the summation over the discrete auxiliary index k can be replaced with the integration over an auxiliary variable running over arbitrary space with a positive measure.

4.2. Gaussian Approximation for Environment Influence

Now we will render concrete the functional $\mathcal{F}$, defined by (6), for some class of situations. Namely, we suppose in what follows that the first system is a macroscopic mechanical system (for example, a moving rigid body) and the second system is a many-body system in the sense of statistical physics (for example, a gas around the rigid body). The influence of the second system on the first one manifests in fluctuations of the motion of the first system. Usually such fluctuations are normally (Gaussian) distributed random variables. The well-known central limit theorem of the probability theory suggests a reason for this special kind of distribution, namely, the integral effect can be considered as the sum of many (almost) independent contributions. The mathematical machinery is simple; the Fourier transform for the probability measure is the product of many functions:

$$f(\lambda) = \prod f_k(\lambda) = exp \sum ln\ f_k(\lambda) \approx exp \sum (ia_k\lambda - \frac{1}{2}b_k\lambda^2),$$

where $a_k = -i\frac{d}{d\lambda}\Big|_{\lambda=0} ln\ f_k(\lambda)$, $b_k = -\frac{d^2}{d\lambda^2}\Big|_{\lambda=0} ln\ f_k(\lambda)$. And it is important that $ln\ f(\lambda)$ admits a quadratic approximation in much broader domain than $f(\lambda)$ itself.

Similarly we argue that $ln\ \mathcal{F}(a,b)$ admits a quadratic approximation in much broader domain than $\mathcal{F}(a,b)$ itself. Instead of the classical notion of independent random summands we use the quantum notion of independent systems described by the tensor product of Hilbert spaces, W_2 being factorized into the tensor product of density matrices and $A_2(t)$ being expanded into the sum of subsystem observables. Then $\mathcal{F}(a,b)$ becomes factorized into the product of functionals corresponding to

subsystems, and the quadratic approximation is obtained. In addition, it can be shown that $\mathcal{F}(a,b)$ is the Fourier transform of "quantum distribution for fluctuations", in a sense.

It is more simple to find coefficients for the quadratic approximation for $ln\ \mathcal{F}(a,b)$; in fact, this is an usual calculation up to the second order in $a(t)$, $b(t)$ considered as small parameters in (6), giving

$$ln\ \mathcal{F}(a,b) \approx -\frac{i}{\hbar}\int dt f_1(t)(a(t)-b(t)) -$$
$$- \frac{1}{2\hbar^2}\iint dt_1 dt_2 f(t_1,t_2)(a(t_1)-b(t_1))(a(t_2)-b(t_2)) + \qquad (8)$$
$$+ \frac{i}{2\hbar}\iint_{t_2>t_1} dt_1 dt_2 g(t_1,t_2)(a(t_1)-b(t_1))(a(t_2)+b(t_2)),$$

where

$$f_1(t) = \langle A_2(t) \rangle_{W_2},$$
$$f(t_1,t_2) = \langle A_2(t_1) \circ A_2(t_2) \rangle_{W_2} - \langle A_2(t_1) \rangle_{W_2} \cdot \langle A_2(t_2) \rangle_{W_2},$$
$$g(t_1,t_2) = \left\langle \frac{i}{\hbar}[A_2(t_1), A_2(t_2)] \right\rangle_{W_2} \qquad (9)$$

Of course, $A \circ B$ denotes $(AB+BA)/2$.

We will see that (8) implies that fluctuations form a Gaussian random process. It is very likely that the applicability domain for (8) coincides with the class of situations where fluctuations are (almost) Gaussian, regardless of how convincing the above reasoning with many independent subsystems seems. From here on we suppose that (8) holds.

4.3. What We Understand by Macroscopic Mechanical System

The canonical quantization of a macroscopic Hamiltonian mechanical system leads to a quantum system which reconstructs the original classical system as a result of well-known passage to the limit for $\hbar \to 0$. A wave packet contracts into a point which moves along a classical trajectory. The passage to the limit needs some conditions, for example, that all used observables result from quantization of such functions on the phase space which are smooth and do not depend on $\hbar$. However, we want apply such arguments to the world where $\hbar$ is a finite constant rather than infinitesimal. The small parameter is not $\hbar$ itself but its value in some "macroscopical" system of units suitable for a given situation. This idea can be formalized as follows. Let us introduce constants $\bar{S}$, $\bar{q}$, $\bar{p}$, $\bar{E}$, $\bar{t}$ having dimensions of the action, length, momentum, energy and time respectively, obeying

$$\bar{q}\cdot\bar{p} = \bar{E}\cdot\bar{t} = \bar{S} \gg \hbar.$$

It was said in Sect. 4.2 that the first system is supposed to be a macroscopic mechanical system. Now we can define what we understand by this. Prior to the canonical quantization we have a classical Hamiltonian system, we restrict ourselves to a bounded domain Ω of its phase space, and we assume the following condition:

$$\left|\frac{\partial^{k+l} h(q,p)}{\partial q^k \partial p^l}\right| \le \frac{\bar{E}}{\bar{q}^{|k|}\cdot\bar{p}^{|l|}} \qquad (10)$$

for all $(q,p)\in\Omega$, $|k|+|l| = 1,2,3$; here $h(q,p)$ is the Hamilton function and k, l are multiindices, that is, $k=(k_1,\dots,k_n)$, n is the number of degrees of freedom, q^k denotes $q_1^{k_1}\dots q_n^{k_n}$, $|k|$ denotes $k_1+\dots+k_n$, and the same for l. The inequality (10) is nothing but the C^3-smoothness with the constant 1 (in Ω) of the Hamilton function in the choosen system of units. Three of derivatives are essential in this paper; perhaps, in other cases a different number of derivatives will be necessary. First-order derivatives must satisfy (10) "without a wide margin", that is, the following relation must be violated:

$$\left|\frac{\partial^{k+l} h(q,p)}{\partial q^k \partial p^l}\right| \ll \frac{\bar{E}}{\bar{q}^{|k|}\cdot\bar{p}^{|l|}} \quad \text{for all } (q,p)\in\Omega,\ |k|+|l| = 1;$$

otherwise the Hamilton function would have been almost constant. It is possible to generalize the condition (10) to the case of a symplectic manifold as the phase space, but we do not dwell on this matter. Note that (10) implies that the considered system have no characteristic time less than $\bar{t}$; so it cannot perform many oscillations during $\bar{t}$, and trajectories cannot diverge very much during $\bar{t}$.

So the macroscopicity condition is formulated for an isolated classical system. The corresponding isolated quantum system is defined uniquely by the canonical quantization using the Weyl correspondence (mentioned in Sect. 3 above), accordingly we have no need to impose more conditions on quantum dynamics, but we require certain localization of a quantum state, because our technique is local. Later we will argue that such localization results from environment influence, but at first we accept the following condition: each considered quantum state is localized in some domain of the phase space, whose size is

$$\ll \bar{q}\cdot(h/\bar{S})^{1/3} \quad \text{and} \quad \ll \bar{p}\cdot(h/\bar{S})^{1/3} \qquad (11)$$

for the coordinate and momentum dimensions respectively. Of course, our conclusions can be extended to arbitrary mixtures of such states. The localization of a state in a domain means fast decreasing of its Weyl symbol (= Wigner distribution) outside of this domain. The bounds stated by (11) arise from our technique used further. Note that they permit a quantum state to occupy many phase cells, namely

$$\sim \frac{\bar{q}\cdot(h/\bar{S})^{1/3}\cdot\bar{p}\cdot(h/\bar{S})^{1/3}}{h} = (\bar{S}/h)^{1/3} \gg 1$$

for each degree of freedom. So the domain is macroscopically small but microscopically large.

Further, we require some conditions on environment in-

fluence; it must conform with the macroscopic nature of the system itself. We impose a condition on the operators $A_1(t)$ introduced in Sect. 4.1. However, the interaction representation was used in 4.1, hence the Hamiltonian motion of the first system itself was included into explicit time dependence of $A_1(t)$, whereas here we take $A_1(t)$ in Schrödinger representation, hence this explicit time dependence does not include the Hamiltonian motion (and in conservative cases $A_1(t)$ does not depend on t). We accept the following smoothness condition:

$$\left| \frac{\partial^{k+l} a(q,p,t)}{\partial q^k \, \partial p^l} \right| \leq \frac{\bar{a}}{\bar{q}^{|k|} \cdot \bar{p}^{|l|}} \qquad (12)$$

for all $(q,p)\in\Omega$, $|k|+|l| = 1,2$; here $a(q,p,t)$ is the Weyl symbol for $A_1(t)$, $\bar{a}$ is some constant such that first-order derivatives satisfy (12) "without a wide margin"; cf. with (10) above.

However, the operators $A_1(t)$ describe the interaction only together with the operators $A_2(t)$, and we need a condition on $A_2(t)$ or rather on the functional $\mathcal{F}$ defined by (6) to ensure that the environment influence is not too large, namely, that it deflects the first system from a classical trajectory within the bounds stated by (11). We will see in the next section that deflecting properties depend on the Fourier transform of $\mathcal{F}$ and that the formulation below is relevant. The functional $\mathcal{F}$ can be approximated, as well as any functional at all, by some linear combinations of functionals $\mathcal{F}_{c,d}$ of the form

$$\mathcal{F}_{c,d}(a,b) = exp\left(-\frac{i}{\hbar}\int dt\, a(t)c(t) + \frac{i}{\hbar}\int dt\, b(t)d(t) \right). \qquad (13)$$

We assume that our functional $\mathcal{F}$ admits such approximation with the following restriction on pathes c,d:

$$\left| \int dt\, \bar{a}c(t) \right| \ll \bar{S}\cdot(\hbar/\bar{S})^{1/3} \quad \text{and the same for d;} \qquad (14)$$

here the integral is taken over arbitrary time interval of duration $\leq\bar{t}$.

4.4. Canonical Commutation Relations near Classical Trajectory

In this section we will show that locally (near a given classical trajectory) the operators $A_1(t)$ can be approximated with some operators satisfying the canonical commutation relations (CCR) in the sense that their commutators are some "c-numbers", that is, operators of the form $\lambda\cdot\mathbb{1}$ Here $A_1(t) = A_1^{int}(t)$ are operators in the interaction representation; the corresponding operators in the Schrödinger representation will be denoted by $A_1^{Sch}(t)$, so that

$$A_1^{int}(t) = exp\left(\frac{i}{\hbar}tH_1\right)\cdot A_1^{Sch}(t)\cdot exp\left(-\frac{i}{\hbar}tH_1\right),$$

where H_1 is the Hamiltonian for the first system. (It was noted that $A_1^{Sch}(t)$ did not depend on t in conservative cases). The approximation will ensure the possibility to replace the operators under $\mathcal{F}$ in formula (5).

We use the following statement (apparently belonging to the folk-lore) showing that a quadratic Hamiltonian generates a linear transform of the phase space, quantum dynamics being reduced to a transfer of Weyl symbols by this transform, that is, along classical trajectories.

4.4.1. Statement. Let a real-valued function h(q,p,t) be a polinomial of degree ≤ 2 in variables $q = (q_1,\dots,q_n)$ and $p = (p_1,\dots,p_n)$ (however its dependence on t remains arbitrary). For each t we consider the operator H(t) whose Weyl symbol is h(q,p,t). Let $U(t) = T\exp\left(-\frac{i}{\hbar}\int_0^t ds H(s)\right)$, and further $W(t) = U(t)WU^*(t)$, $A(t) = U^*(t)AU(t)$ for any density matrix W and any observable A. Then the Weyl symbols w(q,p,t) for W(t), a(q,p,t) for A(t) satisfy

$$w(q(t),p(t),t) = w(q(0),p(0),0),$$
$$a(q(0),p(0),t) = a(q(t),p(t),0)$$

for any classical trajectory q(t), p(t) defined by $\dot q = \frac{\partial h}{\partial p}$, $\dot p = -\frac{\partial h}{\partial q}$. ●

One important corollary: if a(q,p,0) is a linear function, then a(q,p,t) is linear for each t.

Let $q_{cl}(t)$, $p_{cl}(t)$ be a classical trajectory remaining in the domain Ω during the considered time interval. We have for any fixed t the following expansion for the Hamilton function h (which is the Weyl symbol for H_1):

$$h(q,p) = h(q_{cl}(t),\, p_{cl}(t)) + dh(q-q_{cl}(t),\, p-p_{cl}(t)) +$$
$$+ \frac{1}{2}d^2h(q-q_{cl}(t),\, p-p_{cl}(t)) + \bar E\cdot\mathcal{O}\left(\left|\frac{q-q_{cl}(t)}{\bar q}\right|^3,\ \left|\frac{p-p_{cl}(t)}{\bar p}\right|^3\right);$$

the first differential dh is a linear form, the second one d^2h is a quadratic form; of course, the differentials are taken at the point $q_{cl}(t)$, $p_{cl}(t)$. Rejecting the remainder term, we obtain a polynomial $\tilde h(q,p,t)$ of degree ≤ 2 in q,p; it is the Weyl symbol for some operator $\tilde H_1(t)$. Similarly,

$$a(q,p,t) = a(q_{cl}(t),\, p_{cl}(t),\, t) + da(q-q_{cl}(t),\, p-p_{cl}(t)) +$$
$$+ \bar a\cdot\mathcal{O}\left(\left|\frac{q-q_{cl}(t)}{\bar q}\right|^2,\ \left|\frac{p-p_{cl}(t)}{\bar p}\right|^2\right),$$

where a(q,p,t) is the Weyl symbol for $A_1^{Sch}(t)$; rejecting the remainder term, we obtain a polynomial $\tilde a(q,p,t)$ of degree ≤ 1

in q, p and the corresponding operator $\tilde{A}_1^{sch}(t)$. We define approximating operators for $A_1^{int}(t)$ as $\tilde{A}_1^{int}(t) = \tilde{U}^*(t)\tilde{A}_1^{sch}(t)\tilde{U}(t)$, where $\tilde{U}(t) = Texp\left(-\frac{i}{\hbar}\int_0^t ds\tilde{H}_1(s)\right)$. Statement 4.4.1 or rather its corollary shows that the operators $\tilde{A}_1^{int}(t)$ are linear combinations of the Q,P operators and hence satisfy CCR. It should be proved that the replacement of A_1 with $\tilde{A}_1^{int}$ under $\mathcal{F}$ in formula (5) results in a small error provided W_1 is localized near the considered trajectory, that is, the point $q_{cl}(0)$, $p_{cl}(0)$ belongs to the domain appeared in (11). Taking (13) into account we see that it is sufficient to prove that the replacement of A_1^{int} with $\tilde{A}_1^{int}$ in

$$\left(Texp\left(-\frac{i}{\hbar}\int ds A_1^{int}(s)c(s)\right)\right)\cdot W_1\cdot\left(Texp\left(-\frac{i}{\hbar}\int ds A_1^{int}(s)d(s)\right)\right)^*$$

results in a change which is small, that is, of norm $\ll 1$ in $L_1(H)$, provided c,d satisfy (14). In Schrödinger representation $A_1^{sch}(s)$ is replaced with $\tilde{A}_1^{sch}(s)$ and at the same time H_1 with $\tilde{H}_1(s)$ in

$$\left(Texp\left(-\frac{i}{\hbar}\int ds(H_1+A_1^{sch}(s)c(s))\right)\right)W_1\left(Texp\left(-\frac{i}{\hbar}\int ds(H_1+A_1^{sch}(s)d(s))\right)\right)^*.$$

We introduce an "interpolating" operator, depending on an auxiliary parameter t, by taking $\tilde{A}_1^{sch}(s)$ and $\tilde{H}_1(s)$ for $s<t$ but $A_1^{sch}(s)$ and H_1 for $s>t$. It is enough to estimate the derivative of the "interpolating" operator with respect to t. This derivative is the sum of two similar terms ("the right" and "the left"); we consider one of them. Taking into account that the norm of an operator in $L_1(H)$ remains unchanged if the operator is multiplied (from the right or from the left) by any unitary operator, we see that it is enough to estimate the following:

$$\left\|(H_1-\tilde{H}_1(t)+(A_1^{sch}(t)-\tilde{A}_1^{sch}(t))c(t))\left(Texp\left(-\frac{i}{\hbar}\int_0^t ds(\tilde{H}_1(s)+\tilde{A}_1^{sch}(s)c(s))\right)\right)\right.$$

$$\left.\cdot W_1\cdot\left(Texp\left(-\frac{i}{\hbar}\int_0^t ds(\tilde{H}_1(s)+\tilde{A}_1^{sch}(s)d(s))\right)\right)^*\right\|_{L_1(H)}.$$

We use again the possibility of multiplying by any unitary operator to make a little trick, namely, to replace d(s) to c(s); in fact we reject the whole unitary factor containing d(s) and instead we write the similar factor with c(s). Now we have the product of two operators. The first has the Weyl symbol

$$\bar{E}\mathcal{O}\left(\left|\frac{q-q_{cl}(t)}{\bar{q}}\right|^3,\left|\frac{p-p_{cl}(t)}{\bar{p}}\right|^3\right)+\mathcal{O}\left(\left|\frac{q-q_{cl}(t)}{\bar{q}}\right|^2,\left|\frac{p-p_{cl}(t)}{\bar{p}}\right|^2\right)\bar{a}c(t).$$

The second results from the evolution of W under the "perturbated" Hamiltonian $\tilde{H}_1(s)+\tilde{A}_1^{sch}(s)c(s)$, which is again quadratic; hence the statement 4.4.1 is applicable, and we see that this second operator is localized in the domain

$$\left|\frac{q-q_{cl}(t)}{\bar{q}}\right| \ll (h/\bar{S})^{1/3}, \qquad \left|\frac{p-p_{cl}(t)}{\bar{p}}\right| \ll (h/\bar{S})^{1/3}.$$

These estimations together with the smoothness of the Weyl symbol for the first operator show that the norm of the product is $\ll \bar{E}\cdot(h/\bar{S}) + (h/\bar{S})^{2/3}\cdot\bar{a}c(t)$. Using (14) and integrating over time we obtain that the norm of the difference is $\ll (t\bar{E}h/\bar{S}+\bar{S}h/\bar{S})/h$, that is, $\ll 1$.

4.5. Gaussian Approximation for Operation

We had seen in 4.4 that the operators $A_1(t)$ in the interaction Hamiltonian $H_{int}(t) = A_1(t)\otimes A_2(t)$ can be assumed locally satisfying the canonical commutation relations. However $A_2(t)$ can be assumed CCR too. Indeed, only the functional $\mathcal{F}$ resulting from $A_2(t)$ according to (6) is essential, and this functional is Gaussian, that is, the exponential of a quadratic polynomial, according to (8). It is easy to see that any CCR operators result in a Gaussian functional provided the state (W_2) is Gaussian. And conversely, we can construct CCR-operators and a Gaussian state for a given Gaussian functional $\mathcal{F}$ provided that the latter satisfies the condition (1) or rather its infinite-dimensional version. But this condition always holds for a functional resulting from some observables and state according to (8,9).

Thus we accept that $A_1(t)$ and $A_2(t)$ are both CCR (and in addition that W_2 is Gaussian). Then the bilinear Hamiltonian $H_{int}(t) = A_1(t)\otimes A_2(t)$ can be considered as a special case of the quadratic Hamiltonian and the statement 4.4.1 is applicable. But it was formulated for finite dimension only, whereas operators $A_2(t)$ can generate infinite dimensional CCR; however, this is in fact of no importance here. The mentioned statement reduces quantum dynamics to classical. A classical trajectory for the couple of systems is the pair of functions $X_1(t)$, $X_2(t)$ valued in the phase space of the first and the second system respectively. The operator $A_1(t)$ has its Weyl symbol $a_1(x,t)$ which is a linear function of x running over the phase space of the first system, and the same for $A_2(t)$, $a_2(x,t)$. The classical equations of motion under the bilinear Hamiltonian $a_1(x,t)\cdot a_2(x,t)$ can be written (in the interaction representation) in the form

$$\frac{d}{dt}b_1(X_1(t)) = \{a_1(t),\ b_1\}\cdot a_2(X_2(t),t),$$

$$\frac{d}{dt}b_2(X_2(t)) = \{a_2(t),\ b_2\}\cdot a_1(X_1(t),t);$$

here b_1 is an arbitrary linear function on the phase space of the first system, b_2 - of the second one; the Poisson brackets for two linear functions result in a constant function, therefore the argument x is omitted inside these brackets. It follows from the second equation

$$a_2(X_2(t),t) = a_2(X_2(0),t) + \int_0^t ds\frac{d}{ds}a_2(X_2(s),t) =$$

$$= a_2(X_2(0),t) + \int_0^t ds\{a_2(s),\ a_2(t)\}\cdot a_1(X_1(s),s);$$

substitution in the first equation leads to

$$\frac{d}{dt}b_1(X_1(t)) = \{h_{eff}(x,t),\ b_1\}, \qquad \text{where}$$

$$h_{eff}(x,t) = a_1(x,t)\cdot\left(a_2(X_2(0),t)+\int_0^t ds\{a_2(s),\ a_2(t)\}\cdot a_1(X_1(s),s)\right).$$

We see that the influence of the second system on the first one can be described (on the classical level for the time being) by means of an "effective Hamiltonian" $h_{eff}(x,t)$ for the first system. Of course, it is not a true Hamiltonian, because it depends not only on x and t, but also on the history $X_1(s)$, $s\le t$, and on the initial state of the second system $X_2(0)$. But this quasi-Hamiltonian description provides a convenient transition from the interaction representation used above to the Schrödinger one; one must only add $h_{eff}(x,t)$ to the Hamiltonian of the first system.

It was shown in Sect. 4.2 that the three functions f_1, f, g are sufficient to describe the influence of the second system. Let us express h_{eff} in terms of these functions. We have

$$g(t_1,t_2) = \langle\frac{i}{\hbar}[A_2(t_1),A_2(t_2)]\rangle_{W_2} = \{a_2(t_1),\ a_2(t_2)\};$$

$$f_1(t) = \langle A_2(t)\rangle_{W_2} = \int dx a_2(x,t)w_2(x),$$

$$f(t_1,t_2)+f_1(t_1)f_1(t_2)=\langle A_2(t_1)\circ A_2(t_2)\rangle_{W_2}=\int dx a_2(x,t_1)a_2(x,t_2)w_2(x),$$

where $w_2(x)dx$ is the Wigner distribution for W_2, which is a Gaussian measure. Accordingly, the function $\zeta(t) = a_2(X_2(0),t)$ can be considered as a Gaussian random process with the mean $f_1(t)$ and the covariance $f(t_1,t_2)$, if $X_2(0)$ is considered as a random point with the distribution $w_2(x)dx$. We obtain

$$h_{eff}(x,t) = a_1(x,t)\cdot\left(\zeta(t)+\int_0^t ds g(s,t)a_1(X_1(s),s)\right).$$

And the summation over the auxiliary index ν, which was omitted but meant from Sect. 4.1 up to here, can be now eliminated together with the functions $a_1(x,t)$. Namely, we put $\zeta(x,t) = \sum_\nu \zeta_\nu(t)a_{1\nu}(x,t)$ and the same for f_1,

$f(x_1,t_1;\ x_2,t_2) = \sum_{\mu\nu} f_{\mu\nu}(t_1,t_2)a_{1\mu}(x_1,t_1)a_{1\nu}(x_2,t_2)$
and the same for g; then

$$h_{eff}(x,t) = \zeta(x,t)+\int_0^t ds g(X_1(s),s;\ x,t). \qquad (15)$$

It is clear that $\zeta(x,t)$ is a Gaussian random process with the mean f_1 and the covariance f. By the way, these new functions f_1, f, g can be expressed by means of operators $A_2(x,t) = \sum_\nu a_{1\nu}(x,t)A_{2\nu}(t)$. In fact from the very beginning we could have treated the interaction in terms of $A_2(x,t)$ rather than of $A_1(t)\otimes A_2(t)$.

Quantum dynamics can now be reconstructed due to the statement 4.4.1. Namely,

$$< M >_{O(W_1)} = < U^*MU >_{W_1\otimes W_2} = \mathbb{E}\int dx_0 m(X_1(t,x_0))w_1(x_0)$$

for any initial state W_1 of the first system and any observable M for the first system, which is measured at a moment t; here m is the Weyl symbol for M, w_1 - for W_1; $X_1(t,x_0)$ denotes the classical trajectory starting from an initial point x_0 and governed by the Hamiltonian including h_{eff}; it depends on a path of the random process ζ, and $\mathbb{E}$ denotes the averaging on ζ.

Often we will describe dynamics in terms which seem to be classical (for example, the force of friction); however we will mean quantum dynamics corresponding to classical as stated here.

Now we abandon the second system itself (which was called also the environment) because its influence on the first system is taken into account by means of h_{eff}.

4.6. What We Understand by Quasi-Classical Approximation

Traditionally one treats the quasi-classical approximation as short-wave asymptotics for the Schrodinger equation. We propose a wider comprehension in the spirit of Wigner's [6] ideas, including some environment influence taken in the approximation given in 4.5 above. Besides the Hamiltonian, three functions are used:

$$f_1(x,t),\quad f(x_1,t_1;\ x_2,t_2),\quad g(x_1,t_1;\ x_2,t_2),$$

x,x_1,x_2 running over the phase space. These functions provide a reduced description of an environment influence. An "evolution operation" describes quantum dynamics rather than a unitary evolution operator. The presence of two time arguments in f, g reflects possible memory effects in the environment, through which the quantum evolution is generally non-markovian. The markovian property and a semi-group of operations arises if f, g are localized near $t_1=t_2$; this case is of special inte-

rest. The opposite case, when a characteristic time for the environment is much greater than that for the system, falls outside of this paper; apparently it is connected with concepts of dynamical chaos (classical and quantum). Our approach can be used to investigate the long-time behavior of a system only in case when stochastic effects prevail over quantum ones for moderate time (and all the more for large time) and besides an environment gives no long-time memory effects. But this is the case for a wide class of macroscopic systems.

Due to the decay of memory effects we can consider classical dynamics for $t\in(-\infty,+\infty)$ rather than for $t\in(0,+\infty)$.

The functions f_1, f, g are assumed small and smooth on a macroscopic scale. This is why the evolution operation is locally Gaussian. Naturally, the corresponding coefficients are smoothly varying on the macroscopic scale. By the way, the functions f, g satisfy some inequality: the skew bilinear form defined by the kernel g is bounded by the positive symmetric bilinear form defined by the kernel f. This shows that a non-Hamiltonian (in particular, dissipative) evolution is indispensably accompanied by a noise. But we do not dwell on this matter, because we are mostly interested in such cases when the noise is much greater than its fundamental lower bound (due to non-zero temperature; see below).

4.7. Using Fluctuation-Dissipation Relations

From here on we suppose that the second system (the environment) is a heat bath. At the same time we suppose that the interaction does not depend explicitly on t. So A_1^{sch}, A_2^{sch} do not depend on t, and $[H_2, W_2]=0$ (H_2 is the Hamiltonian for the second system), hence

$$f_1(x,t) = f_1(x), \quad f(x_1,t_1; x_2,t_2) = f(x_1,x_2,t_1-t_2),$$
$$g(x_1,t_1; x_2,t_2) = g(x_1,x_2,t_1-t_2).$$

We suppose that W_2 is the Gibbs state for a positive temperature T or, more generally (to comprehend the case of continuous spectrum of H_2), the Kubo-Martin-Schwinger (KMS) state. We do not touch upon fundamental problems of statistical physics about the origin of such state, and we leave out of account the known non-Gibbsian terms. We accept the following fluctuation-dissipation relation connecting the Fourier transforms (with respect to time) for f and g:

$$\hat{g}(x_1,x_2,\omega) = -i\hat{f}(x_1,x_2,\omega)\ th\ \frac{h\omega}{2kT} \qquad (16)$$

In terms of f,g themselves this is a convolution relation:

$$g(x_1,x_2,t) = \int_{-\infty}^{+\infty} ds \left(\frac{h}{kT}\ sh\ \frac{\pi kT}{h}(t-s)\right)^{-1} f(x_1,x_2,s). \qquad (17)$$

Note that the time kernel is exponentially small outside the domain $t-s=\mathcal{O}(h/kT)$. We suppose that $h/kT=\mathcal{O}(\bar{t})$ and that both

functions $f(x_1,x_2,\tau)$, $g(x_1,x_2,\tau)$ are negligible outside the domain $\tau=\mathcal{O}(\bar{t})$. This is suited for continuous spectrum of H_2; for models with a discrete spectrum one can use decreasing in a domain which is finite but large enough.

It follows from (17) that $\int_{-\infty}^{+\infty}d\tau\, g(x_1,x_2,\tau)=0$.

Using $g(x_1,x_2,\tau)=-g(x_2,x_1,-\tau)$ we have $\int_0^{+\infty}d\tau\, g(x_1,x_2,\tau) = \int_0^{+\infty}d\tau\, g(x_2,x_1,\tau)$ and hence

$$\frac{\partial}{\partial x^k}\Big|_{x=x_0}\int_0^{+\infty}d\tau\, g(x,x_0,\tau) = \frac{1}{2}\,\frac{\partial}{\partial x^k}\Big|_{x=x_0}\int_0^{+\infty}d\tau\, g(x,x,\tau);$$

this will be used below.

An environment influence results in a trend and fluctuations. The trend is determined by the differential of the "effective Hamiltonian" (15) averaged on ζ:

$$\frac{\partial}{\partial x^k}\Big|_{x=X(t)}\mathbb{E}h_{\text{eff}}(x,t) = \frac{\partial}{\partial x^k}\Big|_{x=X(t)}\Big(f_1(x,t)+\int_{-\infty}^{t}ds\,g(X(s),s;\,x,t)\Big) =$$

$$= \frac{\partial}{\partial x^k}\Big|_{x=X(t)}f_1(x) - \frac{\partial}{\partial x^k}\Big|_{x=X(t)}\int_0^{+\infty}d\tau\, g(x,X(t-\tau),\tau) =$$

$$= \frac{\partial}{\partial x^k}\Big|_{x=X(t)}f_1(x) - \frac{\partial}{\partial x^k}\Big|_{x=X(t)}\int_0^{+\infty}d\tau\, g(x,X(t),\tau) -$$

$$- \frac{\partial}{\partial x^k}\Big|_{x=X(t)}\int_0^{+\infty}d\tau\,(g(x,X(t-\tau),\tau)-g(x,X(t),\tau)) =$$

$$= \frac{\partial}{\partial x^k}\Big|_{x=X(t)}\Big(f_1(x) - \frac{1}{2}\int_0^{+\infty}d\tau\, g(x,x,\tau)\Big) -$$

$$- \int_0^{+\infty}d\tau\,\frac{\partial}{\partial x^k}\Big|_{x=X(t)}\int_0^{\tau}d\theta\,\frac{d}{d\theta}g(x,X(t-\theta),\tau); \qquad (18)$$

the first term is a total derivative without any path dependence; hence it brings in a Hamiltonian correction to the motion. A non-Hamiltonian correction is introduced by the second term. It can be written as

$$\iint_{0<\theta<\tau<\infty} d\tau\, d\theta\, \dot{X}^l(t-\theta)\,\frac{\partial^2}{\partial x_1^k\,\partial x_2^l}\Big|_{x_1=X(t),\,x_2=X(t-\theta)} g(x_1,x_2,\tau),$$

and it is clear that the knowledge of this correction for all pathes $X(s)$ is equivalent to the knowledge of the function $\frac{\partial^2}{\partial x_1^k\,\partial x_2^l}g(x_1,x_2,\tau)$. Due to the fluctuation-dissipation relation this is equivalent to the knowledge of the function $\frac{\partial^2}{\partial x_1^k\,\partial x_2^l}f(x_1,x_2,\tau) = Cov(\zeta_{,k}(x_1,\tau),\,\zeta_{,l}(x_2,0))$, that is, of the covariance function for fluctuation forces. The non-Hamiltonian correction can be expressed through the fluctuation covariances as follows:

$$- Cov\left(\int_{-\infty}^{+\infty} d\tau \zeta_{,k}(X(t),t+\tau)\frac{1}{\pi}ln\left|cth\frac{\pi kT}{2h}\tau\right|,\ \int_{-\infty}^{t} ds\dot{X}^l(s)\zeta_{,l}(X(s),s)\right).$$

But this is too general for our purposes.

4.8. Markovian Approximation

It was supposed in 4.7 that the relaxation time for the heat bath is $\mathcal{O}(t)$. Starting from here we suppose that it is $\ll t$. Correspondingly, the evolution becomes local in time, that is, markovian, described by differential equations rather than by integro-differential ones. A force of friction arises often which is proportional to the velocity. Sometimes it is proportional to the third derivative (for example, the radiative friction). It may occur a linear combination of different derivatives. Non-linear friction is also possible (for example, one proportional to the cube of velocity), but we do not examine such cases here.

So we linearize the non-Hamiltonian correction (18):

$$- \frac{\partial}{\partial x^k}\bigg|_{x=X(t)} \int_0^{+\infty} d\tau (g(x,X(t-\tau),\tau)-g(x,X(t),\tau)) \approx$$

$$\approx - \frac{\partial}{\partial x^k}\bigg|_{x=X(t)} \int_0^{+\infty} d\tau (X^l(t-\tau)-X^l(t))\frac{\partial}{\partial y^l}\bigg|_{y=X(t)} g(x,y,\tau) =$$

$$= -\int_0^{+\infty} d\tau \frac{\partial^2}{\partial x_1^k \partial x_2^l}\bigg|_{x_1=X(t),x_2=X(t)} g(x_1,x_2,\tau)\cdot \sum_n \frac{(-1)^n}{n!}\tau^n \frac{d^n}{ds^n}\bigg|_{s=t} X^l(s) =$$

$$= -\sum_n C_{kl}^n(X(t))\frac{d^n}{ds^n}\bigg|_{s=t} X^l(s),$$

where $C_{kl}^n(x)=\frac{(-1)^n}{n!}\int_0^{+\infty} d\tau\cdot\tau^n\cdot \frac{\partial^2}{\partial x_1^k \partial x_2^l}\bigg|_{x_1=x,x_2=x} g(x_1,x_2,\tau)$;

the sum over n is regarded as an asymptotic expansion rather than as a convergent series. On the other hand, (16) gives

for $\omega \ll kT/h$ $\quad \hat{g}(x,y,\omega) \approx -\frac{h}{2kT} i\omega \hat{f}(x,y,\omega)$, hence

$$\int_{-\infty}^{+\infty} d\tau \phi(\tau) g(x,y,\tau) \approx \frac{h}{2kT}\int_{-\infty}^{+\infty} d\tau \phi'(\tau) f(x,y,\tau)$$

for any function ϕ whose Fourier transform is negligible outside the domain $\omega=\mathcal{O}(kT/h)$. However

$$\int_{-\infty}^{+\infty} d\tau \phi(\tau) g(x,y,\tau) = \int_{-\infty}^{+\infty} d\tau g(x,y,\tau)\sum_n \frac{1}{n!}\phi^{(n)}(0)\tau^n =$$

$$= \sum_n \frac{1}{n!}\phi^{(n)}(0)\cdot\left(\int_0^{+\infty} d\tau g(x,y,\tau)\cdot\tau^n + \int_0^{+\infty} d\tau g(x,y,-\tau)\cdot(-\tau)^n\right) =$$

$$= \sum_n \frac{1}{n!}\phi^{(n)}(0)\cdot\left(\int_0^{+\infty} d\tau g(x,y,\tau)\cdot\tau^n - (-1)^n\int_0^{+\infty} d\tau g(y,x,\tau)\cdot\tau^n\right),$$

hence

$$\frac{\partial^2}{\partial x_1^k \partial x_2^l}\bigg|_{x_1=x,x_2=x} \int_{-\infty}^{+\infty} d\tau \phi(\tau) g(x,y,\tau) = \sum_n \left((-1)^n C_{kl}^n(x) - C_{lk}^n(x)\right)\phi^{(n)}(0),$$

so we obtain

$$\int_{-\infty}^{+\infty} d\tau \phi'(\tau) Cov(\zeta_{,k}(x,\tau),\ \zeta_{,l}(x,0)) \approx \frac{2kT}{h} \sum_n \left[(-1)^n C^n_{kl}(x) - C^n_{lk}(x)\right] \phi^{(n)}(0),$$

that is,

$$Cov(\zeta_{,k}(x,\tau),\ \zeta_{,l}(x,0)) \approx -\frac{2kT}{h} \sum_n \left[C^{n+1}_{kl}(x) + (-1)^n C^{n+1}_{lk}(x)\right] \delta^{(n)}(t).$$

We see that the same coefficients $C^n_{kl}(x)$ determine both the non-Hamiltonian correction to dynamics and the correlation of fluctuations. The last is found only for coincided points x; however it can be shown that one may put

$$Cov(\zeta_{,k}(x(t_1),t_1),\ \zeta_{,l}(x(t_2),t_2)) \approx$$

$$\approx -\frac{2kT}{h} \sum_n \left[C^{n+1}_{kl}\left(x\left(\frac{t_1+t_2}{2}\right)\right) + (-1)^n C^{n+1}_{lk}\left(x\left(\frac{t_1+t_2}{2}\right)\right)\right] \delta^{(n)}(t_2-t_1)$$

with an error of the same order as the error introduced above by the linearization.

Let us note some restrictions for $C^n_{kl}(x)$ arising from the positive definiteness of a covariance function. Namely, the following matrix must be positive semi-definite for all ω, both positive and negative:

$$\frac{h}{2kT} \frac{\partial^2}{\partial x_1^k \partial x_2^l}\Big|_{x_1=x, x_2=x} \hat{f}(x_1,x_2,\omega) \approx -\sum_n (i\omega)^n \left[C^{n+1}_{kl}(x) + (-1)^n C^{n+1}_{lk}(x)\right] =$$

$$= -(C^1_{kl} + C^1_{lk}) - i\omega(C^2_{kl} - C^2_{lk}) + \omega^2(C^3_{kl} + C^3_{lk}) - \dots .$$

It is clear that the first non-vanishing term here must be a term with an even power of ω, and this leading term must contain a positive semi-definite matrix.

The most simple and most widespread case is that of force of friction which is proportional to the velocity: $F_{fric} = -K_1 \dot{q}$. Such case can be described by means of C^n_{kl} with n=1, and we arrive at the corresponding fluctuation forces F_{fluc} with the following covariation:

$$Cov(F_{fluc}(s),\ F_{fluc}(t)) = 2kTK_1\delta(s-t). \qquad (19)$$

The next case is "radiative-type" force of friction which is proportional to the third derivative: $F_{fric} = K_3 q^{(3)}$; here C^3_{kl} is used giving

$$Cov(F_{fluc}(s),\ F_{fluc}(t)) = -2kTK_3\delta''(s-t). \qquad (20)$$

The same fluctuations conform to more general friction $F_{fric} = K_2 q^{(2)} + K_3 q^{(3)} + K_4 q^{(4)}$, because the symmetric part of the matrix C^n_{kl} with an even n makes no contribution to the low-frequency part of fluctuation spectrum. Note also that $K_3 q^{(3)}$ is in fact an approximation for some expression which is non-local in time. This approximation is valid for trajectories smooth enough, and invalid for "self-accelerating trajectories" containing the factor $exp(mt/K_3)$.

5. TIME OF CLASSICAL FACTORIZATION

5.1. Definition and General Relations

It was shown in section 4 that the operation describing the evolution of a quasi-classical system is locally Gaussian, and a method was given to find its parameters. Therefore results of section 3 can be applied to decide whether this operation is classically factorizable. We define the *time of classical factorization* τ_f for a quasi-classical system as such time that the operation describing the evolution during a time t is classically factorizable for $t>\tau_f$ but it is not for $t<\tau_f$. This is treated locally, that is, near a classical trajectory. We regret to note that the theorem 3.4 yields only a sufficient condition for classical factorizability and we do not know whether it is necessary. Through this fact true values for τ_f can disagree with the values obtained below. However the ratio of this values is bounded with some constant of mathematical nature, so our calculations yields acceptable estimations.

Let us calculate the time of classical factorization for translational degrees of freedom for a body of mass m whose interaction with an environment results in a small force of friction $F_{fric}=-K_1\dot{q}$, other forces being absent. The friction coefficient K_1 interests us so far as it enables to find fluctuation forces. We neglect a damping effect of the friction because in considered cases τ_f is much less than the time $\tau_1=m/K_1$ characterizing the damping effect. Considering one degree of freedom, assuming zero initial coordinate and velocity and applying (19) we obtain the following (co-)variances:

$$D(p(t)) = 2kTK_1t; \qquad D(q(t)) = 2kTK_1t^3/3m^2;$$
$$Cov(q(t), p(t)) = 2kTK_1t^2/2m.$$

Calculating the criterion (2) we find that $\tau_f = \left(\sqrt{3}\frac{h}{kT}\cdot\frac{m}{K_1}\right)^{1/2}$. So

$$\tau_f^2 \sim \tau_{therm}\tau_1, \qquad (21)$$

where $\tau_{therm}=h/kT$ is a characteristic period of thermal fluctuations, $\tau_1=m/K_1$.

The second case $F_{fric}=K_3q^{(3)}$ may seem completely similar but in fact it is not. In the first case the fluctuation force as a function of time behaves as the white noise, hence the momentum behaves as the Wiener process having continuous pathes, therefore the instant values of the momentum are well-defined. However in the second case the momentum behaves as the white noise and the coordinate - as the Wiener process (like the situation for the Brownian motion), hence the instant values of the momentum have too large variances. But this mo-

mentum averages to a small value during a long time. One can speak about intensive exchange of momentum between the body and the surroundings media where a momentum passed to the body returns progressively to the media. So it should be given an averaging time τ_{aver}, specifying indirectly thickness of a boundary layer of the media which is implicitly included in the considered quasi-classical system. Then we obtain

$$\tau_f \sim \tau_{aver}\left(\frac{\tau_{therm}}{\tau_3}\right)^2, \qquad (22)$$

where $\tau_{therm}=\hbar/kT$ as above, $\tau_3=K_3/m$.

Two effects were considered above. However their combination needs a special consideration. Indeed, if one effect results in large fluctuations of the momentum and small fluctuations of the coordinate, but the opposite for the other effect, then their combination leads to a time of classical factorization which is much less than that for each effect separately. We obtain

$$\tau_f \sim min\left(\tau_1^{1/2}\tau_{therm}^{1/2}, \tau_1^{1/2}\tau_3^{-1/2}\tau_{therm}, \tau_1^{1/3}\tau_3^{-1/3}\tau_{aver}^{1/3}\tau_{therm}^{2/3}, \tau_3^{-2}\tau_{aver}^{2}\tau_{therm}^{2}\right). \qquad (23)$$

The above formulas are valid for an almost free body. For a body in a non-linear potential specifying by a time τ_{mech} (it may be a period of oscillation or a time of divergence of trajectories) the following two terms must be included under the minimum sign in (23):

$$\tau_1^{1/6}\tau_3^{-1/6}\tau_{therm}^{1/3}\tau_{mech}^{2/3}, \quad \tau_3^{-1/2}\tau_{therm}^{1/2}\tau_{mech}. \qquad (24)$$

However if the minimum is $\gg \bar{t}$ then we known that $\tau_f \gg \bar{t}$ and nothing more.

5.2. Several Examples

Let us consider a macroscopic body interacting only with a thermal electromagnetic radiation and with a rarefied gas. Its translation motion is accompanied by the following force of friction:

$$F_{fric} \sim -\left(\frac{v}{c}\cdot P_{rad} + \frac{v}{v_0}\cdot P_{gas}\right)\cdot S,$$

where v, c, v_0 are the velocities of the body, the light and the gas molecules respectively, P_{rad} (resp. P_{gas}) is the specific pressure of the radiation (resp. the gas), S is the sectional area of the body. Substituting $P_{rad}\sim(kT)^4(hc)^{-3}$, $P_{gas}\sim\rho v_0^2$, $v_0\sim(kT/m_0)^{1/2}$ and $S\sim l^2$ we obtain $F_{fric}=-K_1 v$ with $K_1 \sim \hbar^{-3}c^{-4}k^4 l^2 T^4 + k^{1/2}m_0^{-1/2}\rho l^2 T^{1/2}$; here T is the temperature of the radiation and the gas (as-

suming equal), m_0 is the mass of the gas molecules, ρ is the specific density of the gas and l is the size of the body. Hence $\tau_1 = m/K_1 \sim min\ (h^3c^4k^{-4}ml^{-2}T^{-4},\ k^{-1/2}m_0^{1/2}\rho^{-1}ml^{-2}T^{-1/2})$. We take for example $m \sim 1$ g, $l \sim 1$ cm, $T \sim 1$ K, $\rho \sim 10^{-26}$ kg/m^3, $m_0 \sim 10^{-27}$ kg; then $\tau_1 \sim min(10^{24},10^{25}) \sim 10^{24}$ sec, $\tau_{therm} \sim 10^{-11}$ sec and (21) gives $\tau_f \sim 10^6$ sec.

Let the body have an electric charge Q and consider the radiative friction

$$F_{fric} = (6\pi\varepsilon_0)^{-1}c^{-3}Q^2q^{(3)};$$

here $q^{(3)}$ denotes the third derivative of the body coordinate; the coefficient is adapted to the SI system of units. We have $K_3 \sim \varepsilon_0^{-1}c^{-3}Q^2$ and $\tau_3 = K_3/m \sim \varepsilon_0^{-1}c^{-3}Q^2m^{-1}$. We take for example $Q \sim 10^{-9}$ C, then $\tau_3 \sim 10^{-31}$ sec and (22) gives $\tau_f \sim 10^{40}\tau_{aver}$. We see that the classical factorization caused by the radiative friction is practically impossible. The combined account of $\tau_1 \sim 10^{24}$ sec and $\tau_3 \sim 10^{-31}$ sec through (23) shows that τ_3 is negligible even for $\tau_{aver} \sim 10^{-12}$ sec.

Now let the body be in the Coulomb field of another body with a charge Q_2 at a distance r. Calculating the second derivative of the potential with respect to r we find $\tau_{mech} \sim \varepsilon_0^{1/2}m^{1/2}r^{3/2}Q^{-1/2}Q_2^{-1/2}$. We take for example $r \sim l$, $Q_2 \sim Q$, then $\tau_{mech} \sim 1$ sec. The six terms form τ_f (see (23), (24)), and the leading term is $\tau_1^{1/6}\tau_3^{-1/6}\tau_{therm}^{1/3}\tau_{mech}^{2/3} \sim 10^5$ sec, which is however not too different from the first term $\tau_1^{1/2}\tau_{therm}^{1/2} \sim 10^6$ sec. Since all the terms are $\gg \tau_{mech}$, one concludes that $\tau_f \gg \tau_{mech} \sim 1$ sec.

But the motion in a potential causes a varying deformation and hence a dissipation provided that the potential is not quadratic. Let the body be a dielectric crystal (say, quartz or sapphire). The dissipation in the body can be described by the relation $Q = (\omega\tau_{dis})^{-1}$, where Q is the quality of mechanical oscillations of the frequency ω and τ_{dis} is a parameter characterizing the material. Sapphire at $T \sim 1$ K has $\tau_{dis} \sim 10^{-15}$ sec; see [6], Chap. 2. For the case of the body in the Coulomb potential the dissipation through deformations can be described by a force of friction proportional to the velocity with $\tau_1 \sim E\tau_{dis}^{-1}r^{-2}l^{-1}m^{-1}\tau_{mech}^{-4}$, where E is the modulus

of elasticity. Taking $E \sim 10^{11}$ N/m^2 we obtain $\tau_1 \sim 10^{24}$ sec.

At last we consider the dissipation caused by the heat exchange between the body and its surrounding when the body is subjected to varying deformations. The thermodynamic cycle is almost adiabatic if $\varkappa c^{-1}\rho^{-1}l^{-2}\tau_{mech} \ll 1$ and almost isothermal in the opposite case; here $\varkappa$ is the heat conductivity for the material of the body, c is its specific heat and ρ is its specific density. For sapphire at T ~ 1 K we have $\varkappa c^{-1}\rho^{-1} \sim 10^4$ m^2/sec, this is why we consider only the almost isothermal case. The dissipation can be described again by a force of friction proportional to the velocity with $\tau_1 \sim \varkappa\alpha^{-2}T^{-1}r^2l^{-3}m^{-1}\tau_{mech}$, where α is the temperature coefficient of expansion. For sapphire at T ~ 1 K we have $\varkappa\alpha^{-2}T^{-1} \sim 10^{22}$ m·kg·sec^{-3} and again $\tau_1 \sim 10^{25}$ sec.

The mechanisms of dissipation have been considered which are the most difficult ones for being eliminated, and we see that some extraordinary but surely attainable isolation of mechanical degrees of freedom makes it possible to retain quantum correlations in macroscopic motion during a macroscopically considerable time which may be enough to execute rather a complicated movement. At the same time we see that even under such extraordinary circumstances a macroscopic delocalization of the wave function is unattainable. On the other hand, consider a macroscopic system under usual circumstances, say, the pointer of a galvanometer, which moves at T ~ 300 K with a friction such that $\tau_1 \sim 3$ sec. Then (21) gives $\tau_f \sim \tau_1^{1/2}\tau_{therm}^{1/2} \sim 10^{-6}$ sec. Of course, making use of (21) in such a way is questionable because the interaction is not weak. Nevertheless it is very likely that the time necessary for a human observer to read an instrument is in any case much greater than the time of classical factorization for the corresponding degrees of freedom of the pointer.

6. DISCUSSION AND CONCLUSION

6.1. About Accuracy of Mechanical Description

There is the judgement that any talk about the coordinate of a macroscopic body with the accuracy, say, 10^{-20} m, is a mathematical abstraction obviously devoid of any physical sense regardless of which mechanics are used, classical or quantum. To prove it they think enough to remind that a macroscopic body is nothing more than a bounded state of many particles described actually by a field theory, and the body is non-ideal in many respects, each of which is enough to introduce a fuzziness much greater than 10^{-20} m.

However a macroscopic body is a collective phenomenon

and it is naturally much less fuzzy than its small parts. The accuracy intrinsic to coordinates and momenta of a macroscopic body is the accuracy up to which the corresponding mechanical equations hold. And it is a fundamental limit for the accuracy of our measurements. If this limit is not attained, it becomes a challenge for the measurements technology rather than for the theory, unless some fundamental barrier is revealed.

For the extraordinary but attainable situation considered in 5.2 the mechanical equations hold during $t \sim 1$ sec with the following accuracy:

$$(D(p(t)))^{1/2} \sim \frac{t}{\tau_f}\sqrt{\frac{\hbar m}{t}} \sim 10^{-24} \text{ kg} \cdot \text{m/sec} \quad \text{for the momenta,}$$

$$(D(q(t)))^{1/2} \sim \frac{t}{\tau_f}\sqrt{\frac{\hbar t}{m}} \sim 10^{-21} \text{ m} \quad \text{for the coordinates.}$$

Of course, it is a matter of quantum mechanical equations, since $D(p(t)) \cdot D(q(t)) \ll \hbar^2$. (This does not contradict to the uncertainty principle, because these parameters are related to an operation rather than a state.) We try to suggest some two-valued measurement with such accuracy. Unfortunately it destroys the quantum state and cannot be used to prepare a state localized as much. Thus, we place a body with a positive electric charge near the unstable stationary point between two fixed bodies with negative charges. Choosing the parameters as in 5.2 we have $\tau_{mech} \sim 1$ sec. Several tens of seconds later an obvious deviation will occur toward one of the two attractors. So we can measure the following two-valued observable:

$$sign\left(Q + \frac{\tau_{mech}}{m} \cdot P + \zeta\right),$$

where ζ is a random variable with $(D(\zeta))^{1/2} \sim 10^{-21}$ m.

6.2. About Bell Inequalities for Macroscopic Bodies

Is it possible to observe a violation of a Bell inequality in some macroscopic experiment? This intriguing problem was discussed in our paper [1] and will be discussed in what follows but remains still unsolved. The formal solution given in [1] leaves the three open questions: (a) how to prepare the needed initial state; (b) how to eliminate thermal fluctuations; (c) how to execute the needed measurements. The question (a) remains open; (b) is answered in Sect. 5.2; (c) will be now discussed. According to [1] and Sect. 2 here we need a way to measure (alternatively, of course) $sign(P+\zeta)$ or $sign(Q+\eta)$, where ζ, η are normally distributed random variables such that $D(\zeta) \cdot D(\eta) \ll \hbar^2$. Due to the unitary equivalence the pointed operators can be replaced with $sign(Q+aP+\zeta)$ and $sign(Q+bP+\eta)$ with arbitrary but essentially different a,b (the initial state being replaced accordingly). And such measurements were

proposed in Sect. 6.1; τ_{mech} being the classical parameter like the famous orientation of a spin-measuring device.

Remark. Maybe, it is the preparation which is crucial here. Maybe, only the states with positive Wigner distributions (cf. with [8]) can be prepared for macroscopic systems during an acceptable time. This seems an interesting topic for mathematical examination.

6.3. About Hidden Variables

A "dotted-in-time" description with hidden variables was promised in the introduction and will be developed here. Consider a quasi-classical system with a finite time of classical factorization τ_f. One can partition the time axis into intervals of the duration τ_f (or more); then an aggregate of hidden variables (or, what is the same, a classical state) per interval arises as follows.

The quantum state change during one interval is described by some operation O of the form $O = O^{qc}O^{cq}$, where O^{cq} transforms quantum states into classical ones, and O^{qc} does the reverse. The evolution during a time $\gg\tau_f$ can be described by the iteration of O:

$$...OOO... = ...(O^{qc}O^{cq})(O^{qc}O^{cq})(O^{qc}O^{cq})...;$$

rearranging the brackets we obtain

$$...(O^{cq}O^{qc})(O^{cq}O^{qc})(O^{cq}O^{qc})...,$$

non-coupled boundary terms being omitted. However, $O^{cq}O^{qc}$ is nothing else but a Markov transition function; correspondingly, its iteration describes a Markov chain

$$... \rightarrowtail \lambda_i \rightarrowtail \lambda_{i+1} \rightarrowtail ...$$

which accompanies the quantum evolution; each λ_i is an aggregate of hidden variables. If τ_f is small then the Markov chain approximates a continuous-time Markov process. This is the case for immediately perceivable motion (say, the pointer of a galvanometer). The opposite case arises for extraordinary circumstances considered in Sect. 5.2. Executing quantum measurements of short duration in partitioning instants we obtain a Markov chain of the form

$$... \rightarrowtail \lambda_i \rightarrowtail (\lambda_i, x_i) \rightarrowtail \lambda_{i+1} \rightarrowtail (\lambda_{i+1}, x_{i+1}) \rightarrowtail ...$$

where λ_i is the aggregate of hidden variables for the time interval preceding the measurement which gives the outcome x_i. Omitting one half of terms, one obtains again $... \rightarrowtail \lambda_i \rightarrowtail \lambda_{i+1} \rightarrowtail ...$ but the transition probabilities are changed by the measurements. Omitting the other half of terms one obtains a two-component Markov chain $... \rightarrowtail (\lambda_i, x_i) \rightarrowtail (\lambda_{i+1}, x_{i+1}) \rightarrowtail ...$ with the second component ob-

servable but the first one not. Note that the random sequence $\ldots x_i, x_{i+1}, \ldots$ is not Markov in general; this is connected with the non-ideal nature of our measurements. Of course, in many cases one and the same interaction process forms the basis of both the classical factorization and the measurement. In this case one can take $\lambda_i = x_i$, and the hidden variables become open.

6.4. About Quasi-Classical Limit

Instead of the well-known quasi-classical limit for $h \to 0$ we propose a new limit: for $h \to 0$, $\lambda \to 0$ and $\lambda^2/h \to \infty$; here λ is the coupling constant of the interaction with the environment. The classical factorization leads to a Markov chain in the phase space. The condition $\lambda^2/h \to \infty$ ensures that $\tau_f \to 0$, hence the time discreteness vanishes in the limit. And the stochastics vanish too; the Markov pathes tend to classical trajectories.

This (h,λ)-limit is more adequate than the traditional h-limit for describing situations occuring usually for actual macroscopic systems, like a billiard-ball rolling on a billiard-cloth, or a planet rotating around the sun, or the pointer of a galvanometer moving along the indicator.

It is possible to calculate for given finite h and λ the corresponding Δq, Δp, Δt indicating the limits for the classical description.

In addition, the long-time behavior for finite λ is quite different from that for $\lambda=0$.

Table. Modes of the behavior of a macroscopic mechanical system.

isolation from environment \ accuracy of measurements	usual	extraordinarily high
usual	classical mechanics with dissipation and small fluctuations	hidden variables
extraordinarily high	classical mechanics	quantum mechanics

6.5. Information Propagation as Cause of Classical Factorization

Let us return to the operators $A_2(x,t)$ mentioned in Sect. 4.5 and put

$$U(x) = \mathrm{T}\exp\left(-\frac{i}{h}\int dt A_2(x,t)\right)$$

for an arbitrary point x of the phase space of a quasi-classical system taken in the Heisenberg representation. Due to (6),

(8), (9) we obtain

$$h^{-2}D(\zeta(x_2)-\zeta(x_1)) \approx -2\ ln\ |\ \langle U^*(x_2)U(x_1)\rangle_{W_2}\ |.$$

Define the Riemann metric on the phase space by taking the above right-hand side as the square of the distance between infinitesimally close points x_1, x_2. Then the criterion (2) of classical factorizability can be written as

$$h\cdot\{f,\ g\} \le |df|\cdot|dg|$$

for any smooth functions f,g on the phase space; here $|df|$ denotes the norm of the covariant vector df on the Riemann space. For the case of one degree of freedom this means that the phase volume of the ellipse, which is the unit circle for this Riemann metric, does not exceed πh.

In the case of a pure state W_2 we have

$$\langle U^*(x_2)U(x_1)\rangle_{W_2} = \langle\psi(x_1),\ \psi(x_2)\rangle,$$

where $\psi(x)=U(x)\psi$, ψ is the state vector for W_2. The state of the second system after the interaction is close to $\psi(x)$ provided the state of the first system is localized near x, namely, in a neighborhood where $\psi(x)$ is almost constant. However, this condition is impracticable in the case of classical factorization. We realize that our Riemann metric characterizes the interaction as "information propagation"; the second system reacts to the first one through the transition $\psi \to \psi(x)$, neighboring points being almost indistinguishable. Of course, we take some liberty with the notion of information; we should rather to talk about quantum correlation setting capable to lead to an indirect quantum measurement...

The classical factorizability means nothing else but the inevitability of such information propagation. This leads to a new point of view on actions which are to be taken in order to prevent the classical factorization (see Sect. 5.2). The high vacuum is needed to prevent the information propagation to the surrounding gas. (But one may say equally right that it is needed to reduce the fluctuations.) The cooled sapphire is needed to prevent the propagation of the information about mechanical movements of the body through its deformations into the phonon gas inside the body or generally into internal degrees of freedom. And so on. The following idea arises: maybe it is better to do the experiment in a superconducting chamber to prevent the information propagation through the electromagnetic radiation? And this is really the case. This is a way to reduce the radiative friction and the friction on the thermal electromagnetic radiation.

We see that the "information propagation" point of view is heuristic. However, its inadequate use leads to discouraging conclusions. It is senseless to drown (to overrun) the radiation of the body with some noise. Speaking about information in this context we are perfectly indifferent towards the

availability of this "information". Perhaps this kind of information should be named "hidden information". Note that it is undestroyable (cf. with the classical law of the conservation of the phase volume).

Remark. Our approach, being local, allows us to estimate $\langle U^*(x_2)U(x_1)\rangle_{W_2}$ only for near points x_1, x_2. We assume that it is small all the more if these points are distant. We have no reason for doubt, however we would like to have a proof.

6.6. Several Remarks

The essence of the Everett's approach is an irreversible propagation of quantum correlations as the cause of the classical behavior. But as to "many worlds", this is rather Kolmogorov's approach. Indeed, the Kolmogorov's axiomatics of the probability theory ($\approx$ 1930) treats any random variable as a function of a universal auxiliary variable ω. Enlarged to the whole physical world this leads immediately to one "possible world" per each value of ω. And Everett warned from the very beginning: worlds co-exist as possibilities rather than as physical systems; see [9], Note ‡ on page 459-460.

Our technique has a common part with that of Stapp [10], namely the Gaussian transform of quantum states into classical ones (see O_1 in the proof of Theorem 3.4 above). It enables to overcome the known continuum problem in the quantum measurement theory. On the other hand, we are not inclined to share Stapp's determination to postulate an exceptional ontological status for the light or for any physical system whatsoever. But we could have entitled this paper like the Stapp's one: "Heat as Foundation of Being". Indeed, just the thermal fluctuations are the main cause of the classical factorization; the latter would occur much slower at zero temperature. And the nature of the heat bath is of less importance. A case is possible where the electromagnetic radiation to the infinity is negligible and the classical factorization results mainly from the phonon radiation into the body.

Maybe there are such cases where the classical behavior results from some mechanism other than proposed here.

We are not inclined to insist on a completely sharp classical description. If we consider this description as secondary, derived from the quantum one, then we should not be surprised at some its fuzziness. However, we are glad that it satisfies the Bell inequalities exactly. Further, why the sole classical realm can be derived? We have no reason to contend its non-uniqueness, but we admit this possibility. If it is really sole according to Deutsch's [11] trend, then it is interesting, but not vital. Indeed, nobody believes it vital for a physical theory to predict the sole planet inhabited by people in the world. The exceptional status of some realm with res-

pect to us people is rather natural.

Maybe a reader is displeased that we use "second-grade" notions of thermodynamic nature for foundations of the "first--grade" quantum theory. However, if we admit the possibility that the God is kindly disposed to dice, why not to heat also?..

Of course, we do not pretend for having solved here any problem of statistical physics. On the contrary, we need solutions for some of them for the foundation of our approach.

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