Contests:
A Unique, Mixed Strategy Equilibrium

Michael Landsberger
Department of Economics
Haifa University, Haifa, Israel
e-mail:lmichael@econ.haifa.ac.il

Boris Tsirelson
School of Mathematical Sciences
Tel Aviv University, Tel Aviv, Israel
e-mail: tsirel@math.tau.ac.il

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Abstract

Actions in games that address economic environments such as auctions and oligopoly games are typically costly, and signals, or types, are interdependent. Consequently, such games may not have equilibria supported by monotone strategies, *monotone equilibria,* see Landsberger and Tsirelson (1999, 2000). Derivation of pure strategy non-monotone equilibria when types are interdependent and continuously distributed has not been explored in the literature, and it is certainly a non trivial matter; mixed-strategy equilibria pose additional problems. We address these issues by considering a contest game that can be interpreted as competition for research funds or jobs.

Assuming a multinormal distribution of signals, we were able to establish a mixed strategy equilibrium and prove that this is the *only* equilibrium in the class of all equilibria. Various properties of this equilibrium were established. We showed some pitfalls one may encounter by trying to impose a-priori (reasonable) restrictions on strategies, or trying to conduct the analysis assuming a general class of signal distribution.

Introduction

A negative result established in Landsberger and Tsirelson (1999, 2000) claims that a large class of symmetric games, including many types of auctions as well as the model considered in this paper, *do not have monotone equilibria,* unless the correlation between signals, bidding cost or the number of players is sufficiently small. This result poses serious questions about equilibria in this class of games. Since non-monotone equilibria in games under incomplete information, continuously distributed and interdependent signals (which is the class considered) is an unexplored terrain, it is not clear how to proceed if one wishes to establish uniqueness of equilibrium and investigate which strategies support it.

Groping our way in the dark, it is prudent not to proceed with full generality, but advance gradually to get an insight, first. To implement this advice, we note that the non-monotonicity established by L&T originates in participation. Therefore, we chose to investigate non-monotone equilibria in models that focus on participation, leaving aside bidding strategies. To further simplify the analysis, we give away the idea of considering a large class of signal distributions, and assume the multinormal distribution with a positive correlation coefficient ρ. This is certainly the most natural and well-investigated distribution for correlated random variables.[1] Restricting attention to this 'simple' setting do we end up with a trivial problem; we leave it to the reader to pass the judgement.

Consequently, we consider a setting in which n players privately observe signals (types), and must then decide whether they want to participate in a game. Participation is costly, and signals are truly disclosed to the principal who runs the contest. Participants in the game are players that were willing to incur participation cost, c. The participant with the highest signal gets a prize, C. Signals are interdependent and the prize is constant. Such a model is different from an auction since there is no bidding strategy; participants are truthful. This assumption is well supported by some real life situations noted below. Consequently, the only strategic variable is participation.

There are many situations that correspond to this model to which we refer as a *contest.* Bidding for research funds is one example; players must decide if they want to participate, which is costly. Each player submits his signal that corresponds to his research record and the player with the highest signal is the winner and gets the prize. Competing for academic, or other, jobs is another example; the job is the prize. In all these cases it is assumed that

[1] Whereas in the case of one dimensional signals one could easily suggest other distributions, this is not an option when dealing with many *correlated* signals.

types are given and are truthfully reported, being supported by verifiable and documented facts, like a curriculum vitae. The setting we consider does not apply to many contests that take place in reality in which participants can affect their types (signals) by exerting effort, see Fullerton and McAffee (1999).

The rationale for signal dependence in situations referred to above, is that although signals such as, say, research performance could be obtained independently, the distribution that generated them is not completely known to players. One can think that each player knows the distribution up to an unknown parameter θ (state of nature). Players cope with uncertainty about θ by assuming it is a random variable with a known distribution. The uncertainty about θ induces signal dependence. Such situations are sometimes referred to as 'conditional independence'.

We prove that:

- Among all symmetric equilibria, our game has a *unique* equilibrium and it is in *mixed* strategies. It is described by a function $\mathbf{p}(\cdot)$ such that a player with signal s participates with probability $\mathbf{p}(s)$.

- There exists a sufficiently high signal, s_{high}, such that signals $s \in (s_{\text{high}}, \infty)$ always participate; namely, $\mathbf{p}(s) = 1$ for all $s \in (s_{\text{high}}, \infty)$.

- The participation strategy $\mathbf{p}(\cdot)$ is discontinuous at s_{high}; namely, $\mathbf{p}(s_{\text{high}}+) = 1$ but $\mathbf{p}(s_{\text{high}}-) = \rho$; that is, $\mathbf{p}(s)$ goes to the correlation coefficient ρ when s goes to s_{high} from the left.

- There exists a low signal $s_{\text{low}} < s_{\text{high}}$ such that on $s \in (s_{\text{low}}, s_{\text{high}})$ the equilibrium participation strategy dictates randomization everywhere; namely, $0 < \mathbf{p}(s) < 1$.

- We do not know, whether $s_{\text{low}} = -\infty$ or not. But if $s_{\text{low}} > \infty$, then the limit $\mathbf{p}(s_{\text{low}}+)$ exists and is equal to 0 or 1.

- The equilibrium participation strategy, $\mathbf{p}(\cdot)$ is continuous on $(s_{\text{low}}, s_{\text{high}})$.

The equilibrium function $\mathbf{p}(\cdot)$ is a solution of a nonlinear integral equation, that, most likely, does not have an explicit analytic solution. In addition to rigorously proven results, we have some observations based on numerical computations for which no generality is claimed. These results can give us indications as to some typical (if not general) properties of equilibrium that we could not address analytically.

- $s_{\text{low}} \neq -\infty$.

- The function $\mathbf{p}(\cdot)$ is non-monotone on (s_{low}, ∞). It is increasing on $(s_{\min}, \infty)$, where $s_{\min}$ is the minimizer of $\mathbf{p}(\cdot)$, if the latter has a single minimum, or the rightmost minimizer if there are several local minima. If $\mathbf{p}(\cdot)$ increases on $(s_{\text{low}}, s_{\text{high}})$ then $s_{\min} = s_{\text{low}}$. Typically, however, $s_{\min} \neq s_{\text{low}}$.

- The interval $(s_{\min}, \infty)$ contains most of the distribution of signals.

- If $s_{\text{low}} < s_{\min}$, the function $\mathbf{p}(\cdot)$ has from 1 to 3 intervals of monotonicity on the interval $(s_{\text{low}}, s_{\min})$.

Unfortunately, we were unable to establish any results for $(-\infty, s_{\text{low}})$.

An important structural property that emerges when participation is a strategic variable, is that strategies and the probability of winning are interdependent. This did not happen in auction models that ignored participation and had monotone equilibria. Such settings made it possible to explain the rationale of the equilibrium strategy in terms of players' assessments of their probability of winning, given their signals only (and irrespective of strategies). From a game theoretic point of view this may sound strange, but under monotonicity, it worked. The fact that this convenient property is absent from our model, is a source of much of the complexity we had to deal with.

To assess the importance of the numerical analysis note the following. Although, eventually we proved analytically that the equilibrium strategy has a 'jump' at ρ, this was done only after this property was revealed by a numerical analysis. Without that, the chances were that the analysis would have been restricted to continuous strategies. Such a restriction is common in the literature; it is based, probably, on the hope that in a model where everything is continuous, so are the equilibrium strategies. Moreover, such an assumption simplifies the mathematics. Well, had we done so, our conclusion would have been that our model does not have an equilibrium. This is so, since the model has a unique equilibrium and this equilibrium is supported by a strategy that is discontinuous.

Having established that there is only one equilibrium, and it is in mixed strategies, we might have been tempted to prove that the equilibrium strategy $\mathbf{p}(s)$ is monotone in s; namely, that players with higher signals participate with a higher probability. This would have been our 'intelligent' guess. The numerical analysis saved us the frustration of trying to prove an incorrect conjecture. Strictly speaking, there is no monotonicity, however, the numerical analysis indicates that, typically, $\mathbf{p}(\cdot)$ is monotone in s for a large segment of the distribution, including the right tail. This is an important result since this is where the winners come from.

These results illustrate, once again, the need to avoid, as much as possible, a-priori restrictions on equilibrium strategies, and also the need to do some numerics before trying to obtain general results when considering complex models.

It is worthwhile to mention that having established a unique mixed strategy equilibrium is somewhat puzzling. One might think that our model is basically a special case of an auction, except that it has a trivial bidding strategy and the winner is not required to make any payments. Therefore (?), one could hope that it allows a monotone equilibrium whenever an auction does. This, however, turns out not to be true since as shown in L&T (1999), a second price auction can have a monotone equilibrium when ρ, c or n are sufficiently small.

From a game theoretic point of view, to the best of our knowledge, this is the first model for which an equilibrium in mixed strategies was established in a game of incomplete information, with continuous types, and was shown to be the only equilibrium in the class of all symmetric equilibria. Our method of establishing a unique equilibrium may be indicative about the strings one may wish to pull to get an insight of how to solve other problems in which monotone equilibria do not exist.

Relating our paper to the existing literature, the closest is Krishna and Morgan (1997) who were the first to consider a model with interdependent signals and costly actions. In considering an all-pay auction with affiliated signals they derived conditions that are sufficient to obtain a monotone equilibrium. They were aware of the fact that when equilibria are not monotone, the currently used techniques can not be applied and it is not clear how to proceed. Applying results established in L&T (1999) and (2000), it follows that for a large number of players any conditions that are sufficient for existence of a monotone equilibrium inevitably force *weak* dependence between signals.

A few papers considered the case when bidders may get more information about signals at a cost, see Matthews (1984) and a more general setting in Jackson (1999). This is certainly a realistic aspect of bidding situations, but these papers concentrated on aggregation of information and did not address participation. In similarity to our paper, they considered interdependent signals, which is a crucial element.

There are more papers that investigated the effect of bidding cost, but they did it in a setting of independent signals. Such settings allow monotone equilibria, and are therefore less related to our paper. Moreover, almost all these papers focused on problems of efficiency and not so much on participation. The papers take different approach to the timing when participation cost must be incurred, and auction rules considered. In Samuelson (1985) bidding cost take place when potential bidders know their valuations, as in our paper. McAfee&McMillan (1987) and Tan (1992) consider the case where cost are incurred before potential bidders know their types. Paying at the stage of complete ignorance, has the natural interpretation of learning about ones type, which does not remove the reasons for the existence of bidding cost. Obviously, both cost may take place in the same auction; one may be willing to pay to improve the quality of his signal, and to have to incur bidding cost.

In similarity to McAfee&McMillan and Tan, Levin&Smith (1994) assume that participation cost are paid before players know their types. They were the first to model participation as part of equilibrium, and addressed the problem of efficiency and market size. Stegeman (1996) considered a more general setting; in particular, he did not restrict attention to symmetric games as do the other papers, including ours. He derived a version of the revelation principle for models where participation cost are accounted for, but, he too did not focus on participation. Lixin (2001) considered the case where players had initially private information about their types, as in Samuelson, but were allowed to further augment it, at a cost. But he, too, assumed independence of information and had therefore no problems with monotonicity. Lixin's paper concentrated on a special auction design; indicative bidding.

The plan of the paper is as follows. In Section 1 we present, very briefly, the essentials of the model. In Section 2 we derive the equilibrium equation. From here on, we assume multinormal signals. A unique, always participate, equilibrium was established in Section 3 for the case where the minimal participating signal was set sufficiently high. In Section 4 we establish a sequence of lemmas that are needed to isolate the equilibrium strategy. Using results established in Section 4, equilibrium is derived in Section 5; we prove its uniqueness and establish some properties. To get a certain feeling as to the properties of equilibrium, we present in Section 6 results obtained from numerical calculations. Such an analysis adds much to our understanding of the properties of equilibrium, in providing partial answers to important questions that can not be addressed analytically. In Section 7 we prove that if signals are independent there exists a unique equilibrium in monotone strategies. However, this pure strategy equilibrium disappears even at the slightest deviation from independence.

1 The Model

We consider a game of n risk neutral and, initially, identical players who observe their private one-dimensional signals $S_1, \dots, S_n$. Player k is allowed to participate if and only if his signal S_k exceeds a reserve level r. In that case the player chooses an action A_k, either 1 (participate) or 0 (quit). All players allowed to participate choose their actions simultaneously and independently.[2] If no one participates, all get nothing. Otherwise, each participant pays an entry cost c, and the one with the highest signal $S_k = \max_l \{ S_l \,|\, S_l > r, A_l = 1 \}$ is the winner and gets the prize, C.[3] Parameters C, c, r and the joint distribution of signals are common knowledge. Naturally, $0 < c < C < \infty$.

The joint distribution of signals is assumed to be of the form

$$\mathbf{F}_n(s_1, \dots, s_n) = \int \mathbf{F}(s_1|\theta) \dots \mathbf{F}(s_n|\theta)\, \mathbf{G}(d\theta)\,, \tag{1.1}$$

where $\mathbf{F}_n(s_1, \dots, s_n) = \mathbb{P}\left(S_1 \le s_1, \dots, S_n \le s_n \right)$ is the n-dimensional (cumulative) distribution function, $\big(\mathbf{F}(\cdot|\theta)\big)$ is a family of one-dimensional distributions parametrized by θ, and the parameter θ is drawn from its distribution $\mathbf{G}$.[4] In other words, we have $n+1$ random variables $\Theta, S_1, \dots, S_n$ such that

Θ is distributed $\mathbf{G}$,
every S_k is distributed $\mathbf{F}_1$,
$S_1, \dots, S_n$ are conditionally independent given Θ,
given $\Theta = \theta$, each S_k is distributed $\mathbf{F}(\cdot|\theta)$;

here $\mathbf{F}_1(s) = \int \mathbf{F}(s|\theta)\, \mathbf{G}(d\theta)$, which is just (1.1) for $n = 1$.

A strategy (possibly mixed) of player k is described by a function $\mathbf{p}_k(\cdot)$ such that

$$\mathbf{p}_k(s_k) = \mathbb{P}\left(A_k = 1 \,\middle|\, S_k = s_k \right) \tag{1.2}$$

for $s_k \in (r, \infty)$; that is, $\mathbf{p}_k(s_k)$ is the participation probability for player k possessing signal s_k. A pure strategy corresponds to the case where $\mathbf{p}_k(\cdot)$ takes on two values only, 0 and 1. In general, $0 \le \mathbf{p}_k(s_k) \le 1$. Nothing like monotonicity, continuity etc. is assumed; $\mathbf{p}_k(\cdot)$ is just a function, defined on (r, ∞) almost everywhere w.r.t. the distribution $\mathbf{F}_1$ of a signal, measurable

[2] We do not consider correlated equilibria.

[3] Ties will be excluded by assuming nonatomic distributions of signals.

[4] The parameter θ need not be one-dimensional, this is why we prefer the notation $\mathbf{G}(d\theta)$ to $d\mathbf{G}(\theta)$.

w.r.t. the same distribution, and taking on values on $[0,1]$.[5] We leave $\mathbf{p}_k(\cdot)$ undefined on $(-\infty, r)$, since such signals exclude participation.

Actions are conditionally independent given the signals, that is,

$$\begin{aligned}
&\mathbb{P}\left(A_1 = a_1, \dots, A_n = a_n \,\middle|\, S_1 = s_1, \dots, S_n = s_n \right) = \\
&\qquad = \mathbb{P}\left(A_1 = a_1 \,\middle|\, S_1 = s_1 \right) \cdot \ldots \cdot \mathbb{P}\left(A_n = a_n \,\middle|\, S_n = s_n \right) = \\
&= \big(a_1 \mathbf{p}_1(s_1) + (1-a_1)(1-\mathbf{p}_1(s_1))\big) \cdot \ldots \cdot \big(a_n \mathbf{p}_n(s_n) + (1-a_n)(1-\mathbf{p}_n(s_n))\big)\,;
\end{aligned}$$

here and henceforth $s_1, \dots, s_n$ run over (r, ∞).[6]

2 Probability of winning and a description of equilibrium

The payoff Π_k of player k is a random variable (in fact, a function of $S_1, \dots, S_n$, $A_1, \dots, A_n$) taking on three values: 0 in the case of no participation, $-c$, participation but no win, and $C-c$, participation and win. In order to calculate the expected payoff given the signal, $\mathbb{E}\left(\Pi_k \,\middle|\, S_k \right)$, we need first the probability of winning, given participation and signal. To this effect, we start by conditioning also on the common (random) factor Θ. To exclude ties, we assume that each $\mathbf{F}(\cdot|\theta)$ is continuous. Just for notational convenience, assume for a while that $k = n$ and all signals are positive. Then, the desired probability is given by

$$\begin{aligned}
(2.1)\quad &\mathbb{P}\left(\Pi_n = C - c \,\middle|\, A_n = 1, S_n = s_n, \Theta \right) = \\
&\quad = \mathbb{P}\left(A_1 S_1 < s_n, \dots, A_{n-1} S_{n-1} < s_n \,\middle|\, A_n = 1, S_n = s_n, \Theta \right) = \\
&\qquad = \prod_{k=1}^{n-1} \mathbb{P}\left(A_k S_k < s_n \,\middle|\, A_n = 1, S_n = s_n, \Theta \right) = \\
&\qquad\qquad = \mathbb{P}\left(A_1 S_1 < s_n \,\middle|\, \Theta \right) \cdot \ldots \cdot \mathbb{P}\left(A_{n-1} S_{n-1} < s_n \,\middle|\, \Theta \right)
\end{aligned}$$

due to conditional independence.

Note that since we do not restrict strategies to be monotone, and not even pure, the probability of winning involves not just signals, S_i, but also actions A_i of the other potential players; consequently, the relevant random

[5] Or rather, an equivalence class of such functions, where equivalence is treated as equality almost everywhere w.r.t. $\mathbf{F}_1$. It is nothing but another form of a distributional strategy for the case of an action space of two points only. Uniqueness of equilibrium will be treated up to equivalence.

[6] However, $S_1, \dots, S_n$ are *not* conditioned to stay in (r, ∞), unless it follows from explicit conditions $S_k = s_k$.

variables are products of signals and actions, $A_i S_i$, see (2.1). This is how participation strategy of players $i = 1, \ldots, n-1$ is needed in order to derive an expression for the probability of winning of player n. This is a natural result in a strategic situation; it was avoided in most of auction models because attention was restricted to models that had equilibria supported by monotone strategies.

Going back to (2.1), we can replace the actions A_i by strategies that induced them, $\mathbf{p}(\cdot)$, to obtain,

$$\mathbb{P}\left(A_1 S_1 < s_n \,\middle|\, \Theta = \theta \right) = 1 - \mathbb{P}\left(A_1 = 1, S_1 > s_n \,\middle|\, \Theta = \theta \right) = 1 - \int_{s_n}^{\infty} \mathbf{p}_1(s_1)\, d\mathbf{F}(s_1|\theta)\,.$$

So,

$$\mathbb{P}\left(\Pi_n = C - c \,\middle|\, A_n = 1, S_n = s_n, \Theta \right) = \\ = \left(1 - \int_{s_n}^{\infty} \mathbf{p}_1(s_1)\, d\mathbf{F}(s_1|\theta)\right) \cdot \ldots \cdot \left(1 - \int_{s_n}^{\infty} \mathbf{p}_{n-1}(s_{n-1})\, d\mathbf{F}(s_{n-1}|\theta)\right).$$

The auxiliary assumption that signals are positive may be discarded now. Considering a *symmetric* equilibrium, we restrict ourselves to the symmetric case

$$\mathbf{p}_1(\cdot) = \cdots = \mathbf{p}_n(\cdot) = \mathbf{p}(\cdot)\,. \tag{2.2}$$

Then

$$\mathbb{P}\left(\Pi_k = C - c \,\middle|\, A_k = 1, S_k = s_k, \Theta = \theta \right) = \left(1 - \int_{s_k}^{\infty} \mathbf{p}(s)\, d\mathbf{F}(s|\theta)\right)^{n-1},$$

which was derived for $k = n$, but holds equally well for all $k = 1, \ldots, n$.

Assuming that distributions $\mathbf{F}(\cdot|\theta)$ have densities $\mathbf{F}'(\cdot|\theta)$ we use Bayes formula for densities. The conditional distribution of Θ given $S_k = s_k$ is

$$\frac{\mathbf{F}'(s_k|\theta)}{\mathbf{F}'_1(s_k)}\, \mathbf{G}(d\theta)\,,$$

where $\mathbf{F}'_1$ is the density of the unconditional distribution of a signal, $\mathbf{F}'_1(s) = \int \mathbf{F}'(s|\theta)\, \mathbf{G}(d\theta)$. Therefore the winning probability

$$\mathbf{W}(s_k) = \mathbb{P}\left(\Pi_k = C - c \,\middle|\, A_k = 1, S_k = s_k \right) \tag{2.3}$$

is equal to

$$\int \mathbb{P}\left(\Pi_k = C - c \,\middle|\, A_k = 1, S_k = s_k, \Theta = \theta\right) \frac{\mathbf{F}'(s_k|\theta)}{\mathbf{F}_1'(s_k)}\,\mathbf{G}(d\theta) = \tag{2.4}$$
$$= \int \left(1 - \int_{s_k}^{\infty} \mathbf{p}(s)\, d\mathbf{F}(s|\theta)\right)^{n-1} \frac{\mathbf{F}'(s_k|\theta)}{\mathbf{F}_1'(s_k)}\,\mathbf{G}(d\theta)\,,$$

which is the probability of winning of a player who decided to participate and whose signal is s_k. Note that the probability depends on strategy of players with higher signals.

The effect of θ is described by the 'winning measure' $\mathbf{w}(\cdot|s)$ defined by

$$\mathbf{w}(d\theta|s) = \left(1 - \int_{s}^{\infty} p(s_1)\, d\mathbf{F}(s_1|\theta)\right)^{n-1} \frac{\mathbf{F}'(s|\theta)}{\mathbf{F}_1'(s)}\,\mathbf{G}(d\theta)\,. \tag{2.5}$$

We have

$$\mathbf{W}(s_k) = \int \mathbf{w}(d\theta|s_k)\,; \tag{2.6}$$

in fact, the probability measure $(1/\mathbf{W}(s))\mathbf{w}(d\theta|s)$ is nothing but the conditional distribution of Θ given that $S_k = s_k$ and player k wins (for a given k). So, expected profit of a participant with signal s_k, is given by,

$$\mathbb{E}\left(\Pi_k \,\middle|\, A_k = 1, S_k = s_k\right) = -c + C\mathbf{W}(s_k)\,. \tag{2.7}$$

Note that the expected profit vanishes if $\mathbf{W}(s_k) = c/C$, in which case the player is indifferent between participating and quitting. We are in a position to describe equilibria.

2.8 Lemma. A participation strategy $\mathbf{p}(\cdot)$ supports a symmetric equilibrium if and only if $\mathbf{F}_1$-almost all $s \in (r, \infty)$ belong to one of the following three cases:[7]

(a) $0 < \mathbf{p}(s) < 1$ and $\mathbf{W}(s) = c/C$;
(b) $\mathbf{p}(s) = 0$ and $\mathbf{W}(s) \le c/C$;
(c) $\mathbf{p}(s) = 1$ and $\mathbf{W}(s) \ge c/C$.

Proof. Follows immediately from (2.7) and the definition of a Nash equilibrium. □

[7] Different s may belong to different cases.

From lemma 2.8 we obtain the equation of a symmetric equilibrium; $\mathbf{W}(s) = c/C$, that is, $\int \mathbf{w}(d\theta|s) = c/C$ with $\mathbf{w}$ substituted from (2.5). However, (b) and (c) are relevant as well.

A few words are in order to identify mathematical properties of the equilibrium condition which is a non-standard *integral equation.* Such an identification may help readers to understand the relation between our equation and well-known integral equations, and corresponding approaches.

The equation $\mathbf{W}(s) = c/C$ may be written in the form

$$\int_{-\infty}^{+\infty} \mathcal{K}_1(s,\theta)\mathcal{F}\biggl(\int_s^\infty \mathcal{K}_2(\theta,s_1)\mathbf{p}(s_1)\,ds_1\biggr)\,d\theta = f(s)\,.$$

Compare it with a Volterra equation of the first kind (φ is the unknown function)

$$\int_s^\infty \mathcal{K}_2(s,s_1)\varphi(s_1)\,ds_1 = f(s)\,,$$

a Volterra equation of the second kind,

$$\varphi(s) = \int_s^\infty \mathcal{K}_2(s,s_1)\varphi(s_1)\,ds_1 + f(s)\,,$$

nonlinear Volterra equation

$$\varphi(s) = \int_s^\infty \mathcal{G}\bigl(s,s_1,\varphi(s_1)\bigr)\,ds_1 + f(s_1)\,,$$

Fredholm equation

$$\varphi(s) = \int_{-\infty}^{+\infty} \mathcal{K}_1(s,\theta)\varphi(\theta)\,d\theta + f(s)\,,$$

and a nonlinear equation of the Hammerstein type

$$\varphi(s) = \int_{-\infty}^{+\infty} \mathcal{K}_1(s,\theta)\mathcal{F}\bigl(\theta,\varphi(\theta)\bigr)\,d\theta$$

(see Tricomi (1957), Sect. 1.3, 1.13, 2.1, and 4.5). Our equation is more complicated, it contains two integrals, external (with a fixed domain of integration, like Fredholm and Hammerstein equations) and internal (with a variable domain of integration, like Volterra equations); it is also nonlinear (like Hammerstein equation). Successive approximations are applicable to these equations, except for the Volterra equation of the first kind (see Tricomi (1957), Sect. 1.3, 1.13, 4.5).

3 Equilibrium strategy of agents with high signals

As noted in the previous section, not restricting attention to monotone strategies only, makes the probability of winning dependent on participation strategy, which makes the a-priori assessments of probability of winning difficult, if not impossible.

It is possible that the simple strategy 'never quit, always participate' (whenever allowed) supports an equilibrium, that is, the constant function, $\mathbf{p}(s) = 1$ for all $s \in (r, \infty)$, satisfies 2.8. One would think that such an equilibrium is reasonable at least for some, very high values of r. And in fact that happens if and only if $\int \mathbf{w}(d\theta|s) \ge c/C$ for all $s \in (r, \infty)$, where

$$\mathbf{w}(d\theta|s) = \mathbf{F}^{n-1}(s|\theta) \frac{\mathbf{F}'(s|\theta)}{\mathbf{F}'_1(s)} \, \mathbf{G}(d\theta) \, ,$$

which is (2.5) for the case of $\mathbf{p}(\cdot) = 1$. The corresponding integral

$$\mathbf{h}(s) = \int \mathbf{F}^{n-1}(s|\theta) \frac{\mathbf{F}'(s|\theta)}{\mathbf{F}'_1(s)} \, \mathbf{G}(d\theta) \tag{3.1}$$

(nothing but $\mathbf{W}(s)$ for the case of $\mathbf{p}(\cdot) = 1$) has a clear meaning, it is the probability of being the maximal signal,

$$\mathbb{P}\left(S_k = \max(S_1, \dots, S_n) \,\middle|\, S_k \right) = \mathbf{h}(S_k) \, . \tag{3.2}$$

The following example illustrates that in a general class of distributions such an equilibrium may not be possible because of strong irregularities (pathologies) of the $\mathbf{h}(\cdot)$ function, that need not be monotone, nor continuous, nor $\mathbf{h}(-\infty) = 0$, nor $\mathbf{h}(+\infty) = 1$. Here is the example. Let θ run over integers, that is, $\mathbf{G}(m-) < \mathbf{G}(m+)$ and $\mathbf{G}(m+) = \mathbf{G}((m+1)-)$ for $m = \dots, -2, -1, 0, 1, 2, \dots$ Further, let $\mathbf{F}(s|\theta) = s - \theta$ for $s \in (\theta, \theta+1)$; that is, $\mathbf{F}(\cdot|\theta)$ is the uniform distribution on $(\theta, \theta+1)$. Then $\mathbf{h}(\cdot)$ is a discontinuous *periodic* function. Namely, $\mathbf{h}(m+\alpha) = \alpha^{n-1}$ for all $\alpha \in (0,1)$ and integer m. Such an erratic behavior of $\mathbf{h}(\cdot)$ prevents $\mathbf{p}(\cdot)$ from being equal to 1 on (r, ∞), no matter how large is r. By the way, these $S_1, \dots, S_n$ are affiliated.

This example shows that in models where participation in a game is a strategic variable (due to the fact that actions are costly), affiliation plays no role, since it can coexist with a pathology as the one we showed above. The example also shows that trying to obtain results that hold for all distributions is not constructive because in the class of 'all distributions' there are pathologies we do not want to be restricted by. Restricting the analysis to distributions for which the $\mathbf{h}(\cdot)$ function, see (3.1), is monotone seems very reasonable.

We believe that restricting attention to equilibria where players with high signals always participate is reasonable, since it means that participants with such signals make positive profits if they win.

From now on, we assume that signals follow the *multinormal distribution*[8] with parameters a, σ, ρ ($\sigma > 0$, $0 < \rho < 1$);

$$\begin{aligned} \mathbb{E} S_k &= a \,, \\ \operatorname{Var}(S_k) &= \sigma^2 \,, \\ \operatorname{Cov}(S_k, S_l) &= \rho\sigma^2 \end{aligned} \tag{3.3}$$

whenever $1 \le k \le n$, $1 \le l \le n$, $k \ne l$. It is of the form (1.1), namely, one may take $n+1$ independent $N(0,1)$ random variables $\Theta, \xi_1, \dots, \xi_n$ and form

$$S_k = a + \sigma\sqrt{\rho}\,\Theta + \sigma\sqrt{1-\rho}\,\xi_k \,, \tag{3.4}$$

then the joint distribution of $S_1, \dots, S_n$ is multinormal,[9] and (3.3) is verified by a simple calculation. Thus,

$$\mathbf{F}(s|\theta) = \Phi\left(\frac{s - \sigma\sqrt{\rho}\,\theta - a}{\sigma\sqrt{1-\rho}} \right) , \tag{3.5}$$

$$\mathbf{G}(\theta) = \Phi(\theta) \,, \tag{3.6}$$

$$\mathbf{F}_1(s) = \Phi\left(\frac{s-a}{\sigma} \right) , \tag{3.7}$$

where $\Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} e^{-u^2/2}\,du$ is the standard normal distribution function. Parameters a, σ are useful in applications, but in the general theory we may restrict ourselves to the case

$$a = 0 \,, \qquad \sigma = 1 \,; \tag{3.8}$$

it means a linear transformation of all signals, harmless when determining the winner, and simplifying formulas. The only parameter that matters is ρ.

[8]The multinormal distribution is the finite-dimensional case of a Gaussian measure. "The modern theory of Gaussian measures lies at the intersection of the theory of random processes, functional analysis, and mathematical physics and is closely connected with diverse applications in quantum field theory, statistical physics, financial mathematics, and other areas of sciences." V.I. Bogachev, "Gaussian measures", American Mathematical Society 1998 (see Preface, p. xi).

[9]A linear transformation always sends a multinormal distribution into a multinormal distribution.

The relation $S_k = \sqrt{\rho}\,\Theta + \sqrt{1-\rho}\,\xi_k$ shows that the conditional distribution of S_k, given $\Theta = \theta$, is $N(\sqrt{\rho}\,\theta, 1-\rho)$. Similarly,[10] the conditional distribution of Θ given $S_k = s$ is $N(\sqrt{\rho}\,s, 1-\rho)$. In other words,[11] [12]

$$\frac{\mathbf{F}'(s|\theta)}{\mathbf{F}'_1(s)}\, d\mathbf{G}(\theta) = d_\theta \Phi\Big(\frac{\theta - \sqrt{\rho}\, s}{\sqrt{1-\rho}} \Big)\,, \tag{3.9}$$

which can be verified also by a straightforward but more tedious calculation. We get from (3.1), (3.5), (3.8) and (3.9)

$$\begin{aligned} \mathbf{h}(s) &= \int \Phi^{n-1}\Big(\frac{s - \sqrt{\rho}\,\theta}{\sqrt{1-\rho}} \Big)\, d_\theta \Phi\Big(\frac{\theta - \sqrt{\rho}\, s}{\sqrt{1-\rho}} \Big) = \\ &= \int \Phi^{n-1}\big(\sqrt{1-\rho}\, s - \sqrt{\rho}\, u \big)\, d\Phi(u) \end{aligned} \tag{3.10}$$

by a change of the variable: $u = (\theta - \sqrt{\rho}\,s)/\sqrt{1-\rho}$; $\theta = \sqrt{\rho}\,s + \sqrt{1-\rho}\,u$; $(s - \sqrt{\rho}\,\theta)/\sqrt{1-\rho} = (s-\rho s)/\sqrt{1-\rho} - \sqrt{\rho}\,u$. For every u the integrand is continuous, and increases strictly, from 0 to 1, when s increases from $-\infty$ to $+\infty$. Therefore $\mathbf{h}(\cdot)$ is a continuous strictly increasing function on $(-\infty, +\infty)$, and $\mathbf{h}(-\infty) = 0$, $\mathbf{h}(+\infty) = 1$. It follows that

$$\mathbf{h}(s_{\text{high}}) = \frac{c}{C} \tag{3.11}$$

for one and only one $s_{\text{high}} \in (-\infty, +\infty)$. Consequently, if $r \ge s_{\text{high}}$, then the strategy "never quit" supports an equilibrium; if $r < s_{\text{high}}$, it does not.

3.12 Lemma. If $r \ge s_{\text{high}}$, there exists one and only one[13] symmetric equilibrium, namely, signals $s \in (r, \infty)$ always participate.

Proof. First, the function $\mathbf{p}(\cdot)$ on (r, ∞), defined by $p(s) = 1$ for all s, is a symmetric equilibrium by Lemma 2.8, since $\int \mathbf{w}(d\theta|s) = \mathbf{h}(s) \ge \mathbf{h}(s_{\text{high}}) = c/C$ for all $s \in (r, \infty)$.

Second, assume that a function $\mathbf{p}(\cdot)$ supports a symmetric equilibrium; we have to prove that $p(s) = 1$ for $\mathbf{F}_1$-almost all $s \in (r, \infty)$. By Lemma 2.8, it suffices to prove that $\mathbf{W}(s) > c/C$ for all $s \in (r, \infty)$. Taking into account that $\int_s^\infty \mathbf{p}(s_1)\, d\mathbf{F}(s_1|\theta) \le 1 - \mathbf{F}(s|\theta)$ (since $\mathbf{p}(s_1) \le 1$), we get from (2.5) $\mathbf{w}(d\theta|s) \ge \mathbf{F}^{n-1}(s|\theta) \frac{\mathbf{F}'(s|\theta)}{\mathbf{F}'_1(s)}\, d\mathbf{G}(\theta)$, thus $\mathbf{W}(s) \ge \mathbf{h}(s) > c/C$. $\square$

[10] The two-dimensional distribution of the pair (Θ, S_k) is equal to that of the pair (S_k, Θ); both are the two-dimensional normal distribution with the mean vector $(0,0)$ and the covariance matrix $\begin{pmatrix} 1 & \sqrt{\rho} \\ \sqrt{\rho} & 1 \end{pmatrix}$.

[11] Now Θ is one-dimensional, and we may write $d\mathbf{G}(\theta)$ rather than $\mathbf{G}(d\theta)$.

[12] The expression $d\Phi\big(\frac{\theta - \sqrt{\rho}\,s}{\sqrt{1-\rho}}\big)$ would be ambiguous. In order to distinguish between $\big(\frac{\partial}{\partial s}\Phi(\cdots)\big)\,ds$ and $\big(\frac{\partial}{\partial\theta}\Phi(\cdots)\big)\,d\theta$ we denote them by $d_s\Phi(\cdots)$ and $d_\theta\Phi(\cdots)$ respectively.

[13] As was noted in Sect. 1, uniqueness of $\mathbf{p}(\cdot)$ is treated up to equivalence.

For $r < s_{\rm high}$, an equilibrium (if any) cannot satisfy $\mathbf{p}(s) = 1$ for (almost) all $s \in (r, \infty)$. However, it must satisfy $\mathbf{p}(s) = 1$ for almost all $s \in (s_{\rm high}, \infty)$, which can be proven similarly to Lemma 3.12 (the second part of the proof). Alternatively, the next lemma can be used.

3.13 Lemma. If a function $\mathbf{p}(\cdot)$ on (r, ∞) supports a symmetric equilibrium, and a number $r_1 \in (r, \infty)$ is given, then the restriction of $\mathbf{p}(\cdot)$ onto (r_1, ∞) supports a symmetric equilibrium for the game with the reserve level r_1.

Proof. A function $\mathbf{W}$ corresponds (by (2.5), (2.6)) to $\mathbf{p}$ and r. Similarly, some $\mathbf{W}_1$ corresponds to $\mathbf{p}_1$ and r_1, where $\mathbf{p}_1$ is the restriction of $\mathbf{p}$ onto (r_1, ∞). We have $\mathbf{p}(s_1) = \mathbf{p}_1(s_1)$ for $s_1 \in (r_1, \infty)$, therefore $\mathbf{W}(s) = \mathbf{W}_1(s)$ for $s \in (r_1, \infty)$, since the right-hand side of (2.5) uses $\mathbf{p}(s_1)$ for $s_1 > s$ only. The pair $(\mathbf{p}, \mathbf{W})$ satisfies the condition of Lemma 2.8 on (r, ∞), therefore the pair $(\mathbf{p}_1, \mathbf{W}_1)$ satisfies the same condition on (r_1, ∞). $\square$

Lemmas 3.12 and 3.13 are essential in deriving the equilibrium. Lemma 3.12 shows that if the reserve is high enough, a unique monotone equilibrium takes place. Lemma 3.13 stipulates that any two equilibria defined on different segments of the form (s, ∞) must coincide on the smaller segment. Consequently, we may start with an equilibrium on any $r > s_{\rm high}$ and augment it for any $r < s_{\rm high}$. The fact that an equilibrium unfolds leftward, is crucial in our proof.

4 The probability of winning and strategies

Extending our analysis to the more interesting and general situation, where $r < s_{\rm high}$, is more complicated since we must consider the equilibrium strategy in conjunction with the probability of winning,

$$\mathbf{W}(s) = \int \mathbf{w}(d\theta|s) \, ,$$

where $\mathbf{w}(d\theta|s)$ is defined, for a given $\mathbf{p}(\cdot)$, by (2.5). We know that $\mathbf{W}(s) = \mathbf{h}(s) > c/C$ for all $s \in (s_{\rm high}, \infty)$. We also know that $s_{\rm high}$ cannot be replaced with a smaller number $s_{\rm high} - \varepsilon$, that is, for every $\varepsilon > 0$ the inequality $\mathbf{W}(s) > c/C$ is violated somewhere on $(s_{\rm high} - \varepsilon, s_{\rm high})$.[14] One may guess that $\mathbf{W}(s) = c/C$ and $\mathbf{p}(s) < 1$ for (almost) all $s \in (s_{\rm high} - \varepsilon, s_{\rm high})$. Note however that it is not proven yet; it could happen that $\mathbf{p}(s_{2n}) < 1$, $\mathbf{W}(s_{2n}) = c/C$, and $\mathbf{p}(s_{2n-1}) = 1$, $\mathbf{W}(s_{2n-1}) > c/C$ for some $s_1 < s_2 < \dots$, $s_n \to s_{\rm high}$.

[14]Otherwise $\mathbf{p}(\cdot) = 1$ on $(s_{\rm high} - \varepsilon, s_{\rm high})$ by Lemma 2.8, which implies $\mathbf{W}(\cdot) = \mathbf{h}(\cdot)$ on $(s_{\rm high} - \varepsilon, \infty)$. However, $\mathbf{h}(\cdot) < c/C$ on $(s_{\rm high} - \varepsilon, s_{\rm high})$.

Such bizarre cases should not be ignored, as far as we pursue an equilibrium unique among *all strategies,* not just some class of well-behaved strategies.

In general, $\mathbf{W}(\cdot)$ need not be continuous (recall the discontinuous periodic function in the example of Sect. 3). For the multinormal case we will see that $\mathbf{W}(\cdot)$ on (r,∞) is continuous and moreover, it is (locally) *absolutely continuous;* it means that the derivative $\mathbf{W}'(\cdot)$ can be defined (almost everywhere w.r.t. Lebesgue measure) such that $\mathbf{W}(b)-\mathbf{W}(a)=\int_a^b \mathbf{W}'(s)\,ds$ whenever $r<a<b<\infty$.[15] In what follows, we investigate properties of $\mathbf{W}'(s)$ for an arbitrary strategy $\mathbf{p}(\cdot)$; equilibrium will appear again only in Lemma 4.15.

Equation (4.1) expresses the intricate relation between strategy and the probability of winning,

$$\mathbf{W}(s)=\int \Big(1-\int_s^\infty \mathbf{p}(s_1)\,d_{s_1}\Phi\Big(\frac{s_1-\sqrt{\rho}\,\theta}{\sqrt{1-\rho}}\Big)\Big)^{n-1} d_\theta\Phi\Big(\frac{\theta-\sqrt{\rho}\,s}{\sqrt{1-\rho}}\Big) \tag{4.1}$$

for $s\in[r,\infty)$ (the formula combines (2.5), (2.6), (3.5), (3.8), (3.9)).

Basically, we differentiate in s Equation (4.1) for $s\in[r,\infty)$. For a continuous $\mathbf{p}(\cdot)$ it is indeed a usual differentiation, but the general case will be treated in a roundabout way (see Lemma 4.12). True, the equilibrium function $\mathbf{p}(\cdot)$ is in fact piecewise continuous; however, the general case is needed when proving uniqueness of the equilibrium.

The variable s occurs twice on the right-hand side of (4.1), namely, at the lower limit of the internal integral, and in $\Phi\big(\frac{\theta-\sqrt{\rho}\,s}{\sqrt{1-\rho}}\big)$. Indeed, a signal plays two roles, 'internal' and 'external'. On one hand, it informs a player about his strength (the internal role). On the other hand, due to interdependence, it informs a player (to some extent) about other players (the external role). In order to separate the two roles we split the signal in two, s^{int} and s^{ext}, and introduce a function of two variables $s^{\text{int}},s^{\text{ext}}$, denoting it by $\mathbf{W}_2$:

$$\mathbf{W}_2(s^{\text{int}},s^{\text{ext}})=\int \Big(1-\int_{s^{\text{int}}}^\infty \mathbf{p}(s)\,d_s\Phi\Big(\frac{s-\sqrt{\rho}\,\theta}{\sqrt{1-\rho}}\Big)\Big)^{n-1} d_\theta\Phi\Big(\frac{\theta-\sqrt{\rho}\,s^{\text{ext}}}{\sqrt{1-\rho}}\Big)\,;$$
$$\mathbf{W}(s)=\mathbf{W}_2(s,s)\,.$$

Clearly, $\mathbf{W}_2(s^{\text{int}},s^{\text{ext}})$ is increasing in s^{int}, which is not surprising since an increase in s^{int} can be given the interpretation that the player can, unilaterally, increase his signal, not affecting others. This, obviously, increases the probability of winning. In contrast, the effect of s^{ext} on $\mathbf{W}_2(s^{\text{int}},s^{\text{ext}})$ is not monotone, in general. A higher s^{ext} may be interpreted as a shift to the right

[15] The derivative need not be continuous, it is just measurable (and integrable); Lebesgue integration is meant (rather than Riemann integration).

of all competitors, which may seem to be bad news. However, if $\mathbf{p}(\cdot)$ does not increase, then sometimes a player with a higher signal may be less threatening, which is good news. If $\mathbf{p}(\cdot)$ is an increasing function, then $\mathbf{W}_2(s^{\text{int}}, s^{\text{ext}})$ decreases in s^{ext} (since the internal integral increases in θ).[16]

In what follows we shall decompose s into s^{int} and s^{ext}; this decomposition may seem unusual. However, this is how we will be able to isolate $\mathbf{p}(s)$. As a byproduct, it formalizes the statement we made before, that a higher signal has two aspects to it: good news and bad news.

Assuming for a while that the usual differentiation rules are justified, we get, denoting the normal density $\Phi'(\cdot)$ by $\varphi(\cdot)$,

(4.2)

$$\mathbf{W}_2(s^{\text{int}}, s^{\text{ext}}) = \\ = \int \Bigl(1 - \int_{s^{\text{int}}}^{\infty} \mathbf{p}(s)\, \varphi\Bigl(\frac{s - \sqrt{\rho}\,\theta}{\sqrt{1-\rho}} \Bigr) \frac{ds}{\sqrt{1-\rho}} \Bigr)^{n-1} \varphi\Bigl(\frac{\theta - \sqrt{\rho}\, s^{\text{ext}}}{\sqrt{1-\rho}} \Bigr) \frac{d\theta}{\sqrt{1-\rho}} \,;$$

(4.3)

$$\frac{\partial}{\partial s^{\text{int}}} \Big|_{s^{\text{int}} = s^{\text{ext}} = s} \mathbf{W}_2(s^{\text{int}}, s^{\text{ext}}) = \\ \int (n-1) \Bigl(1 - \textstyle\int_s^{\infty} \cdots \Bigr)^{n-2} \mathbf{p}(s) \varphi\Bigl(\frac{s - \sqrt{\rho}\,\theta}{\sqrt{1-\rho}} \Bigr) \frac{1}{\sqrt{1-\rho}} \cdot \varphi\Bigl(\frac{\theta - \sqrt{\rho}\, s}{\sqrt{1-\rho}} \Bigr) \frac{d\theta}{\sqrt{1-\rho}} \,;$$

(4.4)

$$\frac{\partial}{\partial s^{\text{ext}}} \Big|_{s^{\text{int}} = s^{\text{ext}} = s} \mathbf{W}_2(s^{\text{int}}, s^{\text{ext}}) = \\ = \int \Bigl(1 - \textstyle\int_s^{\infty} \cdots \Bigr)^{n-1} \varphi'\Bigl(\frac{\theta - \sqrt{\rho}\, s}{\sqrt{1-\rho}} \Bigr) \frac{-\sqrt{\rho}}{\sqrt{1-\rho}} \frac{d\theta}{\sqrt{1-\rho}} \,.$$

Note that the effect of s^{int} is proportional to $\mathbf{p}(s)$, which is natural; an increase of s^{int} matters when crossing a competitor. In the total derivative

$$\frac{d}{ds} \mathbf{W}(s) = \frac{\partial}{\partial s^{\text{int}}} \Big|_{s^{\text{int}} = s^{\text{ext}} = s} \mathbf{W}_2(s^{\text{int}}, s^{\text{ext}}) + \frac{\partial}{\partial s^{\text{ext}}} \Big|_{s^{\text{int}} = s^{\text{ext}} = s} \mathbf{W}_2(s^{\text{int}}, s^{\text{ext}})$$

the first summand is proportional to $\mathbf{p}(s)$, the second summand does not depend on $\mathbf{p}(s)$, thus the sum is linear in $\mathbf{p}(s)$.[17] The linear function is

[16] An increase of θ shifts the distribution to the right, which increases the expectation of any increasing function of the random variable.

[17] The argument is quite informal. The single value $\mathbf{p}(s)$ may be treated as an independent variable, if $\mathbf{p}(\cdot)$ is discontinuous at s which, however, makes the usual differentiation rules inapplicable. A correct interpretation of the argument involves a local continuous variation of the function $\mathbf{p}(\cdot)$ near s, and a suitable limiting procedure. Anyway, the argument is heuristic, it is not used in the formal proof given by Lemma 4.12.

uniquely determined by its values for $\mathbf{p}(s) = 0$ and $\mathbf{p}(s) = 1$; denote these by $\tilde{\mathbf{W}}_0(s)$ and $\tilde{\mathbf{W}}_1(s)$ respectively;

$$\frac{d}{ds}\mathbf{W}(s) = \mathbf{p}(s)\tilde{\mathbf{W}}_1(s) + \big(1-\mathbf{p}(s)\big)\tilde{\mathbf{W}}_0(s)\,, \tag{4.5}$$

$$\tilde{\mathbf{W}}_0(s) = \frac{\partial}{\partial s^{\mathrm{ext}}}\bigg|_{s^{\mathrm{int}}=s^{\mathrm{ext}}=s} \mathbf{W}_2(s^{\mathrm{int}}, s^{\mathrm{ext}})\,, \tag{4.6}$$

$$\tilde{\mathbf{W}}_1(s) = \tilde{\mathbf{W}}_0(s) + \frac{1}{\mathbf{p}(s)}\frac{\partial}{\partial s^{\mathrm{int}}}\bigg|_{s^{\mathrm{int}}=s^{\mathrm{ext}}=s} \mathbf{W}_2(s^{\mathrm{int}}, s^{\mathrm{ext}})\,. \tag{4.7}$$

In order to write explicit formulas for $\tilde{\mathbf{W}}_0(s)$, $\tilde{\mathbf{W}}_1(s)$ more compactly, we introduce

$$\mathbf{w}_m(d\theta|s) = \bigg(1 - \int_s^\infty \mathbf{p}(s_1)\,d_{s_1}\Phi\Big(\frac{s_1-\sqrt{\rho}\,\theta}{\sqrt{1-\rho}}\Big)\bigg)^m d_\theta\Phi\Big(\frac{\theta-\sqrt{\rho}\,s}{\sqrt{1-\rho}}\Big) \tag{4.8}$$

for any m, then $\mathbf{w}(d\theta|s) = \mathbf{w}_{n-1}(d\theta|s)$. Taking into account the property $\varphi'(x) = -x\varphi(x)$ of the normal density φ, we rewrite (4.3), (4.4) as

(4.9)
$$\frac{\partial}{\partial s^{\mathrm{int}}}\bigg|_{s^{\mathrm{int}}=s^{\mathrm{ext}}=s} \mathbf{W}_2(s^{\mathrm{int}}, s^{\mathrm{ext}}) = \mathbf{p}(s)\underbrace{\frac{n-1}{\sqrt{1-\rho}}\int\varphi\Big(\frac{s-\sqrt{\rho}\,\theta}{\sqrt{1-\rho}}\Big)\mathbf{w}_{n-2}(d\theta|s)}_{\tilde{\mathbf{W}}_1(s)-\tilde{\mathbf{W}}_0(s)}\,,$$

$$\frac{\partial}{\partial s^{\mathrm{ext}}}\bigg|_{s^{\mathrm{int}}=s^{\mathrm{ext}}=s} \mathbf{W}_2(s^{\mathrm{int}}, s^{\mathrm{ext}}) = \underbrace{\frac{\sqrt{\rho}}{1-\rho}\int(\theta-\sqrt{\rho}\,s)\mathbf{w}_{n-1}(d\theta|s)}_{\tilde{\mathbf{W}}_0(s)} \tag{4.10}$$

for $s \in [r,\infty)$.

Turning from heuristic to rigorous arguments we use right-hand side of (4.9), (4.10) as the definition of $\tilde{\mathbf{W}}_0(\cdot), \tilde{\mathbf{W}}_1(\cdot)$. Such a definition in terms of integrals does not depend on differentiability of $\mathbf{W}_2$. Note that $\tilde{\mathbf{W}}_0(\cdot), \tilde{\mathbf{W}}_1(\cdot)$ are continuous on (r,∞) (since the right-hand side of (4.8) is continuous in s), and

$$\tilde{\mathbf{W}}_1(s) - \tilde{\mathbf{W}}_0(s) > 0 \quad \text{for all } s\,. \tag{4.11}$$

The following lemma states formally the relation (4.5) obtained before heuristically. It is very useful in proving our major result; Theorem 5.6. It will help to disentangle $\mathbf{p}(s)$.

4.12 Lemma. The function $\mathbf{W}(\cdot)$ is absolutely continuous on (r,∞), and

$$\mathbf{W}'(s) = \mathbf{p}(s)\tilde{\mathbf{W}}_1(s) + (1-\mathbf{p}(s))\tilde{\mathbf{W}}_0(s)$$

for almost all $s \in (r,\infty)$.

(For the proof, see Appendix A.)

Now comes a surprise: a relation that will lead (see Corollary 4.18) to the jump $\mathbf{p}(s_{\text{high}}-) = \rho$, of $\mathbf{p}(\cdot)$ at s_{high}. We consider $\tilde{\mathbf{W}}_0(s_{\text{high}})$ and $\tilde{\mathbf{W}}_1(s_{\text{high}})$ assuming that $\mathbf{p}(\cdot) = 1$ on $(s_{\text{high}}, \infty)$.

4.13 Lemma. $\rho\tilde{\mathbf{W}}_1(s_{\text{high}}) + (1-\rho)\tilde{\mathbf{W}}_0(s_{\text{high}}) = 0$.

The proof, given in Appendix A, is a trick based on $s^{\text{int}}, s^{\text{ext}}, \mathbf{W}_2$. From now on the reader may forget $s^{\text{int}}, s^{\text{ext}}$ and $\mathbf{W}_2$; they are not used in the rest of the paper. In contrast, $\tilde{\mathbf{W}}_0(\cdot), \tilde{\mathbf{W}}_1(\cdot)$ will be used intensively.

Combining Lemma 4.13 and (4.11) we get

$$\tilde{\mathbf{W}}_0(s_{\text{high}}) < 0\,, \quad \tilde{\mathbf{W}}_1(s_{\text{high}}) > 0\,; \tag{4.14}$$

indeed, $\tilde{\mathbf{W}}_0(s_{\text{high}}) = -\rho\big(\tilde{\mathbf{W}}_1(s_{\text{high}}) - \tilde{\mathbf{W}}_0(s_{\text{high}})\big)$ and $\tilde{\mathbf{W}}_1(s_{\text{high}}) = (1-\rho)\big(\tilde{\mathbf{W}}_1(s_{\text{high}}) - \tilde{\mathbf{W}}_0(s_{\text{high}})\big)$.

In this section, the next lemma is needed for $s_0 = s_{\text{high}}$ only; the general case will be used later.

4.15 Lemma. Let $\mathbf{p}(\cdot)$ support an equilibrium, $s_0 \in (r, s_{\text{high}}]$, $\mathbf{W}(s_0) = c/C$, $\tilde{\mathbf{W}}_0(s_0) < 0$, $\tilde{\mathbf{W}}_1(s_0) > 0$. Then there exists $\varepsilon > 0$ such that $\mathbf{W}(\cdot) = c/C$ on $[s_0 - \varepsilon, s_0]$.

(For the proof, see Appendix A.)

4.16 Corollary. Let $\mathbf{p}(\cdot)$, s_0, ε be as in Lemma 4.15; then the equality

$$\mathbf{p}(s) = \frac{-\tilde{\mathbf{W}}_0(s)}{\tilde{\mathbf{W}}_1(s) - \tilde{\mathbf{W}}_0(s)} \tag{4.17}$$

holds for almost all $s \in (s_0 - \varepsilon, s_0)$.

Proof. We have $\mathbf{W}(s) = c/C$, therefore $\mathbf{W}'(s) = 0$. Lemma 4.12 gives $\mathbf{p}(s)\tilde{\mathbf{W}}_1(s) + (1-\mathbf{p}(s))\tilde{\mathbf{W}}_0(s) = 0$, which implies (4.17). □

Considering $s_0 = s_{\text{high}}$, Corollary 4.16 excludes erratic behavior of $\mathbf{p}(\cdot)$ in a left neighborhood of s_{high}. Indeed, the right-hand side of (4.17) is continuous.

4.18 Corollary. If $\mathbf{p}(\cdot)$ supports an equilibrium then $\mathbf{p}(s_{\text{high}}-) = \rho$, maybe after correcting $\mathbf{p}(\cdot)$ on a negligible set.

Proof. Due to Lemma 4.13 and continuity of the right-hand side of (4.17),

$$\frac{-\tilde{\mathbf{W}}_0(s_{\text{high}}-)}{\tilde{\mathbf{W}}_1(s_{\text{high}}-) - \tilde{\mathbf{W}}_0(s_{\text{high}}-)} = \rho\,;$$

it remains to use Corollary 4.16. □

5 Solving the equilibrium equation

5.1 Definition. A pair $(r, \mathbf{p})$ will be called *well-behaved,*[18] if $r \in (-\infty, s_{\mathrm{high}}]$ and $\mathbf{p} : [r, \infty) \to [0, 1]$ is a function such that

(a) $\mathbf{p}(\cdot)$ is continuous on $[r, s_{\mathrm{high}}]$, and $0 < \mathbf{p}(\cdot) < 1$ on $[r, s_{\mathrm{high}}]$, and $\mathbf{p}(\cdot) = 1$ on $(s_{\mathrm{high}}, \infty)$;

(b) $\mathbf{p}(\cdot)$ is an equilibrium strategy.

5.2 Lemma. Let $(r, \mathbf{p})$ be a well-behaved pair. Then the corresponding functions[19] $\mathbf{W}(\cdot)$, $\tilde{\mathbf{W}}_0(\cdot)$, $\tilde{\mathbf{W}}_1(\cdot)$ satisfy

$$\mathbf{W}(r) = c/C\,, \quad \tilde{\mathbf{W}}_0(r) < 0\,, \quad \tilde{\mathbf{W}}_1(r) > 0\,.$$

Proof. The case $r = s_{\mathrm{high}}$ is already understood; here $\mathbf{W}(r) = \mathbf{h}(s_{\mathrm{high}}) = c/C$, recall (3.11), and $\tilde{\mathbf{W}}_0(r) < 0$, $\tilde{\mathbf{W}}_1(r) > 0$, see (4.14).

Let $r < s_{\mathrm{high}}$. We have $0 < \mathbf{p}(\cdot) < 1$ on (r, s_{high}) by 5.1(a); $\mathbf{W}(\cdot) = c/C$ almost everywhere on (r, s_{high}) by Lemma 2.8; taking into account continuity of $\mathbf{W}(\cdot)$ we conclude that $\mathbf{W}(\cdot) = c/C$ everywhere on $[r, s_{\mathrm{high}}]$. So, $\mathbf{W}(r) = c/C$. Further, $\mathbf{W}'(\cdot) = 0$ on (r, s_{high}). By Lemma 4.12, $\mathbf{p}(\cdot)\tilde{\mathbf{W}}_1(\cdot) + (1 - \mathbf{p}(\cdot))\tilde{\mathbf{W}}_0(\cdot) = 0$ almost everywhere on (r, s_{high}); by continuity of $\mathbf{p}(\cdot), \tilde{\mathbf{W}}_0(\cdot), \tilde{\mathbf{W}}_1(\cdot)$, it holds everywhere on $[r, s_{\mathrm{high}}]$. Using (4.11) we get $\tilde{\mathbf{W}}_0(r) = -\mathbf{p}(r)\big(\tilde{\mathbf{W}}_1(r) - \tilde{\mathbf{W}}_0(r)\big) < 0$ and $\tilde{\mathbf{W}}_1(r) = \big(1 - \mathbf{p}(r)\big)\big(\tilde{\mathbf{W}}_1(r) - \tilde{\mathbf{W}}_0(r)\big) > 0$. $\square$

Now, comes uniqueness (assuming existence, for now).

5.3 Lemma. Let $(r, \mathbf{p})$ be a well-behaved pair. Then every equilibrium strategy on $[r, \infty)$ is equal to $\mathbf{p}(\cdot)$ almost everywhere.

Proof. Assume the contrary: $\mathbf{p}_1(\cdot)$ is a different equilibrium strategy on $[r, \infty)$. Consider

$$s_{\mathrm{fork}} = \inf\{s \in (r, s_{\mathrm{high}}] : \mathbf{p}_1(\cdot) = \mathbf{p}(\cdot) \text{ a.e. on } (s, \infty)\}\,;$$

we have $s_{\mathrm{fork}} \in (r, s_{\mathrm{high}}]$. Denote by $\mathbf{p}_0(\cdot)$ the restriction of $\mathbf{p}(\cdot)$ onto $[s_{\mathrm{fork}}, \infty)$, then $(s_{\mathrm{fork}}, \mathbf{p}_0)$ is a well-behaved pair (recall Lemma 3.13). Functions $\mathbf{W}(\cdot)$, $\tilde{\mathbf{W}}_0(\cdot)$, $\tilde{\mathbf{W}}_1(\cdot)$ constructed from $\mathbf{p}_0(\cdot)$ are restrictions onto $[s_{\mathrm{fork}}, \infty)$ of corresponding functions for $\mathbf{p}(\cdot)$, as well as $\mathbf{p}_1(\cdot)$. By Lemma 5.2,

$$\mathbf{W}(s_{\mathrm{fork}}) = c/C\,, \quad \tilde{\mathbf{W}}_0(s_{\mathrm{fork}}) < 0\,, \quad \tilde{\mathbf{W}}_1(s_{\mathrm{fork}}) > 0\,.$$

[18] Of course, the term is auxiliary, intended only for technical use within this section.

[19] Recall (4.1), (4.9), (4.10).

Applying Corollary 4.16 to $\mathbf{p}_1(\cdot)$ and s_{fork} we get $\varepsilon > 0$ such that

$$\mathbf{p}_1(s) = \frac{-\tilde{\mathbf{W}}_0(s)}{\tilde{\mathbf{W}}_1(s) - \tilde{\mathbf{W}}_0(s)}$$

for almost all $s \in (s_{\text{fork}} - \varepsilon, s_{\text{fork}})$; here $\tilde{\mathbf{W}}_0(\cdot), \tilde{\mathbf{W}}_1(\cdot)$ correspond to $\mathbf{p}_1(\cdot)$.

Denoting by $\mathbf{X}(\cdot)$ the restriction of $\mathbf{p}_1(\cdot)$ onto $[s_{\text{fork}} - \varepsilon, s_{\text{fork}}]$ we see that $\mathbf{X}$ is a fixed point of a (nonlinear) operator $\mathcal{A}$ defined by

$$\big(\mathcal{A}(\mathbf{X})\big)(s) = \frac{-\tilde{\mathbf{W}}_0(s)}{\tilde{\mathbf{W}}_1(s) - \tilde{\mathbf{W}}_0(s)}\,,$$

where $\tilde{\mathbf{W}}_0(s)$, $\tilde{\mathbf{W}}_1(s)$ correspond to $\mathbf{p}_1(\cdot)$ (assembled from $\mathbf{X}(\cdot)$ and $\mathbf{p}_0(\cdot)$). The operator $\mathcal{A}$ is defined on the set of all continuous functions $\mathbf{X} : [s_{\text{fork}} - \varepsilon, s_{\text{fork}}] \to [0,1]$ that satisfy the boundary condition $\mathbf{X}(s_{\text{fork}}) = \mathbf{p}_0(s_{\text{fork}})$; $\mathcal{A}$ maps them into continuous functions $[s_{\text{fork}} - \varepsilon, s_{\text{fork}}] \to \mathbb{R}$ (not just $[0,1]$) that satisfy the same boundary condition.

The restriction of $\mathbf{p}(\cdot)$ onto $[s_{\text{fork}} - \varepsilon, s_{\text{fork}}]$ is another fixed point of $\mathcal{A}$, different from $\mathbf{X}$ on $[s_{\text{fork}} - \varepsilon, s_{\text{fork}}]$ and moreover, on $[s_{\text{fork}} - \varepsilon_1, s_{\text{fork}}]$ for every $\varepsilon_1 > 0$. It is enough to prove that $\mathcal{A}$ cannot have two different fixed points, if ε is small enough. The following lemma completes the proof by showing that $\mathcal{A}$ is a contraction. $\square$

5.4 Lemma. Let $(r, \mathbf{p})$ be a well-behaved pair. For any $\varepsilon > 0$ consider the operator $\mathcal{A}$ on continuous functions $\mathbf{X} : [r - \varepsilon, r] \to [0,1]$ satisfying $X(r) = \mathbf{p}(r)$, defined by

$$\big(\mathcal{A}(\mathbf{X})\big)(s) = \frac{-\tilde{\mathbf{W}}_0(s)}{\tilde{\mathbf{W}}_1(s) - \tilde{\mathbf{W}}_0(s)} \quad \text{for } s \in [r - \varepsilon, r]\,,$$

where $\tilde{\mathbf{W}}_0(s)$, $\tilde{\mathbf{W}}_1(s)$ correspond (according to (4.8)–(4.10)) to the function

$$\begin{cases} \mathbf{X}(s) & \text{for } s \in [r - \varepsilon, r], \\ \mathbf{p}(s) & \text{for } s \in (r, \infty). \end{cases}$$

Then there exists $\varepsilon > 0$ such that

$$\big(\mathcal{A}(\mathbf{X})\big)(s) \in (0,1) \quad \text{for all } \mathbf{X} \text{ and } s\,,$$

and

$$\operatorname{dist}\big(\mathcal{A}(\mathbf{X}), \mathcal{A}(\mathbf{Y})\big) \le \frac{1}{2} \operatorname{dist}(\mathbf{X}, \mathbf{Y}) \quad \text{for all } \mathbf{X}, \mathbf{Y}\,,$$

where the metric is defined by

$$\operatorname{dist}(\mathbf{X}, \mathbf{Y}) = \sup_{s \in [r - \varepsilon, r]} |\mathbf{X}(s) - \mathbf{Y}(s)|\,.$$

(For the proof see Appendix B.)

5.5 Lemma. For every well-behaved pair $(r_1, \mathbf{p}_1)$ there exists a well-behaved pair $(r_2, \mathbf{p}_2)$ such that $r_2 < r_1$.

Proof. Lemma 5.4 gives us $\varepsilon > 0$ and an operator $\mathcal{A}$, the operator being a contraction. Every contraction on a complete metric space has a fixed point. Therefore $\mathcal{A}(\mathbf{X}) = \mathbf{X}$ for some $\mathbf{X} : [r_1 - \varepsilon, r_1] \to (0, 1)$. We take $r_2 = r_1 - \varepsilon$ and

$$\mathbf{p}_2(s) = \begin{cases} \mathbf{X}(s) & \text{for } s \in [r_2, r_1], \\ \mathbf{p}_1(s) & \text{for } s \in (r_1, \infty). \end{cases}$$

Functions $\mathbf{W}(\cdot), \tilde{\mathbf{W}}_0(\cdot), \tilde{\mathbf{W}}_1(\cdot)$ constructed from $\mathbf{p}_2(\cdot)$ satisfy $\mathbf{p}_2(s)\tilde{\mathbf{W}}_1(s) + (1 - \mathbf{p}_2(s))\tilde{\mathbf{W}}_0(s) = 0$ for $s \in [r_2, r_1]$, since $\mathcal{A}(\mathbf{X}) = \mathbf{X}$. By Lemma 4.12, $\mathbf{W}$ is absolutely continuous and $\mathbf{W}'(\cdot) = 0$ almost everywhere on (r_2, r_1). Therefore $\mathbf{W}(\cdot) = \text{const}$ on $[r_2, r_1]$. However, $\mathbf{W}(r_1) = c/C$ by Lemma 5.2. Thus, $\mathbf{W}(\cdot) = c/C$ on $[r_2, r_1]$. By Lemma 2.8, $\mathbf{p}_2(\cdot)$ is an equilibrium strategy. So, $(r_2, \mathbf{p}_2)$ is a well-behaved pair. $\square$

5.6 Theorem. There exists $s_{\text{low}} \in [-\infty, s_{\text{high}})$ such that

(a) there exists a function $\mathbf{p} : (s_{\text{low}}, \infty) \to [0, 1]$ such that $\mathbf{p}(\cdot)$ is continuous on $(s_{\text{low}}, s_{\text{high}}]$, and $0 < \mathbf{p}(s) < 1$ for all $s \in (s_{\text{low}}, s_{\text{high}}]$, and $\mathbf{p}(s_{\text{high}}) = \rho$, and $\mathbf{p}(s) = 1$ for all $s \in (s_{\text{high}}, \infty)$, and for every $r \in [s_{\text{low}}, \infty)$ the restriction of $\mathbf{p}(\cdot)$ onto (r, ∞) is an equilibrium strategy;

(b) If $s_{\text{low}} > -\infty$ then the limit $\mathbf{p}(s_{\text{low}}+)$ exists and is equal to 0 or 1;

(c) for every $r \in [s_{\text{low}}, \infty)$, every equilibrium strategy on (r, ∞) is equal to $\mathbf{p}(\cdot)$ almost everywhere on (r, ∞).

Proof. Denote by E the set of all $r \in (-\infty, s_{\text{high}}]$ such that $(r, \mathbf{p}_r)$ is a well-behaved pair for some $\mathbf{p}_r(\cdot)$. By Lemma 5.3, $\mathbf{p}_r(\cdot)$ is uniquely determined by r. By Lemma 3.13, if $r \in E$ and $r_1 \in (r, s_{\text{high}})$ then $r_1 \in E$ and $\mathbf{p}_{r_1}(\cdot)$ is the restriction of $\mathbf{p}_r(\cdot)$ onto $[r_1, \infty)$. By Lemma 5.5, E has no least element. Clearly, $s_{\text{high}} \in E$. So,

$$E = (s_{\text{low}}, s_{\text{high}}]$$

for some $s_{\text{low}} \in [-\infty, s_{\text{high}})$.

Functions $\mathbf{p}_r(\cdot)$ for all $r \in (s_{\text{low}}, s_{\text{high}})$ extend one another; gluing them together we get a function $\mathbf{p} : (s_{\text{low}}, \infty) \to [0, 1]$ such that $\mathbf{p}(\cdot)$ is continuous on $(s_{\text{low}}, s_{\text{high}}]$, and $0 < \mathbf{p}(s) < 1$ for all $s \in (s_{\text{low}}, s_{\text{high}}]$, and $\mathbf{p}(s) = 1$ for all $s \in (s_{\text{high}}, \infty)$, and for every $r \in [s_{\text{low}}, \infty)$, $p|_{(r,\infty)}$ is an equilibrium strategy. Also, $\mathbf{p}(s_{\text{high}}) = \rho$ by Corollary 4.18. Item (a) is proven.

It cannot happen that $s_{\text{low}} > -\infty$ and $0 < \mathbf{p}(s_{\text{low}}+) < 1$, since $s_{\text{low}} \notin E$. Also it cannot happen that $s_{\text{low}} > -\infty$ and the limit $\mathbf{p}(s_{\text{low}}+)$ does not exist, since $\mathbf{p}(\cdot)$ is equal on $(s_{\text{low}}, s_{\text{high}}]$ to the function $\frac{-\tilde{\mathbf{W}}_0(s)}{\tilde{\mathbf{W}}_1(s)-\tilde{\mathbf{W}}_0(s)}$continuous on $[s_{\text{low}}, s_{\text{high}}]$. Item (b) is proven.

Item (c) follows from Lemma 5.3. $\square$

Consequently, we proved existence and uniqueness of equilibrium; in this equilibrium, high types participate with probability 1. Then, there is a discontinuity and we come to the highest type that uses a mixed strategy. We refer to this type as $s_{\text{high}}-$ and prove that he participates with probability ρ. Hence, the lower the interdependence, the bigger the drop from pure to mixed strategy. From then on, there is a segment of signals over which players randomize, and then... the participation probability function hits 1 or 0 (unless $s_{\text{low}} = -\infty$).

The following questions come to mind:

1. How far is s_{low} from s_{high}? Namely, after the drop from $\mathbf{p}(s) = 1$ to $\mathbf{p}(s) = \rho$ what is the length of the interval where $0 < \mathbf{p}(s) < 1$.

2. What is the pattern of the mixed strategy on $(s_{\text{low}}, s_{\text{high}})$; does it oscillate. And if it does, how large are the oscillations.

These are questions that typically can not be answered analytically. A numerical analysis conducted in the next section sheds light on some of the issues.

6 Some numerics

Numerical computation of the equilibrium participation probability function $\mathbf{p}(\cdot)$ for given parameters $n, \rho, c/C$ begins with finding s_{high} by solving (3.11) where $\mathbf{h}(\cdot)$ is given by (3.10).

After that we choose ε and find a fixed point of the contraction operator $\mathcal{A}$ (see Sect. 5); it is a function on $[s_{\text{high}} - \varepsilon, s_{\text{high}}]$. The value of ε ensured by (the proof of) Lemma 5.4 is too small for practical purpose. Instead, we choose by trials and errors ε small enough for iterations $X_{n+1} = \mathcal{A}(X_n)$ to converge quickly. The initial point is the constant function $X_0(s) = \rho$ for $s \in [s_{\text{high}} - \varepsilon, s_{\text{high}}]$. Polynomials of a suitable degree are used instead of arbitrary functions; quadratic or cubic polynomials work nicely. Unfortunately, $\mathcal{A}$ involves double integration (numerical), which goes rather slowly. After finding the fixed point we repeat the process leftward on the new endpoint $s_{\text{high}} - \varepsilon$. And so on. The non-monotone domain needs smaller ε (therefore, longer time).

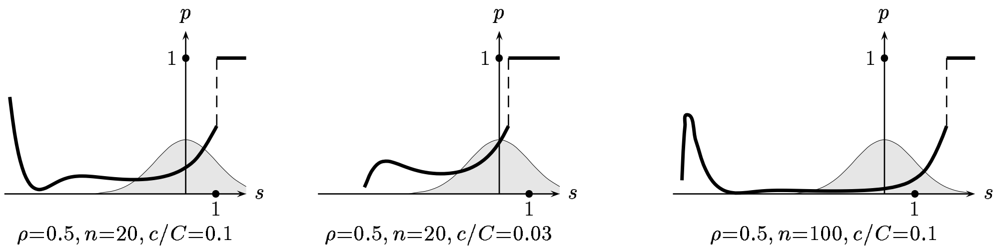

Figure 1: Equilibrium participation probability function computed numerically

Reliability of the process is checked by trying different steps (ε) and different degrees (quadratic or cubic) of polynomials. Computation of a single curve takes about half an hour on a personal computer with “Mathematica”. Some results are shown on Fig. 1.

7 Independent signals

In this section (in contrast to all other sections) we assume that signals are independent, and consider both symmetric and asymmetric equilibria. The game is still symmetric; it is the contest game, and signals are identically distributed.

What matters for a player, is his quantile $\mathbf{F}_1(S_k)$ rather than signal S_k. Indeed, only order relations between signals are used. It means that the distribution $\mathbf{F}_1$ of each signal does not matter, as far as it is nonatomic (that is, the c.d.f. $\mathbf{F}_1$ is continuous). Restricting ourselves to the nonatomic case, we may assume that $\mathbf{F}_1$ is the uniform distribution $\mathrm{U}(0,1)$ or, equally well, the normal distribution $\mathrm{N}(0,1)$. The latter case, $\mathbf{F}_1(s) = \Phi(s)$ for $s \in (-\infty, +\infty)$, is just the special case $\rho = 0$ of the multinormal distribution.[20] However, the former case, $\mathbf{F}_1(s) = s$ for $s \in [0,1]$, is a bit more convenient here, since it allows us to use $A_k S_k$ similarly to (2.1).

The probability $\mathbf{h}(S_k)$ of being the maximal signal (recall (3.2)) is now

$$\mathbf{h}(s) = \mathbf{F}_1^{n-1}(s) = s^{n-1} \,,$$

which is much simpler than (3.1). Equation (3.11), $\mathbf{h}(s_{\mathrm{high}}) = c/C$, is easy to solve:

$$s_{\mathrm{high}} = \sqrt[n-1]{\frac{c}{C}} \,.$$

[20]Also for $\rho > 0$ we could use the uniform distribution, replacing S_k with $\Phi(S_k)$. However, it would be unnatural, since simple linear relations, such as (3.4), would become nonlinear and cumbersome.

Similarly to Lemma 3.12 (but simpler), signals above s_{high} always participate (for every equilibrium).

What happens below s_{high}? Corollary 4.18 suggests that $\mathbf{p}(s_{\text{high}}-) = 0$. However, the complicated machinery of normal correlation is out of place here. The threshold strategy

$$\mathbf{p}(s) = \begin{cases} 1 & \text{for } s > s_{\text{high}}, \\ 0 & \text{for } s < s_{\text{high}} \end{cases} \tag{7.1}$$

supports an equilibrium, since the corresponding winning probability function

$$\mathbf{W}(s) = \max(s^{n-1}, s_{\text{high}}^{n-1}) = \begin{cases} s^{n-1} & \text{for } s \ge s_{\text{high}}, \\ s_{\text{high}}^{n-1} & \text{for } s \le s_{\text{high}} \end{cases}$$

satisfies the condition of Lemma 2.8. Players above s_{high} are interested in participation, since $\mathbf{W}(s) > c/C$. Players below s_{high} are indifferent, since $\mathbf{W}(s) = c/C$.

The indifference opens the way to some asymmetric equilibria. Say, player 1 participates always, while other players adhere to the threshold strategy (7.1). Moreover, player 1 may use an arbitrary function $\mathbf{p}_1(\cdot)$ below s_{high}.

In order to exhaust all equilibria, we have to prove that no more than one player can deviate from the threshold strategy.

The proof is simple. Assume the contrary; say, players 1 and 2 both deviate from (7.1). It means that the support of (the distribution of) the random variable A_1S_1 intersects the open interval $(0, s_{\text{high}})$; and the same for A_2S_2. Moreover, each intersection contains more than one point (due to nonatomicity).

Take $s_1, s_2 \in (0, s_{\text{high}})$, $s_1 \ne s_2$, such that s_1 belongs to the support of A_1S_1, and s_2 belongs to the support of A_2S_2. Assume that $s_1 < s_2$ (the other case is similar). Consider the winning probability function $\mathbf{W}_1(\cdot)$ of player 1. We have $\mathbf{W}_1(s) < \mathbf{W}_1(s_{\text{high}})$ for $s < s_2$; indeed, there is a chance that $\max(A_2S_2, \dots, A_nS_n) = S_2 \in (s, s_{\text{high}})$. So, $\mathbf{W}_1(\cdot) < c/C$ near s_1. Therefore $\mathbf{p}_1(\cdot) = 0$ near s_1 (recall 2.8), and s_1 cannot belong to the support; a contradiction.

Appendix A. Some proofs

Proof of Lemma 4.12. The internal integral $\int_s^\infty \mathbf{p}(s_1)\, d_{s_1}\Phi\Big(\frac{s_1-\sqrt{\rho}\,\theta}{\sqrt{1-\rho}}\Big)$ of (4.1) evidently is absolutely continuous in s, its derivative being equal to

$-\mathbf{p}(s)\varphi\Big(\frac{s-\sqrt{\rho}\,\theta}{\sqrt{1-\rho}}\Big)\frac{1}{\sqrt{1-\rho}}$. The product of absolutely continuous functions is absolutely continuous, and the usual formula $(fg)' = f'g + fg'$ works (almost everywhere). Therefore the integrand[21]

$$f(s,\theta) = \Big(1 - \int_s^\infty \cdots\Big)^{n-1} \varphi\Big(\frac{\theta - \sqrt{\rho}\,s}{\sqrt{1-\rho}}\Big)\frac{1}{\sqrt{1-\rho}}$$

of the right-hand side of (4.1) is absolutely continuous in s, and its derivative is the integrand

$$g(s,\theta) = \frac{\sqrt{\rho}}{1-\rho}(\theta - \sqrt{\rho}\,s)\frac{\mathbf{w}_{n-1}(d\theta|s)}{d\theta} + \mathbf{p}(s)\frac{n-1}{\sqrt{1-\rho}}\varphi\Big(\frac{s-\sqrt{\rho}\,\theta}{\sqrt{1-\rho}}\Big)\frac{\mathbf{w}_{n-2}(d\theta|s)}{d\theta}$$

of the integral (in θ) for $\frac{d}{ds}\mathbf{W}(s) = \big(\frac{\partial}{\partial s^{\text{int}}} + \frac{\partial}{\partial s^{\text{ext}}}\big)\big|_{s^{\text{int}}=s^{\text{ext}}=s}\mathbf{W}_2(s^{\text{int}}, s^{\text{ext}}) = \tilde{\mathbf{W}}_0(s) + \mathbf{p}(s)\big(\tilde{\mathbf{W}}_1(s) - \tilde{\mathbf{W}}_0(s)\big)$. Let $r < a < b < \infty$, then $f(b,\theta) - f(a,\theta) = \int_a^b g(s,\theta)\,ds$ for all θ. Also, $\int f(s,\theta)\,d\theta = \mathbf{W}(s)$ and $\int g(s,\theta)\,d\theta = \tilde{\mathbf{W}}_0(s) + \mathbf{p}(s)\big(\tilde{\mathbf{W}}_1(s) - \tilde{\mathbf{W}}_0(s)\big)$. We have to prove that $\mathbf{W}(b) - \mathbf{W}(a) = \int_a^b\big(\tilde{\mathbf{W}}_0(s) + \mathbf{p}(s)(\tilde{\mathbf{W}}_1(s) - \tilde{\mathbf{W}}_0(s))\big)\,ds$, that is,

$$\int\Big(\int_a^b g(s,\theta)\,ds\Big)\,d\theta = \int_a^b\Big(\int g(s,\theta)\,d\theta\Big)\,ds\,.$$

It remains to note that the corresponding two-dimensional integral converges, since it is dominated by

$$\iint_{(s,\theta)\in(a,b)\times\mathbb{R}} \big(\text{const}_1 \cdot|\theta - \sqrt{\rho}\,s| + \text{const}_2\big)\varphi\Big(\frac{\theta-\sqrt{\rho}\,s}{\sqrt{1-\rho}}\Big)\,ds\,d\theta < \infty\,.$$

□

Proof of Lemma 4.13. For any $s^{\text{int}}, s^{\text{ext}} \in (s_{\text{high}}, \infty)$

$$\mathbf{W}_2(s^{\text{int}}, s^{\text{ext}}) = \int\Big(1 - \int_{s^{\text{int}}}^\infty d_s\Phi\Big(\frac{s-\sqrt{\rho}\,\theta}{\sqrt{1-\rho}}\Big)\Big)^{n-1} d_\theta\Phi\Big(\frac{\theta-\sqrt{\rho}\,s^{\text{ext}}}{\sqrt{1-\rho}}\Big) =$$
$$= \int \Phi^{n-1}\Big(\frac{s^{\text{int}}-\sqrt{\rho}\,\theta}{\sqrt{1-\rho}}\Big)d_\theta\Phi\Big(\frac{\theta-\sqrt{\rho}\,s^{\text{ext}}}{\sqrt{1-\rho}}\Big)\,;$$

the same change of variable as in (3.10) gives

$$\mathbf{W}_2(s^{\text{int}}, s^{\text{ext}}) = \int \Phi^{n-1}\Big(\frac{s^{\text{int}} - \rho s^{\text{ext}}}{\sqrt{1-\rho}} - \sqrt{\rho}\,u\Big)\,d\Phi(u)\,,$$

[21]Please forget auxiliary notations $f(s,\theta)$ and $g(s,\theta)$ after finishing the proof.

and we see that $\mathbf{W}_2(s^{\text{int}}, s^{\text{ext}})$ is constant along lines $s^{\text{int}} - \rho s^{\text{ext}} = \text{const}$, as far as $s^{\text{int}}, s^{\text{ext}} \in (s_{\text{high}}, \infty)$. Therefore

$$\Big(\rho\frac{\partial}{\partial s^{\text{int}}} + \frac{\partial}{\partial s^{\text{ext}}}\Big)\mathbf{W}(s^{\text{int}}, s^{\text{ext}}) = 0$$

and by (4.6), (4.7), taking into account that $\mathbf{p}(\cdot) = 1$ here, we get $\rho\big(\tilde{\mathbf{W}}_1(s) - \tilde{\mathbf{W}}_0(s)\big) + \tilde{\mathbf{W}}_0(s) = 0$ for $s \in (s_{\text{high}}, \infty)$. It remains to use continuity of $\tilde{\mathbf{W}}_0(\cdot)$, $\tilde{\mathbf{W}}_1(\cdot)$. $\square$

Proof of Lemma 4.15. Using continuity of $\tilde{\mathbf{W}}_0, \tilde{\mathbf{W}}_1$ we take $\varepsilon > 0$ such that $\tilde{\mathbf{W}}_0(\cdot) < 0$ and $\tilde{\mathbf{W}}_1(\cdot) > 0$ on $[s_0 - \varepsilon, s_0]$. Almost all $s \in [s_0 - \varepsilon, s_0]$ satisfy

$$\begin{aligned} \mathbf{W}(s) < \frac{c}{C} \quad &\Longrightarrow \quad \mathbf{p}(s) = 0 \quad \Longrightarrow \quad \mathbf{W}'(s) = \tilde{\mathbf{W}}_0(s) < 0\,, \\ \mathbf{W}(s) > \frac{c}{C} \quad &\Longrightarrow \quad \mathbf{p}(s) = 1 \quad \Longrightarrow \quad \mathbf{W}'(s) = \tilde{\mathbf{W}}_1(s) > 0\,. \end{aligned}$$

We have to prove that open sets $\{s \in (s_0 - \varepsilon, s_0) : \mathbf{W}(s) < c/C\}$ and $\{s \in (s_0 - \varepsilon, s_0) : \mathbf{W}(s) > c/C\}$ are empty. Assume the contrary, say, the former set is nonempty (the other case is similar), then there exists $s \in (s_0 - \varepsilon, s_0]$ such that $\mathbf{W}(\cdot) < c/C$ on some left neighborhood of s, and $\mathbf{W}(s) = c/C$.[22] It contradicts to the fact that $\mathbf{W}(\cdot)$ decreases on the neighborhood, since $\mathbf{W}'(\cdot) < 0$ almost everywhere on it. $\square$

Appendix B. The nonlinear contraction operator

Proof of Lemma 5.4. It suffices to prove that

$$\text{(B.1)} \qquad \sup_{s\in[r-\varepsilon,r]} \big|\big(\mathcal{A}(X)\big)(s) - \big(\mathcal{A}(Y)\big)(s)\big| \le C(r,\rho,n) \int_{r-\varepsilon}^{r} |X(s) - Y(s)|\, ds$$

for all $\varepsilon \in (0,1)$ and all continuous functions $X, Y : [r-\varepsilon, r] \to [0,1]$ satisfying $X(r) = Y(r) = \mathbf{p}(r)$; here the constant $C(r,\rho,n)$ may depend on r, ρ, n (but not ε, X, Y). Indeed, having (B.1) we may choose ε such that $C(r,\rho,n)\varepsilon \le 1/2$, then

$$\operatorname{dist}\big(\mathcal{A}(Y), \mathcal{A}(X)\big) \le C(r,\rho,n)\varepsilon \operatorname{dist}(X,Y) \le \frac{1}{2}\operatorname{dist}(X,Y)\,.$$

[22] Every open set consists of open intervals whose endpoints do not belong to the set.

However, we need also $\big(\mathcal{A}(X)\big)(s) \in (0,1)$. Consider the constant function $X_0(s) = \mathbf{p}(r)$ for $s \in [r-\varepsilon, r]$. The corresponding $\mathcal{A}(X_0)$ is a continuous function, and $\big(\mathcal{A}(X_0)\big)(r) = \mathbf{p}(r) \in (0,1)$, therefore there exists $\delta > 0$ such that $\big(\mathcal{A}(X_0)\big)(s) \in (0,1)$ for all $s \in [r-\delta, r]$. We may let $\varepsilon = \delta$, since operators $\mathcal{A}$ on different segments $[r-\varepsilon, r]$ conform to each other. Moreover, we may choose ε such that

$$\big(\mathcal{A}(X_0)\big)(s) \in \big(C(r,\rho,n)\varepsilon, 1 - C(r,\rho,n)\varepsilon\big) \quad \text{for all } s \in [r-\varepsilon, r]\,.$$

Taking into account that

$$\operatorname{dist}\big(\mathcal{A}(X), \mathcal{A}(X_0)\big) \le C(r,\rho,n) \int_{r-\varepsilon}^{r} |X(s) - X_0(s)|\, ds \le C(r,\rho,n)\varepsilon$$

for every X, we get

$$\big(\mathcal{A}(X)\big)(s) \in (0,1) \quad \text{for all } s \in [r-\varepsilon, r]\,.$$

So, he have to prove (B.1). We connect the two functions by a one-parameter family of functions

$$X_u : [r-\varepsilon, r] \to [0,1]\,, \quad X_u(s) = (1-u)X(s) + uY(s)\,, \quad u \in [0,1]\,,$$

then $X_0 = X$, $X_1 = Y$, and

$$\big(\mathcal{A}(Y)\big)(s) - \big(\mathcal{A}(X)\big)(s) = \int_0^1 \frac{\partial}{\partial u}\big(\mathcal{A}(X_u)\big)(s)\, du\,.$$

It suffices to prove that

$$\Big|\frac{\partial}{\partial u}\big(\mathcal{A}(X_u)\big)(s)\Big| \le C(r,\rho,n) \int_{r-\varepsilon}^{r} |Y(s) - X(s)|\, ds\,. \tag{B.2}$$

Each X_u determines the corresponding $\tilde{\mathbf{W}}_{0,u}$ and $\tilde{\mathbf{W}}_{1,u}$ so that $\big(\mathcal{A}(X_u)\big)(s) = -\tilde{\mathbf{W}}_{0,u}(s)/\big(\tilde{\mathbf{W}}_{1,u}(s) - \tilde{\mathbf{W}}_{0,u}(s)\big)$. Thus,

$$\begin{aligned}\frac{\partial}{\partial u}\big(\mathcal{A}(X_u)\big)(s) = \frac{\tilde{\mathbf{W}}_{0,u}(s)}{(\tilde{\mathbf{W}}_{1,u}(s) - \tilde{\mathbf{W}}_{0,u}(s))^2}\frac{\partial}{\partial u}\big(\tilde{\mathbf{W}}_{1,u}(s) - \tilde{\mathbf{W}}_{0,u}(s)\big) - \\ - \frac{1}{\tilde{\mathbf{W}}_{1,u}(s) - \tilde{\mathbf{W}}_{0,u}(s)}\frac{\partial}{\partial u}\tilde{\mathbf{W}}_{0,u}(s)\,.\end{aligned}$$

In order to prove (B.2) it suffices to prove three estimations from above and one estimation from below:

$$\Big|\frac{\partial}{\partial u}\tilde{\mathbf{W}}_{0,u}(s)\Big| \le C_0(r,\rho,n)\delta\,, \tag{B.3}$$

$$\Big|\frac{\partial}{\partial u}\big(\tilde{\mathbf{W}}_{1,u}(s)-\tilde{\mathbf{W}}_{0,u}(s)\big)\Big| \le C_1(r,\rho,n)\delta\,, \tag{B.4}$$

$$|\tilde{\mathbf{W}}_{0,u}(s)| \le C_2(r,\rho,n)\,, \tag{B.5}$$

$$\tilde{\mathbf{W}}_{1,u}(s)-\tilde{\mathbf{W}}_{0,u}(s) \ge c(r,\rho,n)\,, \tag{B.6}$$

where $\delta = \int_{r-\varepsilon}^{r} |Y(s)-X(s)|\,ds$, and C_0, C_1, C_2, c are some constants (depending on r,ρ,n only), $C_0<\infty$, $C_1<\infty$, $C_2<\infty$, $c>0$.

Proof of (B.5) is simple:

$$|\tilde{\mathbf{W}}_{0,u}(s)| \le \frac{\sqrt{\rho}}{1-\rho}\int |\theta-\sqrt{\rho}\,s|\underbrace{\Big(1-\int_s^\infty\dots\Big)^{n-1}}_{\le 1} d_\theta\Phi\Big(\frac{\theta-\sqrt{\rho}\,s}{\sqrt{1-\rho}}\Big) \le$$
$$\le \frac{\sqrt{\rho}}{1-\rho}\int_{-\infty}^{+\infty}|x|\,d\Phi\Big(\frac{x}{\sqrt{1-\rho}}\Big) = C_2(r,\rho,n)$$

(in fact, C_2 depends on ρ only).

Introduce

$$\mathbf{p}_u(s) = \begin{cases} X_u(s) & \text{for } s\in[r-\varepsilon,r],\\ \mathbf{p}(s) & \text{for } s\in(r,\infty);\end{cases}$$

the corresponding $\mathbf{w}_{m,u}(d\theta|s)$ is defined by (4.8) with $\mathbf{p}_u$ in place of $\mathbf{p}$. We have

$$\int_s^\infty \mathbf{p}_u(s_1)\,d_{s_1}\Phi\Big(\frac{s_1-\sqrt{\rho}\,\theta}{\sqrt{1-\rho}}\Big) \le \int_s^\infty d_{s_1}\Phi(\dots) = 1-\Phi\Big(\frac{s-\sqrt{\rho}\,\theta}{\sqrt{1-\rho}}\Big)\,;$$
$$\mathbf{w}_{m,u}(d\theta|s) \ge \Phi^m\Big(\frac{s-\sqrt{\rho}\,\theta}{\sqrt{1-\rho}}\Big)\,d_\theta\Phi\Big(\frac{\theta-\sqrt{\rho}\,s}{\sqrt{1-\rho}}\Big)\,;$$

therefore

$$\tilde{\mathbf{W}}_{1,u}(s)-\tilde{\mathbf{W}}_{0,u}(s) \ge$$
$$\ge \frac{n-1}{\sqrt{1-\rho}}\int \varphi\Big(\frac{s-\sqrt{\rho}\,\theta}{\sqrt{1-\rho}}\Big)\Phi^{n-2}\Big(\frac{s-\sqrt{\rho}\,\theta}{\sqrt{1-\rho}}\Big)\,d_\theta\Phi\Big(\frac{\theta-\sqrt{\rho}\,s}{\sqrt{1-\rho}}\Big)\,.$$

The right-hand side is strictly positive and continuous in s. We may denote its infimum over $s\in[r-\varepsilon,r]$ by $c(r,\rho,n)$, thus proving (B.6).

It remains to prove (B.3) and (B.4). We have for every $s \in [r-\varepsilon, r]$

$$\left|\frac{\partial}{\partial u}\mathbf{p}_u(s)\right| = |Y(s) - X(s)| \cdot \mathbf{1}_{[r-\varepsilon,r]}(s)\,;$$

$$\left|\frac{\partial}{\partial u}\int_s^\infty \mathbf{p}_u(s_1)\,d_{s_1}\Phi\Big(\frac{s_1-\sqrt{\rho}\,\theta}{\sqrt{1-\rho}}\Big)\right| \le$$
$$\le \int_{r-\varepsilon}^r |Y(s_1)-X(s_1)|\frac{1}{\sqrt{1-\rho}}\varphi\Big(\frac{s_1-\sqrt{\rho}\,\theta}{\sqrt{1-\rho}}\Big)\,ds_1 \le \frac{\varphi(0)}{\sqrt{1-\rho}}\delta\,.$$

Thus,

$$\left|\frac{\partial}{\partial u}\mathbf{w}_{m,u}(d\theta|s)\right| \le$$
$$\le m\Big(1-\int_s^\infty \dots\Big)^{m-1}\left|\frac{\partial}{\partial u}\int_s^\infty \mathbf{p}_u(s_1)\,d_{s_1}\Phi(\dots)\right| d_\theta\Phi\Big(\frac{\theta-\sqrt{\rho}\,s}{\sqrt{1-\rho}}\Big) \le$$
$$\le m\frac{\varphi(0)}{\sqrt{1-\rho}}\delta\,d_\theta\Phi\Big(\frac{\theta-\sqrt{\rho}\,s}{\sqrt{1-\rho}}\Big)\,.$$

Therefore

$$\left|\frac{\partial}{\partial u}\tilde{\mathbf{W}}_{0,u}(s)\right| \le \frac{\sqrt{\rho}}{1-\rho}\int |\theta-\sqrt{\rho}\,s|\left|\frac{\partial}{\partial u}\mathbf{w}_{n-1}(d\theta|s)\right| \le$$
$$\le \frac{\sqrt{\rho}}{1-\rho}\int |\theta-\sqrt{\rho}\,s| m\frac{\varphi(0)}{\sqrt{1-\rho}}\delta\,d_\theta\Phi\Big(\frac{\theta-\sqrt{\rho}\,s}{\sqrt{1-\rho}}\Big) =$$
$$= \delta\frac{\sqrt{\rho}}{1-\rho}(n-1)\frac{\varphi(0)}{\sqrt{1-\rho}}\int_{-\infty}^\infty |x|\,d\Phi\Big(\frac{x}{\sqrt{1-\rho}}\Big)\,,$$

which proves (B.3). Finally,

$$\left|\frac{\partial}{\partial u}\big(\tilde{\mathbf{W}}_{1,u}(s)-\tilde{\mathbf{W}}_{0,u}(s)\right| \le \frac{n-1}{\sqrt{1-\rho}}\int \varphi\Big(\frac{s-\sqrt{\rho}\,\theta}{\sqrt{1-\rho}}\Big)\left|\frac{\partial}{\partial u}\mathbf{w}_{n-2,u}(d\theta|s)\right| \le$$
$$\le \frac{n-1}{\sqrt{1-\rho}}\int \underbrace{\varphi\Big(\frac{s-\sqrt{\rho}\,\theta}{\sqrt{1-\rho}}\Big)}_{\le\varphi(0)}(n-2)\frac{\varphi(0)}{\sqrt{1-\rho}}\delta\,d_\theta\Phi\Big(\frac{\theta-\sqrt{\rho}\,s}{\sqrt{1-\rho}}\Big) \le$$
$$\le \delta\frac{(n-1)(n-2)\varphi(0)}{1-\rho}\varphi(0)\underbrace{\int d_\theta\Phi(\dots)}_{\le 1}\,,$$

which proves (B.4). $\square$

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