Bell inequalities and operator algebras

Boris Tsirelson

Problem

Quantum Bell-type inequalities are defined in terms of two (or more) subsystems of a quantum system. The subsystems may be treated either via (local) Hilbert spaces, — tensor factors of the given (global) Hilbert space, or via commuting (local) operator algebras. The latter approach is less restrictive, it just requires that the given operators commute whenever they belong to different subsystems.

Are these two approaches equivalent?

Background

Quantum Bell-type inequalities are discussed in Problem 26 "Bell inequalities holding for all quantum states" and Partial Solutions to Problem 1 "All the Bell inequalities" (and many other texts, including my own [1]) without a clear indication of the approach used.

Some relevant formal definitions (for two subsystems) may be found in [1], Sect. 2 (and Sect. 1). Formulas (2.1–2.3) of [1] formalize the less restrictive approach (via commuting operators), while formulas (2.4–2.6) formalize a more restrictive approach, requiring a tensor product Hilbert space (which could be essential) and some additional restrictions (these are not essential). Unfortunately, it is claimed (without proof) in [1] that the two approaches are equivalent! This is my error. I regret deeply for my irresponsible claim. A failure of my old would-be-proof came to light in May 2006. We must thank Antonio Acin who requested from me a proof.

Three convex sets are considered in [1] and Problem 26, denoted $X_{\text{HDB}} \subset X_{\text{QB}} \subset X_{\text{B}}$ and $C \subset Q \subset P$ respectively. Their elements are called 'behaviors' and 'correlations', respectively. However, in both cases one should consider four sets, say, $C \subset Q' \subset Q'' \subset P$, where Q', Q'' correspond to the two approaches (more restrictive and less restrictive, respectively).

The question is, whether $Q' = Q''$, or not.

If $Q' \neq Q''$ then another (even more important) question arises naturally: is Q' dense in Q'', or not?

If Q' is essentially different from Q'' (that is, not dense) then we should decide, which one is more relevant to physics. I believe that Q' is. Here is why. In physics we deal not just with commuting operators, but with operators belonging to local algebras corresponding to disjoint domains (on a positive distance). True, it is usually believed that local algebras are type III factors. But it is also usually believed that they are separated by type I factors, provided that the domains are disjoint. (This is called "F property" or "funnel property" in the local quantum field theory.) And type I factors mean a tensor product Hilbert space.

Still another question appears if $Q' \neq Q''$: which properties of operator algebras are discerned by Q? One may introduce $Q_{\mathrm{I}}, Q_{\mathrm{II}}, Q_{\mathrm{III}}$ for type I, II, III factors, then $Q_{\mathrm{I}} = Q'$, but what about $Q_{\mathrm{II}}, Q_{\mathrm{III}}$? Are they equal?

Partial Results

If the given (global) Hilbert space H is finite-dimensional and only two subsystems are dealt with, then the two approaches are equivalent. *Proof* (sketch). The given operators of the first subsystem generate an operator algebra $\mathcal{A}_1$. Its center decomposes H into the direct sum $H_1 \oplus \dots \oplus H_n$ of subspaces (sectors) H_k. On each sector, $\mathcal{A}_1$ boils down to a factor. Thus, $H_k = H'_k \otimes H''_k$ and

$$H = H'_1 \otimes H''_1 \oplus \dots \oplus H'_n \otimes H''_n \, .$$

It remains to embed H into $H' \otimes H''$ where

$$H' = H'_1 \oplus \dots \oplus H'_n \, , \quad H'' = H''_1 \oplus \dots \oplus H''_n \, .$$

Remarks

Three decades ago some mathematicians told me that quantum Bell-type inequalities belong to physics, not mathematics, because they are not related to the advanced mathematical theory of operator algebras (classification etc). What now?

If you wonder, where to find commuting operators that do not fit into the tensor product paradigm, here is a candidature. Consider the free group G with (say) two generators a, b. (Elements of G are like this: $b^3 a^{-2} b a b^{-2} a$.) The countable group G acts on the Hilbert space $H = l_2(G)$ (of all square summable functions on G) by left shifts as well as right shifts: $U_x \psi(z) = \psi(xz)$, $V_x \psi(z) = \psi(zx^{-1})$ for $x, z \in G$, $\psi \in H$. Every U_x commutes with every V_y. However, it does not decompose H into $H' \otimes H''$.

Let us choose the simplest state vector ψ, namely, $\psi(x) = 1$ if x is the unit of G, otherwise $\psi(x) = 0$. Strong correlations appear: $\langle U_x V_x \rangle_\psi = 1$. Do they lead to a point of Q'' not approximable by Q'? I do not know.

References

[1] B.S. Tsirelson, *Some results and problems on quantum Bell-type inequalities,* Hadronic Journal Supplement **8**, 329–345 (1993). Available also on my site, here: `http://www.tau.ac.il/~tsirel/download/hadron.html`

Boris Tsirelson
School of Mathematics
Tel Aviv University
Tel Aviv 69978, Israel

`mailto:tsirel@post.tau.ac.il`
`http://www.tau.ac.il/~tsirel/`