# 2 Best response

## 2a Optimal actions and best response

We want to find the best response $\mu_1$ to a given strategy $\mu_2$. The game under consideration is basically the 'very simple auction' game (1b4), but some more general cases will be also treated.

Our profit function $\mathbf{\Pi}_1$ (defined by (1b1)) has two important properties:

(2a1) $\mathbf{\Pi}_1(a_1, s_1; a_2, s_2)$ does not depend on $s_2$ ;

(2a2) $\mathbf{\Pi}_1(a_1, s_1; a_2, s_2)$ is linear in $s_1$ .

Property (2a1) allows us writing $\mathbf{\Pi}_1(a_1, s_1; a_2)$, omitting $s_2$. Property (2a2) may be written as

(2a3) $$\mathbf{\Pi}_1(a_1, s_1; a_2) = (1 - s_1)\mathbf{\Pi}_1(a_1, 0; a_2) + s_1\mathbf{\Pi}_1(a_1, 1; a_2) = \\ = s_1 \cdot \big(\mathbf{\Pi}_1(a_1, 1; a_2) - \mathbf{\Pi}_1(a_1, 0; a_2)\big) + \mathbf{\Pi}_1(a_1, 0; a_2)\,;$$

a linear function is uniquely determined by its values at two points, say, 0 and 1. According to (1b1),

(2a4) $$\mathbf{\Pi}_1(a_1, 1; a_2) - \mathbf{\Pi}_1(a_1, 0; a_2) = \begin{cases} 0 & \text{if } a_1 < a_2, \\ 1/2 & \text{if } a_1 = a_2, \\ 1 & \text{if } a_1 > a_2, \end{cases}$$

just the winning probability (of the first player). Also,

(2a5) $$-\mathbf{\Pi}_1(a_1, 0; a_2) = \begin{cases} 0 & \text{if } a_1 < a_2, \\ \frac{1}{2}a_1 & \text{if } a_1 = a_2, \\ a_1 & \text{if } a_1 > a_2 \end{cases}$$

is the expected loss (payment) of the first player. Thus, (2a3) means[1]

(2a6) $$\mathbf{\Pi}_1(a_1, s_1; a_2) = s_1 \cdot (\text{winning probability}) - (\text{expected loss})\,.$$

However, that case is too simple; $a_2$ is given, and the only source of randomness is tie breaking (occurs for $a_1 = a_2$ only).

Here is a more interesting case: a strategy $\mu_2$ is given (rather than an action $a_2$). Then (recall (1d2))

$$\mathbf{\Pi}_1(a_1, s_1; \mu_2) = \iint_{\mathcal{A}_2 \times \mathcal{S}_2} \mathbf{\Pi}_1(a_1, s_1; a_2, s_2)\, d\mu_2(a_2, s_2)\,,$$

[1] In fact, (expected loss) $=$ $a_1$ $\cdot$ (winning probability), thus $\mathbf{\Pi}_1(a_1, s_1; a_2) = (s_1 - a_1) \cdot$ (winning probability). These relations are specific for *first price* auctions. Whenever possible, I prefer to rely on more general facts, valid for various single unit auctions.

which becomes simpler due to (2a1):

(2a7) $$\mathbf{\Pi}_1(a_1, s_1; P_{A_2}) = \int_{\mathcal{A}_2} \mathbf{\Pi}_1(a_1, s_1; a_2)\, dP_{A_2}(a_2)\,;$$

the distribution $P_{A_2}$ of $A_2$ is relevant, while $S_2$ is not. In other words: the first player is bothered by the action of the competitor, but indifferent to the cause of the action. The best response to $\mu_2$ is rather the best response to its marginal measure $P_{A_2}$.

Integrating (2a3) by $dP_{A_2}(a_2)$ we get

(2a8) $$\mathbf{\Pi}_1(a_1, s_1; P_{A_2}) = s_1 \cdot \big(\mathbf{\Pi}_1(a_1, 1; P_{A_2}) - \mathbf{\Pi}_1(a_1, 0; P_{A_2})\big) + \mathbf{\Pi}_1(a_1, 0; P_{A_2})\,,$$

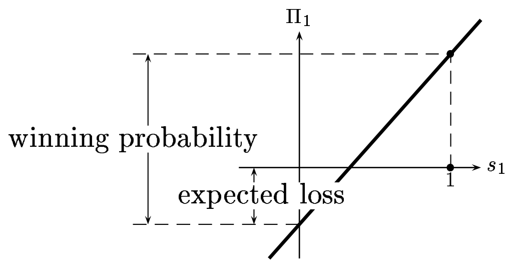
which is interpreted similarly to (2a6):

(2a9) $$\mathbf{\Pi}_1(a_1, s_1; P_{A_2}) = s_1 \cdot (\text{winning probability}) - (\text{expected loss})\,.$$

This time, $A_2$ is a source of randomness (more important than tie breaking), and the winning probability may be any number of $[0,1]$ (not just $0, 1/2, 1$). Still,

(2a10) $$\mathbf{\Pi}_1(a_1, s_1; P_{A_2}) \quad \text{is linear in } s_1\,.$$

Each $a_1$ determines a winning probability $\mathbf{\Pi}_1(a_1, 1; P_{A_2}) - \mathbf{\Pi}_1(a_1, 0; P_{A_2})$ and an expected loss $-\mathbf{\Pi}_1(a_1, 0; P_{A_2})$; thus, $a_1$ determines a linear function $s_1 \mapsto \mathbf{\Pi}_1(a_1, s_1; P_{A_2})$, and its graph, a straight line.




**2a11. Exercise.** Let $A_2 \sim \mathrm{U}(0, 1/2)$, the uniform distribution on $(0, 0.5)$. Then the winning probability is

$$\mathbf{\Pi}_1\big(a_1, 1; \mathrm{U}(0,1/2)\big) - \mathbf{\Pi}_1\big(a_1, 0; \mathrm{U}(0,1/2)\big) = \begin{cases} 2a_1 & \text{if } a_1 \in [0, 1/2], \\ 1 & \text{if } a_1 \in [1/2, \infty); \end{cases}$$

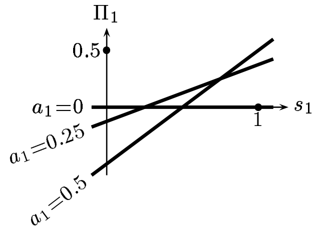
the expected loss is $a_1 \cdot (\text{winning probability})$, that is,

$$-\mathbf{\Pi}_1\big(a_1, 0; \mathrm{U}(0,1/2)\big) = \begin{cases} 2a_1^2 & \text{if } a_1 \in [0, 1/2], \\ a_1 & \text{if } a_1 \in [1/2, \infty); \end{cases}$$

so, the expected payoff is

$$\mathbf{\Pi}_1\big(a_1, s_1; \mathrm{U}(0,1/2)\big) = \begin{cases} s_1 \cdot 2a_1 - 2a_1^2 & \text{if } a_1 \in [0, 1/2], \\ s_1 - a_1 & \text{if } a_1 \in [1/2, \infty) \end{cases}$$

(as was seen in Sect. 1a, page 2).




Show it by integrating (2a4), (2a5).

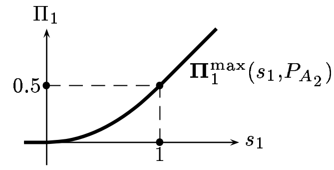
Striving to maximize his profit, the first player should consider the function

$$\mathbf{\Pi}_1^{\max}(s_1; P_{A_2}) = \sup_{a_1 \in \mathcal{A}_1} \mathbf{\Pi}_1(a_1, s_1; P_{A_2})\,; \tag{2a12}$$

its graph is the envelope of straight lines, therefore, the function is convex. Of course, $\mathbf{\Pi}_1^{\max}(s_1, P_{A_2}) < \infty$, since $\mathbf{\Pi}_1(a_1, s_1; P_{A_2}) \le s_1$.

**2a13. Exercise.** Let $A_2 \sim \mathrm{U}(0, 1/2)$ again. Then the maximal profit is

$$\mathbf{\Pi}_1^{\max}\big(s_1; \mathrm{U}(0,1/2)\big) = \begin{cases} 0 & \text{if } s_1 \in (-\infty, 0], \\ \frac12 s_1^2 & \text{if } s_1 \in [0,1], \\ s_1 - \frac12 & \text{if } s_1 \in [1, \infty). \end{cases}$$




Show it. (Find the optimal $a_1$ and substitute it.)

For every strategy $\mu_1$ the expected profit is bounded from above via $\mathbf{\Pi}_1^{\max}$; we just integrate the inequality

$$\mathbf{\Pi}_1(a_1, s_1; P_{A_2}) \le \mathbf{\Pi}_1^{\max}(s_1; P_{A_2}) \tag{2a14}$$

according to (1d3):

$$\mathbf{\Pi}_1(\mu_1; \mu_2) = \iint_{\mathcal{A}_1 \times \mathcal{S}_1} \mathbf{\Pi}_1(a_1, s_1; \mu_2)\, d\mu_1(a_1, s_1) \le$$
$$\le \iint_{\mathcal{A}_1 \times \mathcal{S}_1} \mathbf{\Pi}_1^{\max}(s_1; P_{A_2})\, d\mu_1(a_1, s_1) = \int_{\mathcal{S}_1} \mathbf{\Pi}_1^{\max}(s_1; P_{A_2})\, dP_{S_1}(s_1)\,.$$

That is,

$$\begin{aligned} &\mathbf{\Pi}_1(\mu_1; \mu_2) \le \mathbf{\Pi}_1^{\max}(P_{S_1}; P_{A_2})\,, \quad \text{where} \\ &\mathbf{\Pi}_1^{\max}(P_{S_1}; P_{A_2}) = \int_{\mathcal{S}_1} \mathbf{\Pi}_1^{\max}(s_1; P_{A_2})\, dP_{S_1}(s_1)\,. \end{aligned} \tag{2a15}$$

The same argument in a different language: we substitute random variables $A_1, S_1$ into (2a14) and take expectations,

$$\mathbf{\Pi}_1(\mu_1; \mu_2) = \mathbb{E}\,\mathbf{\Pi}_1(A_1, S_1; \mu_2) \le \mathbb{E}\,\mathbf{\Pi}_1^{\max}(S_1; P_{A_2}) = \mathbf{\Pi}_1^{\max}(P_{S_1}; P_{A_2})\,.$$

Still another form of the same argument: we rewrite (2a14) via conditional expectations,

$$\mathbb{E}\left( \boldsymbol{\Pi}_1(A_1,S_1;A_2,S_2) \,\middle|\, A_1,S_1 \right) \le \boldsymbol{\Pi}_1^{\max}(S_1;P_{A_2}) \tag{2a16}$$

and take unconditional expectation:

$$\begin{aligned}\boldsymbol{\Pi}_1(\mu_1;\mu_2) = \mathbb{E}\,\boldsymbol{\Pi}_1(A_1,S_1;A_2,S_2) = \mathbb{E}\left(\mathbb{E}\left( \boldsymbol{\Pi}_1(A_1,S_1;A_2,S_2) \,\middle|\, A_1,S_1 \right)\right) \le \\ \le \mathbb{E}\left(\boldsymbol{\Pi}_1^{\max}(S_1;P_{A_2})\right) = \boldsymbol{\Pi}_1^{\max}(P_{S_1};P_{A_2})\,.\end{aligned}$$

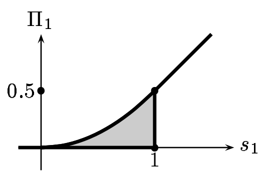
**2a17. Exercise.** Let $A_2 \sim \mathrm{U}(0,1/2)$ again, and $S_1 \sim \mathrm{U}(0,1)$. Then

$$\boldsymbol{\Pi}_1(\mu_1,\mu_2) \le \frac{1}{6}$$

for all $\mu_1$, since

$$\boldsymbol{\Pi}_1^{\max}\big(\mathrm{U}(0,1),\mathrm{U}(0,1/2)\big) = \frac{1}{6}\,.$$

Show it by integrating the result of 2a13.




So, the first player cannot get more than $\boldsymbol{\Pi}_1^{\max}(P_{S_1},P_{A_2})$. The next question is, whether he can reach the upper bound, or not. Note that

$$\begin{aligned}\boldsymbol{\Pi}_1^{\max}(P_{S_1},P_{A_2}) - \boldsymbol{\Pi}_1(\mu_1,P_{A_2}) = \\ = \iint_{\mathcal{A}_1\times\mathcal{S}_1} \big(\boldsymbol{\Pi}_1^{\max}(s_1;P_{A_2}) - \boldsymbol{\Pi}_1(a_1,s_1;P_{A_2})\big)\,d\mu_1(a_1,s_1) = \\ = \mathbb{E}\left(\boldsymbol{\Pi}_1^{\max}(S_1;P_{A_2}) - \boldsymbol{\Pi}_1(A_1,S_1;P_{A_2})\right).\end{aligned}$$

The defect of a strategy is equal to the expectation of a nonnegative random variable, or the integral of a nonnegative function. It vanishes if and only if the random variable vanishes almost surely (that is, with probability 1), or equivalently, the function vanishes almost everywhere (w.r.t. the measure $\mu_1$). We get the following result.

**2a18. Lemma.** The following conditions are equivalent for every strategy $\mu_1$ and every distribution $P_{A_2}$:

(a) $\boldsymbol{\Pi}_1(\mu_1;P_{A_2}) = \boldsymbol{\Pi}_1^{\max}(P_{S_1};P_{A_2})$.

(b) $\mu_1$ is concentrated on the set

$$\{(a_1,s_1)\in\mathcal{A}_1\times\mathcal{S}_1 : \boldsymbol{\Pi}_1(a_1,s_1;P_{A_2}) = \boldsymbol{\Pi}_1^{\max}(s_1;P_{A_2})\}\,.$$

(c) $\boldsymbol{\Pi}_1(A_1,S_1;P_{A_2}) = \boldsymbol{\Pi}_1^{\max}(S_1;P_{A_2})$ almost surely.

Item (b) shows that we should find an optimal action $a_1$ for each signal $s_1$ separately;[2] by an *optimal action* we mean $a_1$ satisfying

$$\mathbf{\Pi}_1(a_1, s_1; P_{A_2}) = \mathbf{\Pi}_1^{\max}(s_1; P_{A_2})\,.$$

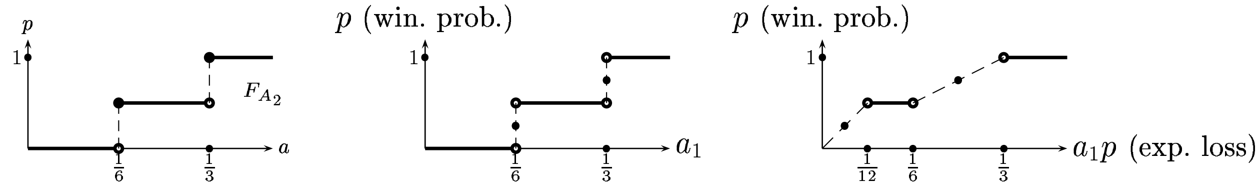
## 2b Atoms, gaps and best response

Each possible action $a_1$ may be visualized as a point on the plane (expected loss, winning probability). We have (recall (2a4)–(2a7))[3]

$$\begin{aligned}&(\text{winning probability}) = \mathbf{\Pi}_1(a_1, 1; P_{A_2}) - \mathbf{\Pi}_1(a_1, 0; P_{A_2}) = \\ &\quad = \mathbb{P}\big(\, A_2 < a_1 \,\big) + \frac{1}{2}\mathbb{P}\big(\, A_2 = a_1 \,\big) = \frac{1}{2}F_{A_2}(a_1-) + \frac{1}{2}F_{A_2}(a_1+)\,, \\ &(\text{expected loss}) = a_1 \cdot (\text{winning probability})\,;\end{aligned}$$

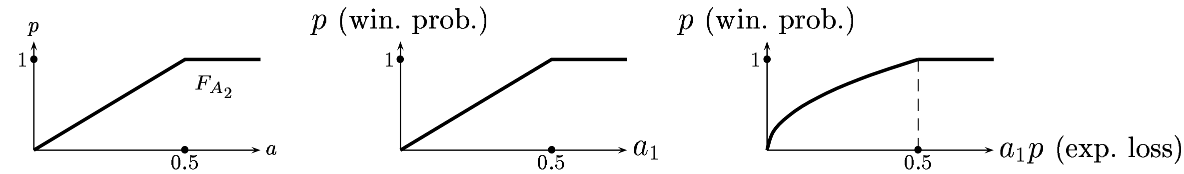
as usual, $F_{A_2}$ is the (cumulative) distribution function of $A_2$.

**2b1. Example.** Let $A_2$ be a discrete random variable, $\mathbb{P}\big(\, A_2 = 1/6 \,\big) = 0.5$, $\mathbb{P}\big(\, A_2 = 1/3 \,\big) = 0.5$.




The last picture shows all possible actions as points on the plane (expected loss, winning probability).

**2b2. Example.** Let $A_2 \sim \mathrm{U}(0, 1/2)$.

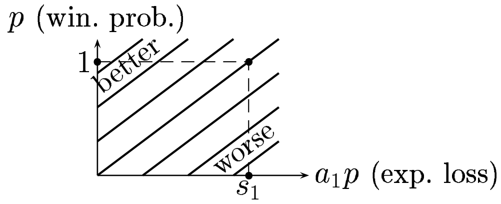



The third picture shows all possible actions as points on the plane (expected loss, winning probability). The parabolic segment $p = \sqrt{2(a_1 p)}$ on the third picture corresponds to the linear segment $p = 2a_1$ on the second picture.

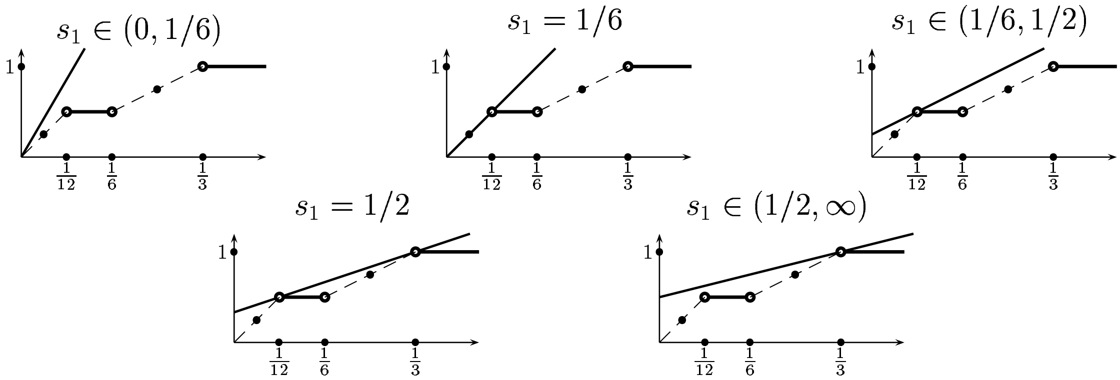
[2]It is quite natural; the first player knowing the actual value $s_1$ of his signal need not bother at all about other possible values of the random variable $S_1$. A strategy may seem to be an unnecessary complication. Do not forget, however, that the considered situation is oversimplified by assuming that the first player knows $P_{A_2}$, the distribution of actions of his competitor. In reality the first player, not knowing $P_{A_2}$, wants to guess thoughts of the competitor. The latter does not know the actual $s_1$ and is thinking about all possible values of $S_1$. This chain explains, why the whole distribution of $S_1$ may bother the first player even after receiving the signal.

[3]Now everything is specialized for *first price* auctions.

An optimal action maximizes the (goal, objective) function $\mathbf{\Pi}_1(a_1, s_1; P_{A_2}) = s_1 \cdot$ (winning probability) − (expected loss) linear on the plane (expected loss, winning probability).




**2b3. Example.** Let $A_2$ be as in 2b1 (discrete).




**Case** $s_1 \in (0, 1/6)$**.** All actions $a_1 \in [0, 1/6)$ are optimal. They all give (winning probability) = 0 and (expected loss) = 0. No other action is optimal.

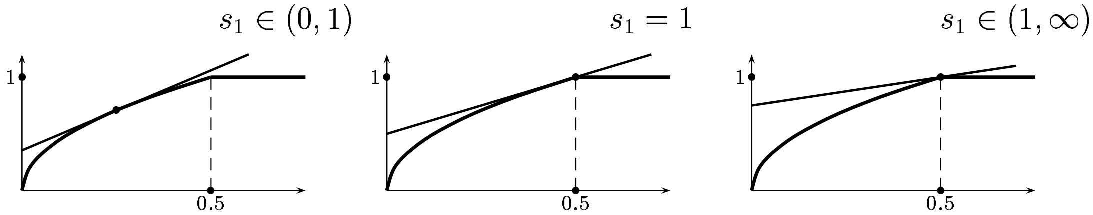
**Case** $s_1 = 1/6$**.** All actions $a_1 \in [0, 1/6)$ are still optimal, and in addition, the action $a_1 = 1/6$ is optimal. It gives (winning probability) = 1/4 and (expected loss) = 1/24. No other action is optimal.

**Case** $s_1 \in (1/6, 1/2)$**.** No action is optimal. The point $(1/12, 1/2)$ could be optimal, however, it does not belong to our set. Rather, it belongs to its closure.

**Case** $s_1 = 1/2$**.** No action is optimal. Points $(1/12, 1/2)$ and $(1/3, 1)$ could be optimal, however, they do not belong to our set.

**Case** $s_1 \in (1/2, \infty)$**.** No action is optimal. The point $(1/3, 1)$ could be optimal, however, it does not belong to our set.

**2b4. Example.** Let $A_2$ be as in 2b2 (nonatomic), namely, $A_2 \sim \mathrm{U}(0, 1/2)$.




**Case** $s_1 \in (0, 1)$**.** The action $a_1 = \frac{1}{2}s_1$ is optimal. It gives (winning probability) = $s_1$ and (expected loss) = $\frac{1}{2}s_1^2$. No other action is optimal. The optimum can be found as follows (though, you may find different ways):

$$\frac{d}{dx}\sqrt{2x} = \frac{1}{s_1}\,; \qquad x = (\text{expected loss})$$
$$\frac{2}{2\sqrt{2x}} = \frac{1}{s_1}\,; \quad \sqrt{2x} = s_1\,; \quad x = \frac{1}{2}s_1^2\,; \quad p = \sqrt{2x} = s_1\,; \quad a_1 = \frac{x}{p} = \frac{1}{2}s_1\,.$$

**Case $s_1 = 1$.** The action $a_1 = \frac{1}{2}$ is optimal. It gives (winning probability) $= 1$ and (expected loss) $= \frac{1}{2}$. No other action is optimal. The value $s_1 = 1$ may be found as follows:

$$\frac{d}{dx}\Big|_{x=1/2}\sqrt{2x} = \frac{1}{s_1}\,. \qquad x = (\text{expected loss})$$

**Case $s_1 \in (1, \infty)$.** The action $a_1 = \frac{1}{2}$ is optimal. It gives (winning probability) $= 1$ and (expected loss) $= \frac{1}{2}$. No other action is optimal.

You see, the continuous case is simpler than the discrete case, which is rather typical in actions theory. This is why theoreticians often prefer continuous models even though real-life bidding is always discrete.

Applying Lemma 2a18 to Example 2b4 we see that, for $P_{A_2} = \mathrm{U}(0, 1/2)$ and an arbitrary $P_{S_1}$, there exists one and only one best response (to $P_{A_2}$). Basically, it is given by

$$A_1 = \frac{1}{2}\min(S_1, 1)\,.$$

More formally, it is the joint distribution $P_{S_1,A_1}$ of random variables $S_1$ (distributed $P_{S_1}$, as required) and $A_1$ (defined as $\frac{1}{2}\min(S_1, 1)$).

In contrast, for the discrete distribution $P_{A_2}$ of Examples 2b1, 2b3 there is no best response, in general. Though, if $P_{S_1}\big([0, 1/6]\big) = 1$, that is, $0 \le S_1 \le 1/6$ almost surely, then there exists a best response; say, $A_1 = 0$ (for all $S_1$). In fact, there are a lot of best responses, since we are pretty free to choose conditional distributions of $A_1$ given $S_1$; we only must keep $A_1$ within $[0, 1/6)$ (or $[0, 1/6]$, when $S = 1/6$). Of course, here the first player is just fooling around; he never wins.

Do not think that a nonatomic $P_{A_2}$ is always like that of Example 2b2.

**2b5. Exercise.** Let $P_{S_1}$ be nonatomic.

(a) Find an example of a nonatomic $P_{A_2}$ giving a non-convex curve on the plane (expected loss, winning probability).

Hint: approximate the discrete distribution of 2b1 by a continuous distribution.

(b) Assuming smoothness,[4] show that the curve is convex if and only if

$$f'_{A_2}(a) \le 2\frac{f^2_{A_2}(a)}{F_{A_2}(a)} \quad \text{for all } a\,;$$

here $f_{A_2}(a) = F'_{A_2}(a)$ is the density.

[4]That is, existence and continuity of all needed derivatives.

Hint: $\frac{d}{dp}(ap) = p\frac{da}{dp} + a = \frac{p}{dp/da} + a$ must increase.
(c) Think, what happens to the best response, if the curve is smooth but not convex.
Hint: $A_1$ becomes a discontinuous function of $S_1$.

Do not think that a best response exists only for nonatomic $P_{A_2}$.

**2b6. Exercise.** Let $P_{S_1}$ be nonatomic. Find an example of $P_{A_2}$ having an atom but still admitting a best response.

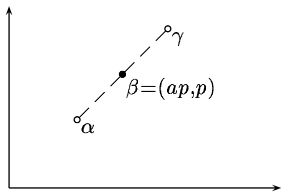
Hint. Make the curve non-convex (as in 2b1, 2b3) and insert a small atom to the region of concavity.

Do not think that the best response cannot create atoms (of $A_1$). For example, consider the best response $A_1 = \frac{1}{2}\min(S_1, 1)$ to $P_{A_2} = \mathrm{U}(0, 1/2)$ and let, say, $S_1 \sim \mathrm{U}(0, 2)$; then $\mathbb{P}\left( A_1 = 1/2 \right) = \mathbb{P}\left( S_1 > 1 \right) = 1/2$.

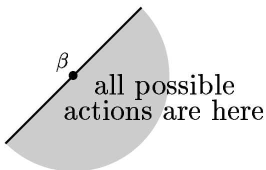
The first player should never place his bid into an atom of $A_2$; he should prefer a larger bid. The following result formalizes the argument.

**2b7. Lemma.** Let $a$ be an optimal action (against $P_{A_2}$) for an arbitrary $s_1$. Then $\mathbb{P}\left( A_2 = a \right) = 0$ or $\mathbb{P}\left( A_2 < a \right) = 0$.

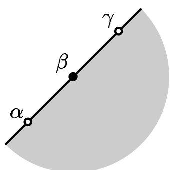
*Sketch of the proof.* Assume the contrary: $\mathbb{P}\left( A_2 < a \right) > 0$, $\mathbb{P}\left( A_2 = a \right) > 0$. Consider the corresponding point $(ap, p)$, where $p = \mathbb{P}\left( A_2 < a \right) + \frac{1}{2}\mathbb{P}\left( A_2 = a \right)$, on the plane (expected loss, winning probability). Being an atom, it is of the form



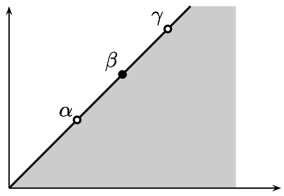

Being an optimal action, it lies on the boundary of a closed half-plane that contains all possible actions.




Points $\alpha$ and $\gamma$ belong to the closed half-plane (they do not belong to the set of possible actions, however, they belong to its closure). Therefore $\alpha$ and $\gamma$ lie on the boundary of the half-plane.



Therefore, the boundary goes through the origin.




However, it contradicts to existence of $a_2 < a$ such that $F(a_2) > 0$. □

**2b8. Exercise.** Let $a$ be an atom of $P_{A_2}$ such that $\mathbb{P}\big(A_2 < a\big) > 0$.

(a) Show that there exists $\varepsilon > 0$ such that actions of $(a - \varepsilon, a)$ cannot be optimal (irrespective of $s_1$).

Hint. The point $\alpha$ is an interior point of the convex hull of all possible actions.

(b) What about $(a, a + \varepsilon)$?

**2b9. Exercise.** Let $a$ be an atom of $P_{A_2}$ such that $\mathbb{P}\big(A_2 < a\big) = 0$. Show that $a$ is an optimal action for $s_1 = a$, and no other $s_1$.

An interval $(x, y)$ is called a *gap* of (the distribution of) a random variable $X$, if $\mathbb{P}\big(x < X < y\big) = 0$, however, $\mathbb{P}\big(x - \varepsilon < X < y\big) > 0$ and $\mathbb{P}\big(x < X < y + \varepsilon\big) > 0$ for every $\varepsilon > 0$. In terms of the (cumulative) distribution function, a gap is a horizontal interval on the graph. A gap of $A_2$, in terms of the plane (expected loss, winning probability), is also a horizontal interval (on the set of possible actions).

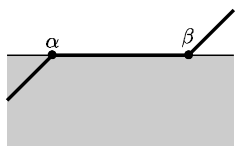
The first player should never place his bid into a gap of $A_2$; he should prefer a smaller bid. Moreover, optimal bids should not be too close to the right end of a gap, according to the following result.

**2b10. Lemma.** Let $(x, y)$ be a gap of $A_2$.

(a) If $y$ is an atom of $A_2$ then an optimal action never belongs to $(x, y]$.

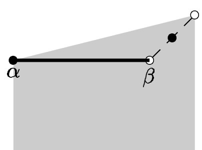
(b) If $y$ is not an atom of $A_2$ then there exists $\varepsilon > 0$ such that an optimal action never belongs to $(x, y + \varepsilon)$.

*Sketch of the proof.* Consider the corresponding horizontal interval on the plane (expected loss, winning probability). The closed half-plane below the interval cannot contain all possible actions.



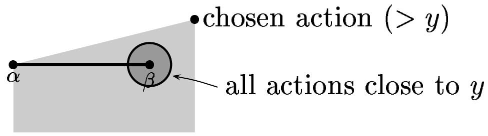

Therefore a point on the interval $(\alpha, \beta)$ cannot be an optimal action. That is, an optimal action never belongs to $(x, y)$.

Case (a): $y$ is an atom.



Every half-plane containing all possible actions contains the atom as an *interior* point. Therefore the atom cannot be an optimal action.

Case (b): $y$ is not an atom. Then actions close to $y$ are represented on our plane by points close to $\beta$, and lie above the horizontal line.




Some neighborhood of $\beta$ is contained in the interior of every half-plane containing all possible actions. Therefore actions close enough to $y$ cannot be optimal. □

**2b11. Exercise.** (a) What about actions of $(x - \varepsilon, x)$?

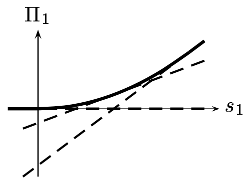
(b) In 2b10(b), can we omit the assumption that $y$ is not an atom?[5]

## 2c Best response is weakly monotone

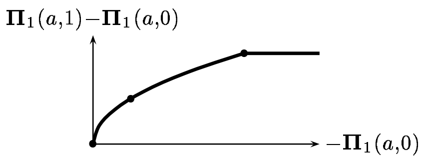
Optimality of an action $a_1$ for a signal $s_1$ means that

$$\mathbf{\Pi}_1(a_1, s_1) = \mathbf{\Pi}_1^{\max}(s_1) = \sup_a \mathbf{\Pi}_1(a, s_1) = \sup_a \big( s_1 \cdot (\mathbf{\Pi}_1(a,1) - \mathbf{\Pi}_1(a,0)) + \mathbf{\Pi}_1(a,0) \big) \, ;$$

here $P_{A_2}$ is suppressed in the notation. Arguments of this subsection are quite general; linearity of $\mathbf{\Pi}_1(a_1, s_1)$ in $s_1$ is all we need. No matter what is the set of all possible actions $a$, and what are the two functions $\mathbf{\Pi}_1(a,0), \mathbf{\Pi}_1(a,1)$ of $a$. In terms of the plane $(s_1, \Pi_1)$ we have just a set of linear functions, and their supremum.



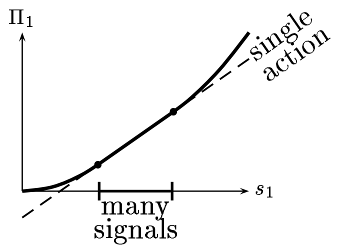

The plane (expected loss, winning probability) becomes now the plane $(-\mathbf{\Pi}_1(a,0), \mathbf{\Pi}_1(a,1) - \mathbf{\Pi}_1(a,0))$, irrespective of any interpretation of these two quantities. On that plane we have a set of points.

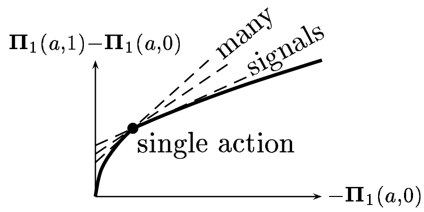



Straight lines on the former plane $(s_1, \Pi_1)$ are represented by points on the latter plane. Both represent (possible) actions. Different actions may sometimes lead to the same lines (points); such actions are equivalent, and may be thought of as a single action.

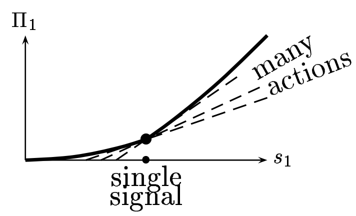
[5] I do not ask, whether the *proof* remains true, or not. I ask, whether the *statement* remains true, or not.

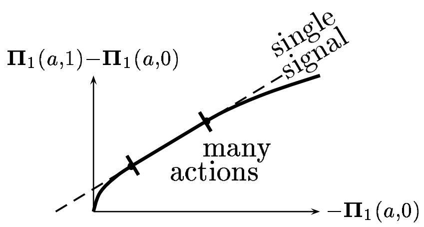
A single action can be optimal for many signals,



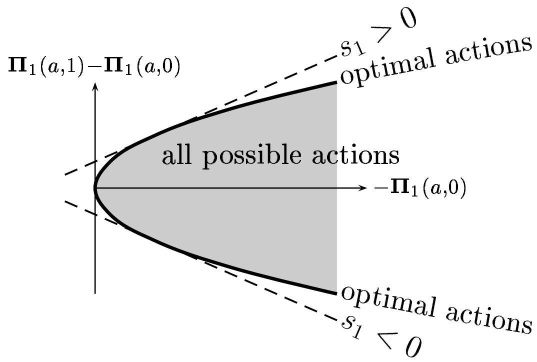




and many actions can be optimal for a single signal,




In principle, possible actions can be a two-dimensional set (region). However, we may guess that optimal actions are a one-dimensional set (curve).




Here, for full generality, both positive and negative signals are stipulated. We guess that optimal actions are linearly ordered according to their $\mathbf{\Pi}_1(a,1) - \mathbf{\Pi}_1(a,0)$. (Ordering according to $-\mathbf{\Pi}_1(a,0)$ could also work if signals are always positive.)

**2c1. Exercise.** Let two actions $a', a''$ have the same $\mathbf{\Pi}_1(\cdot,1) - \mathbf{\Pi}_1(\cdot,0)$.[6] If both actions are optimal (for some signals $s', s''$ respectively, not just the same signal) then the two actions are equivalent.[7]

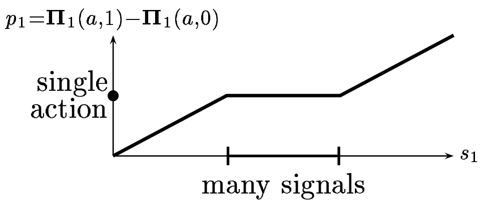
Prove it.

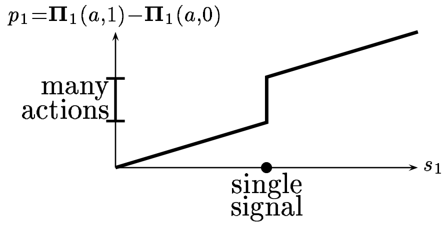
Thus, optimal actions are linearly ordered according to $\mathbf{\Pi}_1(\cdot,1) - \mathbf{\Pi}_1(\cdot,0)$. Consider the relation between a signal $s_1$ and the value $p_1 = \mathbf{\Pi}_1(a_1,1) - \mathbf{\Pi}_1(a_1,0)$ of an optimal action $a_1$

[6] That is, $\mathbf{\Pi}_1(a',1) - \mathbf{\Pi}_1(a',0) = \mathbf{\Pi}_1(a'',1) - \mathbf{\Pi}_1(a'',0)$.

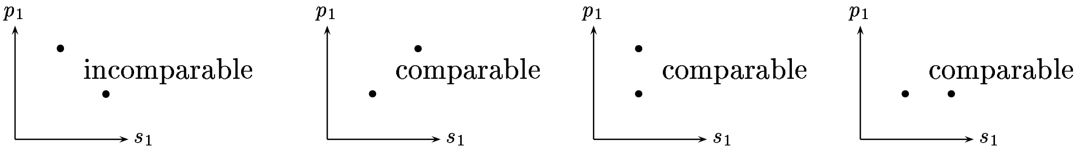
[7] Which means here $\mathbf{\Pi}_1(a',0) = \mathbf{\Pi}_1(a'',0)$.

(for $s_1$). The relation need not be functional (single-valued) in either direction.



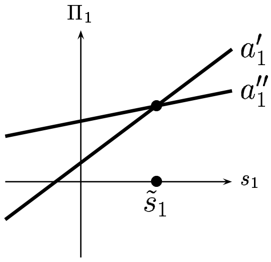




The next result shows that the relation is *weakly monotone,* namely, *weakly increasing,* which means that the set on the $(s_1, p_1)$ plane does not contain two points $(s_1', p_1')$, $(s_1'', p_1'')$ such that $s_1' < s_1''$ but $p_1' > p_1''$. Such points are called incomparable.




**2c2. Lemma.** Let $a_1'$ be an optimal action for $s_1'$, and $a_1''$ — for $s_1''$. If $s_1' < s_1''$ then $p_1' \le p_1''$, where $p_1' = \mathbf{\Pi}_1(a_1', 1) - \mathbf{\Pi}_1(a_1', 0)$, $p_1'' = \mathbf{\Pi}_1(a_1'', 1) - \mathbf{\Pi}_1(a_1'', 0)$.

*Proof.* Assume the contrary: $p_1' > p_1''$. Consider the corresponding two straight lines on the plane $(s_1, \Pi_1)$.[8] Their slopes are $p_1', p_1''$ respectively. We have $p_1' > p_1''$, therefore the two lines intersect at a single point $\tilde{s}_1$, the first line being below the second for $s_1 \in (-\infty, \tilde{s}_1)$ and above it for $s_1 \in (\tilde{s}_1, +\infty)$.[9]




It follows that $s_1' \in [\tilde{s}_1, +\infty)$; otherwise $a_1'$ could not be optimal for $s_1'$. Similarly, $s_1'' \in (-\infty, \tilde{s}_1]$. However, it contradicts to the inequality $s_1' < s_1''$. □

We turn from optimal actions to best response strategies. A strategy is a joint distribution of $S_1$ and $A_1$. It determines a joint distribution of $S_1$ and $\mathbf{\Pi}_1(A_1, 1) - \mathbf{\Pi}_1(A_1, 0)$. Combining Lemmas 2a18 and 2c2 we see that the latter joint distribution is concentrated on a weakly increasing set (that is, a set that does not contain two incomparable points).

A probability distribution on $\mathbb{R}^2$ will be called weakly increasing, if it is concentrated on a weakly increasing subset of $\mathbb{R}^2$. It is easy to see that the closure of a weakly increasing set is also a weakly increasing set (think, why). Thus, a weakly increasing distribution is concentrated on a weakly increasing closed set. Recall that every distribution has its *support,* the least closed set of probability 1. We see that a distribution is weakly increasing if and only if its support is a weakly increasing set.

The next result is thus obtained.

---

[8]Can you use the plane $(s_1, p_1)$ instead?

[9]That is called *single crossing* property.

**2c3. Lemma.** If the joint distribution of $S_1$ and $A_1$ is a best response,[10] then the joint distribution of $S_1$ and $\mathbf{\Pi}_1(A_1, 1) - \mathbf{\Pi}_1(A_1, 0)$ is weakly increasing.

Does it mean that $\mathbf{\Pi}_1(A_1, 1) - \mathbf{\Pi}_1(A_1, 0)$ is uniquely determined by $S_1$? In general, it does not, since a single $s_1$ can correspond to many (nonequivalent) optimal actions. However, such $s_1$ are a finite or countable set (maybe, empty).[11] If $S_1$ is nonatomic then a finite or countable set is negligible.

**2c4. Corollary.** If the joint distribution of $S_1$ and $A_1$ is a best response and $S_1$ is nonatomic, then $\mathbf{\Pi}_1(A_1, 1) - \mathbf{\Pi}_1(A_1, 0)$ is an increasing function of $S_1$.

We return from the general case to first price auctions. Here, $\mathbf{\Pi}_1(A_1, 1) - \mathbf{\Pi}_1(A_1, 0)$ is interpreted as the winning probability (conditioned by $A_1$). Thus, Lemma 2c3 means that the joint distribution of $S_1$ and the winning probability is weakly increasing.

If the support of $A_2$ is the whole $[0, \infty)$ then an action is uniquely determined by its winning probability, moreover, $A_1$ is an increasing function of $\mathbf{\Pi}_1(A_1, 1) - \mathbf{\Pi}_1(A_1, 0)$. Combined with Lemma 2c3 it means that a best response is a weakly increasing strategy.[12] If the support of $A_2$ does not contain 0, then the first player can fool around, when $S_1$ is small enough; recall page 17. All never-winning actions (that is, of winning probability 0) are equivalent. In contrast, if the support of $A_2$ is bounded, it does not mean that all always-winning actions (that is, of winning probability 1) are equivalent. They differ in the expected loss. Only the least among them can be optimal. The same for every gap (if any) of $A_2$.

**2c5. Corollary.** A best response is a weakly increasing strategy except, maybe, never-winning actions.

A strategy $P_{S_1,A_1}$ is called a *pure strategy,* if it is concentrated on the graph of a function,[13] $\{(s_1, a_1) : a_1 = \varphi(s_1)\}$. Otherwise, the strategy is called *mixed.* A pure strategy is called *increasing,* if it is concentrated on the graph of an increasing function.[14] A strategy is an increasing pure strategy if and only if it is both a pure strategy and a weakly increasing strategy (think, why).

If $S_1$ has atoms, it may happen that the player has many equally profitable actions. Otherwise, the winning probability is an increasing function of $S_1$, and the optimal action is an increasing function of the winning probability, except for never-winning actions.

**2c6. Corollary.** If $P_{S_1}$ is nonatomic then a best response is an increasing pure strategy, except maybe for never-winning actions.

[10] To some strategy of the competitor or, more generally, to some given set on the plane $(-\mathbf{\Pi}_1(a, 0), \mathbf{\Pi}_1(a, 1) - \mathbf{\Pi}_1(a, 0))$.

[11] Which follows from the fact that an increasing function has at most countably many jumps.

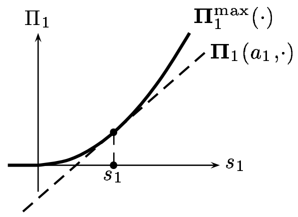
[12] A strategy is a probability distribution on $\mathbb{R}^2$, thus, the definition of weak increase is applicable.

[13] An equivalent definition: if $A_1$ is a function of $S_1$. Another equivalent definition: if the conditional distribution of $A_1$ given $S_1$ is degenerate (to a single atom) almost surely. Still another equivalent definition: if the strategy cannot be represented as the mixture of two different strategies. However, it does not mean that the *support* is the graph of a function (think, why).

[14] Not necessarily strictly increasing.

## 2d Integral of winning probability

Optimality of an action $a_1$ for a signal $s_1$, expressed in terms of the plane $(s_1, \Pi_1)$, looks as follows:




We guess that

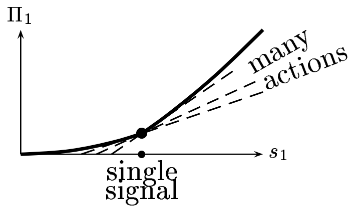
$$\frac{d}{ds}\bigg|_{s=s_1} \mathbf{\Pi}_1^{\max}(s) = \frac{d}{ds}\bigg|_{s=s_1} \mathbf{\Pi}_1(a_1, s) = \mathbf{\Pi}_1(a_1, 1) - \mathbf{\Pi}_1(a_1, 0)\,,$$

that is,

$$\frac{d(\text{expected profit})}{d(\text{signal})} = (\text{winning probability})\,.$$

Can we prove the guess?

First of all, the function $\mathbf{\Pi}_1^{\max}(\cdot)$ need not be differentiable.




Also, if a single signal corresponds to many actions, then its winning probability is ill-defined. However, such points are exceptional; they are jumps of the increasing function $\frac{d}{ds}\mathbf{\Pi}_1^{\max}(s)$, thus, they are at most a countable set. They do not invalidate the formula[15]

$$\mathbf{\Pi}_1^{\max}(s'') - \mathbf{\Pi}_1^{\max}(s') = \int_{s'}^{s''} \frac{d}{ds}\mathbf{\Pi}_1^{\max}(s)\,ds\,.$$

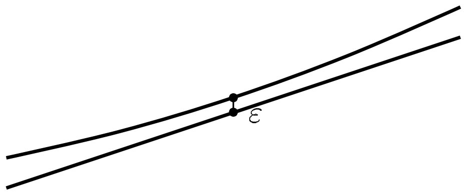
Thus, we may restrict ourselves to points $s_1$ such that $\mathbf{\Pi}_1^{\max}(\cdot)$ is differentiable at $s_1$.[16] Still, it may happen that the supremum is not reached, that is, $\mathbf{\Pi}_1(a_1, s_1) < \mathbf{\Pi}_1^{\max}(s_1)$ for all $a_1$. True, existence of a best response implies that an optimal action exists for almost all signals; however, exceptional signals must be a set of probability 0 according to $P_{S_1}$ rather than a set of Lebesgue measure 0.[17] Especially, the winning probability can be ill-defined in a gap of $S_1$; how to integrate over the gap?

[15] A bounded monotone function on a bounded interval is Riemann integrable, and values at jumps do not influence the integral.

[16] Which means (due to convexity) that the left derivative is equal to the right derivative.

[17] A set of Lebesgue measure 0 may be neglected when considering integrals $\int(\dots)\,ds$.

Whether the supremum is reached or not, anyway, for every $s_1$ and every $\varepsilon > 0$ there exists $a_1$ such that $\mathbf{\Pi}_1(a_1, s_1) > \mathbf{\Pi}_1^{\max}(s_1) - \varepsilon$.




For small $\varepsilon$, slopes of the curve and the line at $s_1$ must be close. That is,

$$\frac{d}{ds}\bigg|_{s=s_1} \mathbf{\Pi}_1^{\max}(s) = \lim_{\mathbf{\Pi}_1(a_1,s_1)\to\mathbf{\Pi}_1^{\max}(s_1)} \big(\mathbf{\Pi}_1(a_1,1) - \mathbf{\Pi}_1(a_1,0)\big)\,,$$

which follows from general properties of convex functions. All nearly optimal actions have nearly the same winning probability. So, we may define the winning probability as

$$p_1(s_1) = \lim_{\mathbf{\Pi}_1(a_1,s_1)\to\mathbf{\Pi}_1^{\max}(s_1)} \big(\mathbf{\Pi}_1(a_1,1) - \mathbf{\Pi}_1(a_1,0)\big)\,;$$

the limit exists for all $s_1$ except for at most a countable set, and is equal to $\frac{d}{ds_1}\mathbf{\Pi}_1^{\max}(s_1)$. And, of course, $p_1(s_1)$ is equal to $\mathbf{\Pi}_1(a_1,1) - \mathbf{\Pi}_1(a_1,0)$ whenever there is an optimal action $a_1$. So,

$$\mathbf{\Pi}_1^{\max}(s'') - \mathbf{\Pi}_1^{\max}(s') = \int_{s'}^{s''} p_1(s_1)\,ds_1$$

for arbitrary $s', s''$. The formula is quite general; linearity of $\mathbf{\Pi}_1(a_1, s_1)$ is all we need; and $p_1(s_1)$ is not necessarily interpreted as winning probability.