5 Random connected components

5a Connected sets among closed sets

Recall some topological notions and facts.

A metrizable space $X$ is connected, if $\text{ Clopen}(X) = \{0, X\}$.

A subset $Y \subset X$ is connected if it is itself a connected space. Note that

\[(5a1) \quad \text{ Clopen}(Y) \supset \{ A \cap Y : A \in \text{ Clopen}(X) \};\]

in general, these are not equal (indeed, it happens routinely that $X$ is connected while $Y$ is not). For a connected $Y$,

\[(5a2) \quad \forall A \in \text{ Clopen}(X) \quad (Y \subset A \vee Y \subset X \setminus A).\]

For arbitrary $Y$, choosing a compatible metric $\rho$ on $X$ and denoting as before $\text{ dist}(x, A) = \inf\{\rho(x, a) : a \in A\}$, we have

\[
\text{ Clopen}(Y) = \{ A \subset Y : \overline{A} \cap (Y \setminus A) = \emptyset \wedge (Y \setminus A) \cap A = \emptyset \} = \\
= \{ A \subset Y : (\forall a \in A \text{ dist}(a, Y \setminus A) > 0) \wedge (\forall b \in Y \setminus A \text{ dist}(b, A) > 0) \}. 
\]

For every $A \in \text{ Clopen}(Y)$ there exist open sets $U, V \subset X$ such that $U \cap V = \emptyset$, $U \cap Y = A$ and $V \cap Y = Y \setminus A$; namely, we may take

\[
U = \bigcup_{a \in A} B^{\rho}(a, 0.5\rho(a, Y \setminus A)), \quad V = \bigcup_{b \in Y \setminus A} B^{\rho}(b, 0.5\rho(b, A)). 
\]

(Here $B^{\rho}(a, r) = \{x : \rho(x, a) < r\}$.) Thus,

\[(5a3) \quad \text{ Clopen}(Y) = \\
= \{ Y \cap U : U, V \text{ are open in } X \wedge U \cap V = \emptyset \wedge U \cup V \supset Y \}. 
\]
5a4 Proposition. For every compact metrizable space $X$ the set
\[
\{ F \in \mathcal{F}(X) : F \text{ is connected} \}
\]
is Borel measurable.

From now on (till 5a9) $X$ is compact, and $(U_n)_n$ is a sequence\(^1\) of open subsets of $X$ such that for all $E, F \in \mathcal{F}(X)$,
\[
5a5 \quad E \cap F = \emptyset \implies \exists m, n \left( U_m \cap U_n = \emptyset \land E \subset U_m \land F \subset U_n \right).
\]

5a6 Core exercise. Prove existence of such $(U_n)_n$.

Note that Clopen($X$) $\subset \{ U_1, U_2, \ldots \}$; and by the way, it shows that Clopen($X$) is at most countable (provided that $X$ is compact).\(^2\)

5a7 Core exercise. For every $F \in \mathcal{F}(X)$,
\[
\text{Clopen}(F) = \{ F \cap U_m : U_m \cap U_n = \emptyset \land U_m \cup U_n \supset F \}.
\]
Prove it.

5a8 Core exercise. A closed set $F \subset X$ is connected if and only if
\[
\forall m, n \left( ( U_m \cap U_n = \emptyset \land U_m \cup U_n \supset F ) \implies ( U_m \cap F = \emptyset \lor U_n \cap F = \emptyset ) \right).
\]
Prove it.

5a9 Core exercise. Prove Prop. 5a4.

Prop. 5a4 fails for Polish (not just compact) spaces. In particular, it fails if $X$ is an infinite-dimensional separable Hilbert space.\(^3\)

Does 5a4 hold for $X = \mathbb{R}^d$? I do not know!\(^4\) If you feel enthusiastic to reduce connectedness of a closed set to some property of its compact subsets, take into account the following instructive example of a connected closed subset of $\mathbb{R}^2$:

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\(^1\)“Countable superbase”, if you like.

\(^2\)For Polish $X$, Clopen($X$) need not be countable (try a discrete space).

\(^3\)This is basically the phenomenon mentioned before 4d10.

\(^4\)I only know that $\{ F \in \mathcal{F}(\mathbb{R}^d) : F \text{ is connected} \}$ is coanalytic, since its complement is the image of the Borel set $\{(F_1, F_2) : F_1 \cap F_2 = \emptyset, F_1 \neq \emptyset, F_2 \neq \emptyset \}$ (recall 4d15) under the Borel map $(F_1, F_2) \mapsto F_1 \cup F_2$. 
**5a10 Remark.** As a palliative we may treat a random closed subset of \( \mathbb{R}^d \) via the one-point compactification \( \mathbb{R}^d \cup \{\infty\} \) (including \( \infty \) into each unbounded closed set). Then all unbounded connected components (if any) are glued together.

**5b Connected components**

Consider two equivalence relations on a metrizable space \( X \): points \( x, y \in X \) are equivalent, when

5b1 \[ x, y \in Y \text{ for some connected } Y \subset X ; \]

5b2 \[ \forall A \in \text{Clopen}(X) \quad (x \in A \iff y \in A) . \]

Clearly, 5b1 implies 5b2 (recall 5a2). In general they are not equivalent; an example:

Equivalence classes for 5b1 are called connected components (of \( X \)); for 5b2 — quasiconnected components. In general, every quasiconnected component decomposes into connected components. But a compact \( X \) is simpler.

**5b3 Lemma.** For a compact \( X \), 5b1 and 5b2 are equivalent.

Proof. Let \( Y \) be an equivalence class for 5b2; we’ll prove that \( Y \) is connected. By 5a3 it is sufficient to prove \( Y \subset U \) or \( Y \subset V \) whenever open \( U, V \subset X \) satisfy \( U \cap V = \emptyset \) and \( U \cup V \supset Y \). Compactness gives us \( A \in \text{Clopen}(X) \) such that \( Y \subset A \subset U \cup V \). Thus, \( A \cap U = A \setminus V \in \text{Clopen}(X) \).

All points of \( Y \) being 5b2-equivalent, we get \( Y \subset A \cap U \) or \( Y \subset X \setminus (A \cap U) \); accordingly, \( Y \subset U \) or \( Y \subset V \).

Note that a compact \( X \) can have uncountably many connected components (try the Cantor set).

**5b4 Proposition.** The following subset of \( \mathbf{F}(X) \times \mathbf{F}(X) \) is Borel measurable, provided that \( X \) is compact:

\[ \{(E, F) : E \text{ is a connected component of } F\} . \]

Choosing a compatible metric \( \rho \) on \( X \) we define for \( E, F \in \mathbf{F}(X) \)

\[ d_\mathcal{Q}(E, F) = \sup_{x \in E} \text{dist}(x, F) = \sup_{x \in E} \inf_{y \in F} \rho(x, y) = \inf\{r > 0 : E \subset F_{+r}\} , \]

\[ d_H(E, F) = \max\{d_\mathcal{Q}(E, F), d_\mathcal{Q}(F, E)\} = \inf\{r > 0 : E \subset F_{+r} \land F \subset E_{+r}\} . \]
(as usual, inf $\emptyset = +\infty$); $d_H$ is a metric on $F(X) \setminus \{\emptyset\}$, — the well-known Hausdorff metric.\footnote{In fact, the Hausdorff metric on a compact $X$ turns $F(X)$ into a compact metric space whose Borel $\sigma$-algebra coincides with that of (4d2). The set of connected components of $X$ need not be closed in $(F(X), d_H)$; an example: $[-1, 0] \cup \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$. For a Polish $X$, the metric space $(F(X), d_H)$ is complete but generally nonseparable (try a discrete $X$). See also Sect. 2 in Beer's article (cited in footnote 5 on page 71).}

5b5 Lemma. $d_\mathcal{E} : F(X) \times F(X) \to [0, \infty]$ is a Borel function.

Proof. We take a sequence $(x_n)_n$ dense in $X$ and note that
\[
d_\mathcal{E}(E, F) = \inf_{r>0} \sup_{n : \text{dist}(x_n, E) \leq \varepsilon} \text{dist}(x_n, F)
\]
and dist$(x_n, \cdot)$ is a Borel function by 4d8(a).

Proof of Prop. 5b4. For $E, F \in F(X)$, by 5b3, $E$ is a connected component of $F$ if and only if
\begin{enumerate}[(a)]  
    
    \item $E$ is connected, and  
    \item $E \subset F$, and $E$ is the intersection of all $A \in \text{Clopen}(F)$ such that $A \supset E$.
\end{enumerate}

Condition (a) leads to a Borel set by Prop. 5a4; we'll prove the same for (b). By compactness, (b) is equivalent to
\begin{enumerate}[(b1)]  
    
    \item for every $\varepsilon > 0$ there exists $A \in \text{Clopen}(F)$ such that $A \supset E$ and $d_\mathcal{E}(A, E) \leq \varepsilon$.
\end{enumerate}

We choose $(U_n)_n$ satisfying (5a3). By (5a7), (b1) is equivalent to
\begin{enumerate}[(b2)]  
    
    \item for every $\varepsilon > 0$ there exist $m, n$ such that $U_m \cap U_n = \emptyset$, $U_m \cup U_n \supset F$, $F \cap U_m \supset E$ and $d_\mathcal{E}(F \cap U_m, E) \leq \varepsilon$.
\end{enumerate}

It remains to check, for given $m, n$ satisfying $U_m \cap U_n = \emptyset$, that each of the following three conditions leads to a Borel set:
\begin{enumerate}[(c)]  
    
    \item $F \subset U_m \cup U_n$,  
    \item $E \subset F \setminus U_n$,  
    \item $d_\mathcal{E}(F \setminus U_n, E) \leq \varepsilon$.
\end{enumerate}

We rewrite (c), (d) as
\begin{enumerate}[(c1)]  
    
    \item $F \cap (X \setminus (U_m \cup U_n)) = \emptyset$,  
    \item $d_\mathcal{E}(F, F \setminus U_n) = 0$.
\end{enumerate}

For (c1) we use 4d12(b). For (d1) and (e) we use 5b5 taking into account that the map $F \mapsto F \setminus U_n = F \cap (X \setminus U_n)$ is Borel measurable by 4d14. \qed

We conclude.
5b6 Theorem. Let $X$ be a compact metrizable space and $S$ a random closed subset of $X$. Then the set of all connected components of $S$ is a random Borel subset of the standard $2$ Borel space $F(X)$.

5c Counting the connected components

5c1 Core exercise. Let $S$ be a random measurable subset of a standard Borel space. Then $\{\omega : |S(\omega)| \leq n\}$ is measurable for every $n$. (Here $|S(\omega)|$ is the number of points in $S(\omega)$.)

Prove it.

By 5c1 (and Theorem 5b6), given a random closed set in a compact metrizable $X$, the number of its connected components is a random variable (with values in $\{0, 1, 2, \ldots\} \cup \{\infty\}$). In this sense we may count the connected components. Given a Borel set $B \subset F(X)$, we may count the connected components belonging to $B$ (think, why). For example we may count the connected components contained in a given ball, or intersecting a given ball, etc.

5c2 Core exercise. Let $S$ be a random closed subset of $\mathbb{R}^d$. Then the set of all bounded connected components of $S$ is a random Borel subset of $F(\mathbb{R}^d)$.

Prove it.

5d Classifying the connected components: random knots

A knot is a subset of $\mathbb{R}^3$ homeomorphic to a circle.

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1I mean the set of all connected components of $S(\omega)$ as a function of $\omega$.
2Recall 4d5.
3The set of $\omega$ such that $S(\omega)$ is uncountable (and therefore of cardinality continuum, see Kechris, Th. (13.6) or Srivastava, Th. 4.3.5) is also measurable, see Kechris, Th. (29.19).
Knots $K_1, K_2$ are of the same type (in other words, equivalent) if $h(K_1) = K_2$ for some homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$.\footnote{See for instance Sect. 1.1 in: R.H. Crowell, R.H. Fox, “Introduction to knot theory”, Springer 1963.} A knot is tame if it is equivalent to a polygonal knot. Or, equivalently, to a smooth knot (continuously differentiable). Otherwise it is a wild knot.\footnote{Crowell and Fox, Sect. 1.2.} All knots that lie in a plane are of the same type (the trivial type; the unknot).\footnote{Crowell and Fox, Sect. 1.2.} This is a deep theorem in general, but relatively simple for tame knots.

All tame knot types are a countable set.\footnote{Crowell and Fox, Chapter I, Exercise 5. In fact, they are a semigroup isomorphic to \{1, 2, 3, ...\} with multiplication. By the way, some wild knots are infinite products of tame knots.}

\textbf{5d1 Proposition.} Each knot type is a universally measurable\footnote{In fact, Borel measurable, but this is harder to prove.} subset of $\mathbf{F}(\mathbb{R}^3)$.

Thus, all tame knots are also a universally measurable set.

By 5c2 and 5d1, given a random closed set in $\mathbb{R}^3$, we may count its connected components that are (a) tame knots; (b) tame knots of a given type; (c) tame knots of a given type that are contained in a given ball; etc.

You may think about a random smooth map $f : \mathbb{R}^3 \to \mathbb{R}^2$ (or $\mathbb{R}^3 \to \mathbb{C}$) such that almost surely, $f$ is regular (that is, rank $df(\cdot) = 2$) at all $x$ satisfying $f(x) = 0$. Then all bounded connected components of the random closed set $f^{-1}(0)$ are tame knots (almost surely).

The group $\text{Homeo}(\mathbb{R}^3)$ of all homeomorphisms $\mathbb{R}^3 \to \mathbb{R}^3$ acts on the standard Borel space $\mathbf{F}(\mathbb{R}^3)$.

Here is a generalization of Prop. 5d1.

\textbf{5d2 Proposition.} For every $F \in \mathbf{F}(\mathbb{R}^3)$ its orbit $\{h(F) : h \in \text{Homeo}(\mathbb{R}^3)\}$ is a universally measurable subset of $\mathbf{F}(\mathbb{R}^3)$.

\textbf{5d3 Core exercise.} The set $C(\mathbb{R}^3 \to \mathbb{R}^3)$ of all continuous maps $\mathbb{R}^3 \to \mathbb{R}^3$ endowed with the $\sigma$-algebra generated by evaluations $f \mapsto f(x)$ is a standard Borel space.

Prove it.

\textbf{5d4 Core exercise.} $f(x)$ is jointly measurable in $f \in C(\mathbb{R}^3 \to \mathbb{R}^3)$ and $x \in \mathbb{R}^3$.

Prove it.
5d5 Core exercise. The composition \( g \circ f \in C(\mathbb{R}^3 \to \mathbb{R}^3) \) is jointly measurable in \( f, g \in C(\mathbb{R}^3 \to \mathbb{R}^3) \).

Prove it.

We embed the set \( \text{Homeo}(\mathbb{R}^3) \) into \( C(\mathbb{R}^3 \to \mathbb{R}^3) \times C(\mathbb{R}^3 \to \mathbb{R}^3) \) as follows:

\[
h \mapsto (h, h^{-1}).
\]

By 5d5 the image \( \{(f, g) : f \circ g = \text{id} \land g \circ f = \text{id} \} \) is a Borel set, therefore a standard Borel space. We endow \( \text{Homeo}(\mathbb{R}^3) \) with the corresponding \( \sigma \)-algebra and observe that

\[
(5d6) \quad \text{Homeo}(\mathbb{R}^3) \text{ is a standard Borel space.}\]

5d7 Extra exercise. \( \text{Homeo}(\mathbb{R}^3) \) is both a Borel subset and a measurable subspace of \( C(\mathbb{R}^3 \to \mathbb{R}^3) \).

Prove it.\(^2\)

(By the way, it follows that the map \( f \mapsto f^{-1} \) is Borel measurable on \( \text{Homeo}(\mathbb{R}^3) \) treated as a subset of \( C(\mathbb{R}^3 \to \mathbb{R}^3) \).)

5d8 Core exercise. Let \( F \subset \mathbb{R}^3 \) be a closed set, then the map

\[
C(\mathbb{R}^3 \to \mathbb{R}^3) \ni f \mapsto f(F) \in F(\mathbb{R}^3)
\]

is Borel measurable.

Prove it.

Thus, the orbit \( \{h(F) : h \in \text{Homeo}(\mathbb{R}^3)\} \) is the image of the standard Borel space \( \text{Homeo}(\mathbb{R}^3) \) under the Borel map \( h \mapsto h(F) \).

Here is a topology-free counterpart of Def. 3e1.

5d9 Definition. A subset of a countably separated measurable space is \textit{analytic} if it is the image of some standard Borel space under some measurable map.

Does 5d9 conflict with 3e1? No, it does not.

5d10 Lemma. For every subset \( A \) of a separable metrizable space, the following two conditions are equivalent:

(a) \( A \) is the image of some Polish space under some continuous map;

(b) \( A \) is the image of some standard Borel space under some Borel map.

\(^1\)It is in fact a Polish group (and therefore, by 4d7, a standard Borel space).

\(^2\)In fact, this follows from standardness by a general theorem; but you are asked to prove it explicitly.
Proof. (a) $\Rightarrow$ (b): the Polish space is a standard Borel space by 4d7, and the continuous map is Borel measurable.

(b) $\Rightarrow$ (a): we may assume that the standard Borel space is a Borel subset of the Cantor set, or even the whole Cantor set (extend the map by a constant...); it remains to use Prop. 3d1.

And here is a counterpart of Theorem 3f22.

5d11 Theorem. Analytic sets in countably separated measurable spaces are universally measurable.

Proof. Let $(X, \mathcal{A})$ be a standard Borel space, $(Y, \mathcal{B})$ a countably separated measurable space, $f : X \to Y$ a measurable map, and $\mu$ a probability measure on $(Y, \mathcal{B})$; we have to prove that $f(X)$ is $\mu$-measurable.

Without loss of generality we may assume that $(Y, \mathcal{B})$ is a Borel space. Proof: first, by 1d35, $(Y, \mathcal{B}_1)$ is a Borel space for some $\mathcal{B}_1 \subset \mathcal{B}$; second, $f$ is measurable from $(X, \mathcal{A})$ to $(Y, \mathcal{B}_1)$; third, every $(\mu|_{\mathcal{B}_1})$-measurable set is $\mu$-measurable.

By 1d40 we may assume that $Y \subset \mathbb{R}$. Moreover, we may assume that $Y = \mathbb{R}$. Proof: we define a probability measure $\nu$ on $\mathbb{R}$ by $\nu(B) = \mu(B \cap Y)$ for $B \in \mathcal{B}(\mathbb{R})$ and observe that every $\nu$-measurable subset of $Y$ is $\mu$-measurable.

It remains to use Lemma 5d10 and Theorem 3f22.

Propositions 5d2 and 5d11 follow. Proposition 5d2 (and its proof) generalizes readily to arbitrary locally compact separable metrizable spaces (in place of $\mathbb{R}^3$).

So, we may count tame knots. What about wild knots?

5d12 Proposition. All knots are a universally measurable subset of $\mathbf{F}(\mathbb{R}^3)$.

Thus, wild knots are also a universally measurable set.

The circle is a compact metrizable space; knots are its homeomorphic images in $\mathbb{R}^3$.

5d13 Core exercise. Let $K$ be a compact metrizable space, and (similarly to 5d8) $C(K \to \mathbb{R}^3)$ the standard Borel space of continuous maps. Then the set of all homeomorphisms $f : K \to f(K) \subset \mathbb{R}^3$ is a Borel subset of $C(K \to \mathbb{R}^3)$.

Prove it.

Similarly to 5d8 the map

$$C(K \to \mathbb{R}^3) \ni f \mapsto f(K) \in \mathbf{F}(\mathbb{R}^3)$$
is Borel measurable. Its image

\[ \{ F : F \text{ is homeomorphic to } K \} \subset F(\mathbb{R}^3) \]

is analytic, thus, universally measurable. Prop. 5.12 follows.

In fact, the set of all knots is Borel measurable.\(^1\)

You may think also about random links, and a lot of other random geometric objects.

\(\text{\footnotesize\textsuperscript{1}}\)The same holds for subsets of a given \(\sigma\)-compact separable metrizable space \(X\), that are homeomorphic to a given compact metrizable space \(K\); see: C. Ryll-Nardzewski, “On a Freedman’s problem”, Fund. Math. 57 (1965), 273–274. The corresponding Borel complexity is not bounded in \(K\) even if \(X = [0,1] \times [0,1]\); see Fact 3.12 in: A. Marcone, “Complexity of sets and binary relations in continuum theory: a survey” (2005). For \(K = S^1\) this Borel set is \(F_{\sigma\delta}\) and not \(G_{\delta\sigma}\); see Lemma 6.2 and Theorem 8.5 in: R. Camerlo, U.B. Darji, A. Marcone, “Classification problems in continuum theory”, Trans. AMS 357:11, 4301–4328 (2005).
Hints to exercises

5a6: consider a countable base and all finite unions of its sets.
5c1: use 3f24, taking into account that the Borel space is countably separated.
5c2: recall 5a10.
5d3: recall 2c10(c).
5d4: similar to (4b4).
5d5: use 5d4.
5d8: use a dense sequence.

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