3 Brownian rotations via stochastic integrals

3a Dimension one: the circle

3a1 Definition. A *Brownian motion* in a topological group[1] G is a sample continuous[2] process $(X_t)_{t\in[0,\infty)}$ in G with stationary independent increments[3] $X_{s,t} = X^{-1}(s)X(t)$.

3a2 Example. Let $(B_t)_t$ be the standard Brownian motion in $\mathbb{R}$, then $(\sigma B_t + vt)_t$ is a Brownian motion in (the additive group of) $\mathbb{R}$, for any $\sigma \in [0,\infty)$, $v \in \mathbb{R}$.[4]

3a3 Definition. A *morphism* of a Brownian motion $(X_t)_t$ in G_1 into a Brownian motion $(Y_t)_t$ in G_2 is a Brownian motion $(Z_t)_t$ in $G_1 \times G_2$, $Z_t = (X'_t, Y'_t)$, such that[5]
$(X'_t)_t$ is distributed like $(X_t)_t$,
$(Y'_t)_t$ is distributed like $(Y_t)_t$,
the σ-field generated by $(Y'_t)_t$ is included into the σ-field generated by $(X'_t)_t$.

3a4 Example. Let $(B_t)_t$ be the standard Brownian motion in $\mathbb{R}$, and $Y_t = e^{iB_t}$. Then $(Y_t)_t$ is a Brownian motion in $\mathbb{T}$, and $(B_t, Y_t)_t$ is a morphism of $(B_t)_t$ into $(Y_t)_t$. Another morphism of $(B_t)_t$ into $(Y_t)_t$ is $(B_t, e^{-iB_t})_t$.

We may guess that $(B_t, e^{i\sigma B_t + ivt})_t$ is the general form of a morphism of $(B_t)_t$ into a Brownian motion in $\mathbb{T}$. How could we prove the guess?

Let $(B_t, Y_t)_t$ be a morphism of $(B_t)_t$ into a Brownian motion in $\mathbb{T}$. Then

$$Y_t = \sum_{n=0}^{\infty} \int\limits_{0<t_1<\dots<t_n<\infty}\!\!\!\cdots\!\!\int f_{n,t}(t_1,\dots,t_n)\, dB_{t_1}\dots dB_{t_n}$$

for some $f_n \in L_2(\Delta_n, \mathbb{C})$. Can we restrict ourselves to $0 < t_1 < \dots < t_n < t$? In other words, is Y_t measurable w.r.t. $\mathcal{F}^B_{0,t}$? Clearly, Y_t is independent of $\mathcal{F}^B_{t,\infty}$; so what?

3a5 Exercise. Let X, Y be two independent random variables, and $Z = f(X,Y)$ be independent of Y. Does it follow that $Z = g(X)$?

Hint. Find an elementary counterexample (say, $X = \pm 1$, $Y = \pm 1$).

3a6 Exercise. (a) Let X, Y be two independent random variables, and the pair $\big(X, f(X,Y)\big)$ be independent of Y. Then $f(X,Y) = g(X)$ for some g.

(b) Let sub-σ-fields $\mathcal{E}, \mathcal{F}, \mathcal{G}$ (on a given probability space) satisfy[6]

$$\mathcal{G} \subset \mathcal{E} \vee \mathcal{F}\,,$$
$$\mathcal{E} \vee \mathcal{G} \text{ and } \mathcal{F} \text{ are independent.}$$

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[1] A topological group G is both a group and a topological space such that the maps $G \ni x \mapsto x^{-1} \in G$ and $G \times G \ni (x,y) \mapsto xy \in G$ are continuous. (We need metrizable G only.)

[2] Recall 1e.

[3] Thus, $X_{r,s}X_{s,t} = X_{r,t}$. In a commutative group the operation is often written additively, in which case $X_{s,t} = X_t - X_s$.

[4] In fact, it is the general form of a Brownian motion in (the additive group of) $\mathbb{R}$.

[5] $G_1 \times G_2 = \{(x,y) : x \in G_1, y \in G_2\}$ is a topological group with the product topology and the operation $(x_1,x_2)(y_1,y_2) = (x_1y_1, x_2y_2)$.

[6] $\mathcal{E} \vee \mathcal{F}$ denotes the sub-σ-field generated by $\mathcal{E}$ and $\mathcal{F}$.

Then $\mathcal{G} \subset \mathcal{E}$.

Prove it.

Hint (to (a)): denote $Z = f(X,Y)$; check that $\mathbb{P}\big(Z \in C \,\big|\, X \in A, Y \in B \big) = \mathbb{P}\big(Z \in C \,\big|\, X \in A \big)$; conclude that $\mathbb{E}\big(Z \,\big|\, X,Y \big) = \mathbb{E}\big(Z \,\big|\, X \big)$. And what about integrability of Z?

Applying 3a6(b) to $\mathcal{E} = \mathcal{F}^B_{0,t}$, $\mathcal{F} = \mathcal{F}^B_{t,\infty}$, $\mathcal{G} = \mathcal{F}^Y_{0,t}$ we get $\mathcal{F}^Y_{0,t} \subset \mathcal{F}^B_{0,t}$. Therefore

$$Y_t = \sum_{n=0}^\infty \int\cdots\int_{0<t_1<\dots<t_n<t} f_{n,t}(t_1,\dots,t_n)\, dB_{t_1}\dots dB_{t_n}$$

for some $f_{n,t} \in L_2(\Delta_n \cap (0,t)^n, \mathbb{C})$. Moreover,

$$Y_s^{-1} Y_t = \sum_{n=0}^\infty \int\cdots\int_{s<t_1<\dots<t_n<t} f_{n,t-s}(t_1-s,\dots,t_n-s)\, dB_{t_1}\dots dB_{t_n}\,,$$

since the joint distribution of the increments of Y and B on (s,t) is the same as on $(0,t-s)$. The equality $Y_t = Y_s(Y_s^{-1}Y_t)$ becomes

$$\text{(3a7)}\quad \sum_{n=0}^\infty \int\cdots\int_{0<t_1<\dots<t_n<t} f_{n,t}(t_1,\dots,t_n)\, dB_{t_1}\dots dB_{t_n} = \\ = \Big(\sum_{n=0}^\infty \int\cdots\int_{0<t_1<\dots<t_n<s} f_{n,s}(t_1,\dots,t_n)\, dB_{t_1}\dots dB_{t_n} \Big) \cdot \\ \cdot \Big(\sum_{n=0}^\infty \int\cdots\int_{s<t_1<\dots<t_n<t} f_{n,t-s}(t_1-s,\dots,t_n-s)\, dB_{t_1}\dots dB_{t_n} \Big)$$

whenever $0 < s < t < \infty$. Can we solve this equation, thus finding $f_{n,t}$?

First of all we want to eliminate dB and rewrite the equation in terms of $f_{n,t}$ only.

3a8 Exercise. Let $0 < s < \infty$, $0 \le k \le n$, $f \in L_2\big(\Delta_k \cap (0,s)^k\big)$, $g \in L_2\big(\Delta_{n-k} \cap (s,\infty)^{n-k}\big)$, then

$$\int\cdots\int_{0<t_1<\dots<t_n<\infty} f(t_1,\dots,t_k) g(t_{k+1},\dots,t_n) \mathbf{1}_{(0,s)}(t_k) \mathbf{1}_{(s,\infty)}(t_{k+1})\, dB_{t_1}\dots dB_{t_n} = \\ = \Big(\int\cdots\int_{0<t_1<\dots<t_k<s} f(t_1,\dots,t_k)\, dB_{t_1}\dots dB_{t_k} \Big) \cdot \\ \cdot \Big(\int\cdots\int_{s<t_1<\dots<t_{n-k}<\infty} g(t_1,\dots,t_{n-k})\, dB_{t_1}\dots dB_{t_{n-k}} \Big)\,.$$

Prove it.

Hint. Both sides are bilinear in f,g; it is enough to prove the equality for $f(t_1,\dots,t_k) = \mathbf{1}_{(a_1,b_1)}(t_1)\dots\mathbf{1}_{(a_k,b_k)}(t_k)$, $g(t_1,\dots,t_{n-k}) = \mathbf{1}_{(a_{k+1},b_{k+1})}(t_1)\dots\mathbf{1}_{(a_n,b_n)}(t_{n-k})$ whenever $0 < a_1 < b_1 < \dots < a_n < b_n < \infty$. Now use 2e6.

3a9 Exercise. The following equality is equivalent to (3a7):

$$f_{n,t}(t_1,\dots,t_n) = \sum_{k=0}^{n} f_{k,s}(t_1,\dots,t_k) f_{n-k,t-s}(t_{k+1}-s,\dots,t_n-s)\mathbf{1}_{(0,s)}(t_k)\mathbf{1}_{(s,\infty)}(t_{k+1})$$

for almost all $(t_1,\dots,t_n)\in\Delta_n$.

Prove it.

Hint: use 3a8.

All that may be written in a more elegant way, by interpreting $\Delta = \Delta_0 \cup \Delta_1 \cup \Delta_2 \cup \dots$ as the set of all finite subsets of $(0,\infty)$, and the sequence of functions $f_{0,s}, f_{1,s},\dots$ as a single function

$$f_s : \Delta_{0,s} \to \mathbb{C},$$

where $\Delta_{0,s} \subset \Delta$ is the set of all finite subsets of $(0,s)$. Note that Δ is equipped with the Lebesgue measure 'mes' (n-dimensional on Δ_n), $\operatorname{mes}\Delta_{0,s} = e^s$, and

$$\|f_s\|_{L_2(\Delta_{0,s})} = \|Y_s\|_{L_2(\Omega)}\,.$$

We get

$$f_t(M) = f_s\big(M\cap(0,s)\big) f_{t-s}\big(M\cap(s,\infty)-s\big) \tag{3a10}$$

for almost all $M\in\Delta$; of course, $M\cap(s,\infty)-s$ means $\{a-s : a\in M\cap(s,\infty)\}$. No need to bother about the case $M\ni s$, since such sets M are a negligible subset of Δ. On the other hand, the point $\emptyset\in\Delta_0\subset\Delta$ is an atom (non-negligible point) of Δ.

The atom is a good starting point for solving (3a10). We have

$$f_t(\emptyset) = f_s(\emptyset) f_{t-s}(\emptyset)\,, \tag{3a11}$$

which means that

$$f_t(\emptyset) = e^{at} \tag{3a12}$$

for some $a\in\mathbb{C}$, unless the multiplicative function $t\mapsto f_t(\emptyset)$ is too bad (discontinuous everywhere, and in fact, non-measurable). However, $f_t(\emptyset) = \mathbb{E}\,Y_t$, which is continuous in t (by the bounded convergence theorem).

Having (3a12) we turn to one-point sets $M=\{u\}$ in (3a10):

$$f_t(\{u\}) = f_s(\{u\}) \underbrace{f_{t-s}(\emptyset)}_{=\exp a(t-s)} \quad \text{for almost all } u\in(0,s)\,;$$

$$f_t(\{u\}) = \underbrace{f_s(\emptyset)}_{=\exp as} f_{t-s}(\{u-s\}) \quad \text{for almost all } u\in(s,t)\,.$$

Roughly speaking, the expression $e^{-at}f_t(\{u\})$ depends only on u, and at the same time, it depends only on $t-u$. It must be a constant! However, what to do with 'almost all'?

For any n we consider the average (like conditional expectation) g_n of the function $u\mapsto f_1(\{u\})$ w.r.t. the partition $(0,1) = (0,2^{-n})\cup(2^{-n},2\cdot2^{-n})\cup\dots\cup(1-2^{-n},1)$,

$$g_n(u) = 2^n\int_{k2^{-n}}^{(k+1)2^{-n}} f_1(\{u\})\,du \quad \text{for } u\in\big(k2^{-n},(k+1)2^{-n}\big)\,.$$

3a13 Exercise. $g_n \to f_1(\{\cdot\})$ (for $n \to \infty$) in $L_2(0,1)$.[7]

Prove it.

Hint. The increasing sequence of partitions leads to an increasing sequence of subspaces in $L_2(0,1)$. Their union is dense in $L_2(0,1)$ (indeed, it approaches any continuous function). And g_n is the orthogonal projection of $f_1(\{\cdot\})$ onto the n-th subspace.

3a14 Exercise. $g_n(u) = c_n \exp(a(1-2^{-n}))$ for all u; here $c_n = 2^n \int_0^{2^{-n}} f_1(\{u\})\,du$.

Prove it.

Hint: use (3a10) and (3a12).

3a15 Exercise. $c_n \to \sigma$ (for $n \to \infty$) for some $\sigma \in \mathbb{C}$.

Prove it.

Hint: $c_n = c_{n+1} \exp(a2^{-(n+1)})$.

3a16 Exercise. $f_1(\{u\}) = \sigma e^a$ for almost all $u \in (0,1)$; here σ is as in 3a15.

Prove it.

Hint: use 3a13, 3a14, 3a15.

Similarly,

$$f_t\big(\{u\}\big) = \sigma e^{at} \quad \text{for almost all } u \in (0,1). \tag{3a17}$$

The whole $\Delta_{0,1}$ may be partitioned (for a given n) according to the numbers[8] $|M \cap (0,2^{-n})|, \dots, |M \cap (1-2^{-n}, 2^{-n})|$. Let g_n be the average of f_1 w.r.t. that partition.

3a18 Exercise. $g_n \to f_1$ in $L_2(\Delta_{0,1})$.

Prove it.

Hint: similar to 3a13.

3a19 Exercise. $g_n(M) = c_n^{|M|} \exp\big(a(1-|M|2^{-n})\big)$ for all n large enough (depending on $M \in \Delta_{0,1}$); here c_n is as in 3a14.

Prove it.

Hint: $|M \cap (k2^{-n}, (k+1)2^{-n})| \le 1$ for all k, provided that n is large enough.

3a20 Exercise. $f_1(M) = \sigma^{|M|} e^a$ for almost all $M \in \Delta_{0,1}$; here σ is as in 3a15.

Prove it.

Similarly,

$$f_t(M) = e^{at}\sigma^{|M|} \quad \text{for almost all } M \in \Delta_{0,t}. \tag{3a21}$$

Finally, $f_{n,t}(t_1,\dots,t_n) = e^{at}\sigma^n$;

$$\begin{aligned} Y_t &= e^{at} \sum_{n=0}^{\infty} \sigma^n \int\!\!\cdots\!\!\int_{0<t_1<\dots<t_n<t} dB_{t_1}\dots dB_{t_n} = e^{at} \sum_{n=0}^{\infty} \sigma^n \frac{t^{n/2}}{n!} H_n\Big(\frac{1}{\sqrt{t}} B_{0,t}\Big) = \\ &= e^{at} \exp\Big(\frac{1}{\sqrt{t}} B_{0,t}\sigma\sqrt{t} - \frac{1}{2}\sigma^2 t\Big) = \exp\big(\sigma B_{0,t} + (a - \tfrac{1}{2}\sigma^2)t\big)\,. \end{aligned} \tag{3a22}$$

[7] In fact, they converge almost everywhere, but we do not need it.

[8] $|M|$ stands for the number of points of M.

However, $|Y_t| = 1$ a.s., thus σ and $a - \frac{1}{2}\sigma^2$ must belong to $i\mathbb{R}$,[9] and our guess is confirmed.

3a23 Theorem. Every morphism of the standard Brownian motion $(B_t)_t$ in $\mathbb{R}$ into a Brownian motion in $\mathbb{T}$ is of the form $(B_t, Y_t)_t$ where

$$Y_t = \exp(i\sigma B_t + ivt)$$

and $\sigma, v \in \mathbb{R}$.

3a24 Exercise. Every morphism of $(B_t)_t$ into a Brownian motion in $(\mathbb{R}, +)$ (that is, the additive group of $\mathbb{R}$) is of the form $(B_t, Y_t)_t$ where

$$Y_t = \sigma B_t + vt$$

and $\sigma, v \in \mathbb{R}$.

(a) Deduce it from Theorem 3a23.

(b) Prove it from scratch, assuming however that $Y_t \in L_2$.

Hint. (a) Apply 3a23 to $(B_t, e^{iY_t})_t$ and note that a continuous function $[0,\infty) \to 2\pi\mathbb{Z}$ is necessarily constant. Or alternatively, consider $e^{i\lambda_1 Y_t}$ and $e^{i\lambda_2 Y_t}$ with λ_1/λ_2 irrational.
(b) The additive counterpart of (3a10) holds.

3a25 Exercise. Every morphism of $(B_t)_t$ into a Brownian motion in $(\mathbb{C} \setminus \{0\}, \cdot)$ (that is, the multiplicative group of non-zero complex numbers) is of the form $(B_t, Y_t)_t$ where

$$Y_t = \exp(\sigma B_t + vt)$$

and $\sigma, v \in \mathbb{C}$.

Prove it.

Hint. First, apply 3a24 to $(B_t, \ln|Y_t|)_t$ and show that $Y_t \in L_2$. Second, use (3a22). Or alternatively, apply 3a23 to $(B_t, Y_t/|Y_t|)_t$.

3a26 Exercise. Generalize 3a25 to $(\mathbb{C}, \cdot)$, the multiplicative semigroup of $\mathbb{C}$. To this end, generalize also 3a3.

Hint. $Y_t \neq 0$. Indeed, $\mathbb{P}\big(Y_t = 0\big) = e^{-ct}$, but only $c = 0$ conforms to 1e1(b).

3b Non-commuting matrices

We turn to the group $\mathrm{SO}(n)$ of rotations of $\mathbb{R}^n$. That is, $\mathrm{SO}(n)$ is the group of all $n \times n$ matrices U such that[10]

$$|\psi U| = |\psi| \quad \text{for all } \psi \in \mathbb{R}^n\,, \quad \text{and} \quad \det U = 1\,.$$

However, $|\psi U|^2 = (\psi U)(\psi U)^* = \psi U U^* \psi$, where U^* is the transpose. Therefore $U \in \mathrm{SO}(n)$ if and only if[11]

$$UU^* = 1\,, \quad \text{and} \quad \det U = 1\,.$$

[9] If $\sigma \in i\mathbb{R}$ but $a - \frac{1}{2}\sigma^2 \notin i\mathbb{R}$ then Y_t leaves the circle *tangentially.* A wonder! It never happens in the smooth analysis. In fact, $d|Y_t|$ contains not only dY_t but also $(dY_t)^2$.

[10] I write ψU treating ψ as a row (rather than a column); thus, the product $U_1 U_2$ means that U_1 acts before U_2.

[11] Conditions $UU^* = 1$, $U^*U = 1$ and $U^* = U^{-1}$ are equivalent.

Note that $\mathrm{SO}(2) = \mathbb{T} = \big\{ \big(\begin{smallmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{smallmatrix} \big) : \alpha \in \mathbb{R} \big\}$ is commutative, but SO(3) is not. Each element of SO(3) is the rotation around some line by some angle. Note especially rotations around the axes:

$$\begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix} \in \mathrm{SO}(3)\,. \tag{3b1}$$

The group $\mathrm{SO}(n)$ lies in the algebra $\mathrm{M}_n(\mathbb{R})$ of all real $n \times n$ matrices.

The general form of a continuous one-parameter semigroup in $\mathrm{M}_n(\mathbb{R})$,

$$\big(\exp(ta)\big)_{t\in[0,\infty)}\,,$$

where $a \in \mathrm{M}_n(\mathbb{R})$, is a well-known fact of linear algebra. Of course, $\exp(a) = \sum_{n=0}^{\infty} \frac{1}{n!} a^n$. The operator

$$a = \lim_{t\to 0+} \frac{1}{t}\big(\exp(ta) - 1\big)$$

is called the *generator* of the semigroup. The following fact is also well-known, but I want to prove it, since the proof will help us later.

3b2 Lemma. The following two conditions for $a \in \mathrm{M}_n(\mathbb{R})$ are equivalent:

(a) $\exp(ta) \in \mathrm{SO}(n)$ for all $t \in [0,\infty)$;

(b) $a^* = -a$.

Proof. (a) $\Longrightarrow$ (b). We have $\big(\exp(ta)\big)\big(\exp(ta)\big)^* = 1$ and $\exp(ta) = 1 + ta + o(t)$ for $t \to 0$. Thus $\big(1 + ta + o(t)\big)\big(1 + ta^* + o(t)\big) = 1$; $ta + ta^* = o(t)$; $a + a^* = o(1)$; $a + a^* = 0$.

(b) $\Longrightarrow$ (a). First, it is sufficient to prove that $\exp(ta)$ is isometric; then $\det \exp(ta) = \pm 1$, and by continuity it must be equal to $+1$. Second, we introduce a function $\varphi : \mathrm{M}_n(\mathbb{R}) \to [0,\infty]$,

$$\varphi(U) = \sup_{|\psi|=1} \big| \ln |\psi U| \big|$$

and note that $\varphi(U_1 U_2) \le \varphi(U_1) + \varphi(U_2)$; therefore

$$\varphi\big(\exp(ta)\big) \le n\varphi\big(\exp(\tfrac{1}{n} ta)\big)\,;$$

it remains to prove that $\varphi\big(\exp(ta)\big) = o(t)$ for $t \to 0$, which is straightforward:

$$\begin{aligned} |\psi \exp(ta)|^2 &= |\psi\big(1 + ta + o(t)\big)|^2 = |\psi + t\psi a + o(t)|^2 = \\ &= |\psi|^2 + t\langle \psi a, \psi\rangle + t\langle \psi, \psi a\rangle + o(t) = |\psi|^2 + t\langle \psi(a + a^*), \psi\rangle + o(t) = |\psi|^2 + o(t)\,. \end{aligned}$$

□

Brownian motions in $\mathrm{SO}(n)$ are defined (recall 3a1), as well as morphisms of the standard Brownian motion $(B_t)_t$ in $\mathbb{R}$ into Brownian motions in $\mathrm{SO}(n)$ (recall 3a3). Similarly to 3a, if we have functions

$$f_t \in L_2\big(\Delta_{0,t}, \mathrm{M}_n(\mathbb{R})\big)$$

satisfying (3a10), then we get a morphism $(B_t, Y_t)_t$,

$$Y_t = \int_{\Delta_{0,t}} f_t(M) \Big(\prod_{s \in M} dB_s \Big) dM = \sum_{n=0}^{\infty} \int\!\!\cdots\!\!\int_{0<s_1<\dots<s_n<t} f_{n,t}(s_1,\dots,s_n)\, dB_{s_1} \dots dB_{s_n} \,, \tag{3b3}$$

where $f_{n,t}(s_1,\dots,s_n) = f_t\big(\{s_1,\dots,s_n\}\big)$. Of course, a stochastic integral of a vector-function (or matrix-function) is treated coordinate-wise. Generally we get (a morphism to) a Brownian motion in $(\mathrm{M}_n(\mathbb{R}), \cdot)$; under some additional conditions on (f_t) we should get a Brownian motion in $\mathrm{SO}(n)$.

Similarly to 3a we find the general form of (f_t) satisfying (3a10) and such that $f_t(\emptyset)$ is continuous in t. Namely,

$$f_t\big(\{s_1,\dots,s_n\}\big) = \exp(s_1 a)\sigma \exp((s_2 - s_1)a)\sigma \dots \exp((t - s_n)a) \,; \tag{3b4}$$

here $a, \sigma \in \mathrm{M}_n(\mathbb{R})$. Unlike 3a, we cannot join the exponentials into $\exp(ta)$ since they need not commute with σ.

3b5 Exercise. Prove (3b4).

Hint. Similar to 3a. Note that $\|e^a - 1\| \le e^{\|a\|} - 1$ (since $\|a^n\| \le \|a\|^n$).

We assume that $Y_t \in \mathrm{SO}(n)$ for all $t \in [0,\infty)$ and try to deduce (necessary) conditions on a, σ in the spirit of "(a)$\Longrightarrow$ (b)" in the proof of 3b2.

3b6 Exercise. Prove that

$$Y_t = \exp(ta) + \int_0^t \exp(sa)\sigma \exp((t-s)a)\, dB_s + O(t) \quad \text{for } t \to 0 \,,$$

that is, the norm of the remainder term in $L_2(\Omega)$ does not exceed $\mathrm{const} \cdot t$.

Hint. Recall the last formula of Sect. 2: $\|X\|^2 = \sum \|f_n\|^2$. Generalize it to vector-functions.

We have for small t

$$\int_0^t \exp(sa)\sigma \exp((t-s)a)\, dB_s = \int_0^t \big(1 + O(t)\big)\sigma\big(1 + O(t)\big)\, dB_s = \sigma \int_0^t dB_s + O(t\sqrt{t}) \,,$$

therefore

$$Y_t = 1 + \sigma B_{0,t} + O(t) \,;$$
$$1 = Y_t Y_t^* = \big(1 + \sigma B_{0,t} + o(\sqrt{t})\big)\big(1 + \sigma^* B_{0,t} + o(\sqrt{t})\big) = 1 + (\sigma + \sigma^*) B_{0,t} + o(\sqrt{t})$$

(note however that the last occurrence of $o(\sqrt{t})$ is interpreted in $L_1(\Omega)$ rather than $L_2(\Omega)$); so, $(\sigma + \sigma^*) B_{0,t} / \sqrt{t} = o(1)$, which means that

$$\sigma^* = -\sigma \,.$$

However, the necessary condition is definitely not sufficient, since it does not include a. We need the next term of the asymptotics:[12]

$$Y_t = \exp(ta) + \int_0^t \exp(sa)\sigma \exp((t-s)a)\,dB_s + \\ + \iint_{0<r<s<t} \exp(ra)\sigma\exp((s-r)a)\sigma\exp((t-s)a)\,dB_r dB_s + O(t^{3/2})\,;$$

$$\iint_{0<r<s<t} \exp(ra)\sigma\exp((s-r)a)\sigma\exp((t-s)a)\,dB_r dB_s = \sigma^2 \underbrace{\iint_{0<r<s<t} dB_r dB_s}_{=\frac12(B_{0,t}^2-t)} + O(t^2)\,;$$

$$Y_t = 1 + \sigma B_{0,t} + ta + \frac12\sigma^2(B_{0,t}^2 - t) + O(t^{3/2})\,;$$

$$Y_tY_t^* = 1 + \underbrace{(\sigma+\sigma^*)}_{=0} B_{0,t} + t(a+a^*) + \frac12\big(\sigma^2 + (\sigma^2)^*\big)(B_{0,t}^2 - t) + \sigma\sigma^* B_{0,t}^2 + O(t^{3/2})\,;$$

$$t(a+a^*) + \sigma^2(B_{0,t}^2 - t) - \sigma^2 B_{0,t}^2 = o(t)\,;$$

$$t(a+a^*-\sigma^2) = o(t)\,;$$

$$a + a^* = \sigma^2\,.$$

We see that the following two conditions are equivalent:

- $\sigma^* = -\sigma$ and $a + a^* = \sigma^2$;
- $Y_tY_t^* = 1 + o(t)$ in $L_1(\Omega)$.

3b7 Exercise. The following two conditions are equivalent:

(a) $Y_t \in \mathrm{SO}(n)$ for all t almost surely;

(b) $\sigma^* = -\sigma$ and $a + a^* = \sigma^2$.

Prove it.

Hint. Use the argument above. (b) $\Longrightarrow$ (a): similarly to 3b2; consider $\mathbb{E}\min(1, \varphi(Y_t))$.

In the case $n = 1$ we get the result of 3a again. Indeed, $\mathbb{C}$ may be treated as a subalgebra of $\mathrm{M}_2(\mathbb{R})$, $\mathbb{C} \ni z \mapsto \begin{pmatrix} \operatorname{Re} z & \operatorname{Im} z \\ -\operatorname{Im} z & \operatorname{Re} z \end{pmatrix} \in \mathrm{M}_2(\mathbb{R})$, and $\mathbb{T}$ becomes $\mathrm{SO}(1)$, $\mathbb{T} \in e^{i\alpha} \mapsto \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \in \mathrm{SO}(1)$. Treating σ and a as complex numbers, we rewrite 3b7(b) as $\overline{\sigma} = -\sigma$ and $a + \overline{a} = \sigma^2$, that is,

$$\sigma \in i\mathbb{R}\,, \quad \operatorname{Re} a = \frac12\sigma^2\,.$$

Compare it with (3a22) and the phrase after it: "σ and $a - \frac12\sigma^2$ must belong to $i\mathbb{R}$". Similarly to 3a, we may replace $\sigma, a - \frac12\sigma^2$ with $i\sigma, iv$, writing a morphism of the standard Brownian motion $(B_t)_t$ in $\mathbb{R}$ into a Brownian motion in $\mathrm{SO}(n)$ as $(B_t, Y_t)_t$,

$$\text{(3b8)}\quad Y_t = \sum_{n=0}^\infty i^n \int\cdots\int_{0<s_1<\dots<s_n<t} \exp\big(s_1(iv - \tfrac12\sigma^2)\big)\sigma\exp\big((s_2-s_1)(iv-\tfrac12\sigma^2)\big)\sigma\dots \\ \dots\exp\big((t-s_n)(iv-\tfrac12\sigma^2)\big)\,dB_{s_1}\dots dB_{s_n}\,,$$

where $\sigma, v \in \mathrm{M}_n(\mathbb{R})$ satisfy $\sigma^* = \sigma$, $v^* = v$. That is the general form of such morphisms.

[12] Do not forget: in stochastic analysis a point can leave the circle (or another manifold) tangentially!

3b9 Exercise.

$$Y_1 = \lim_{n\to\infty} \left(\exp\left(i\sigma B_{0,\frac1n} + iv\frac1n\right) \dots \exp\left(i\sigma B_{\frac{n-1}{n},1} + iv\frac1n\right) \right)$$

(convergence in $L_2(\Omega)$).

Prove it.

Hint: $Y_t = \exp(i\sigma B_{0,t} + ivt) + O(t^{3/2})$ for $t \to 0$.

Here is a convenient notation for (3b8):

$$Y_t = \operatorname{Texp}\left(\int_0^t (i\sigma\, dB_s + iv\, ds) \right) ;$$

here "Texp" is called "chronological exponential". The idea is that the integral is the sum of infinitesimal terms, but they need not commute; we multiply their exponentials in the chronological order.[13]

We may also introduce a Brownian motion $(X_t)_t$ in $\big(\mathrm{M}_n(\mathbb{R}), +\big)$ (the *additive* group of matrices),

$$X_t = \sigma B_t + vt \,,$$

and consider a morphism of $(X_t)_t$ into $(Y_t)_t$,

$$Y_t = \operatorname{Texp}\left(i \int_0^t dX_s \right) .$$

3c Finite dimension

3c1 Exercise. (a) The general form of a morphism of the standard Brownian motion $(B_t)_t$ in $\mathbb{R}$ into a Brownian motion in $\mathbb{R}^n$ is $(B_t, Y_t)_t$ where

$$Y_t = \sigma B_t + vt$$

and $\sigma, v \in \mathbb{R}^n$.

(b) There is no morphism of the standard Brownian motion in $\mathbb{R}$ into the standard Brownian motion in $\mathbb{R}^2$.

Prove it.

Hint. (a): apply 3a24 to each coordinate of Y_t; (b): follows from (a).

Striving to describe arbitrary Brownian motions in $\mathrm{SO}(n)$ we should turn from $(B_t)_t$ in $\mathbb{R}$ to m independent copies of $(B_t)_t$, that is, the standard m-dimensional Brownian motion $\big(B_1(t), \dots, B_m(t)\big)_t$. Generalization of stochastic integrals is straightforward. The general form of a random variable $X \in L_2(\mathcal{F}_{0,\infty}^{B_1,\dots,B_m})$ is

$$X = f_0 + \int_0^\infty \sum_{k=1}^m f_{1,k}(t)\, dB_k(t) + \iint_{0<s<t<\infty} \sum_{k,l} f_{2,k,l}(s,t)\, dB_k(s) dB_l(t) + \dots =$$
$$= \sum_{n=0}^\infty \int \cdots \int_{0<s_1<\dots<s_n<\infty} \sum_{k_1,\dots,k_n \in \{1,\dots,m\}} f_{n,k_1,\dots,k_n}(s_1,\dots,s_n)\, dB_{k_1}(s_1) \dots dB_{k_n}(s_n) ;$$

[13]In some sense, $\exp(i\sigma\, dB + iv\, dt) = 1 + i(\sigma\, dB + v\, dt) - \frac12(\sigma\, dB + v\, dt)^2 = 1 + i(\sigma\, dB + v\, dt) - \frac12\sigma^2\, dt = 1 + i\sigma\, dB + (iv - \frac12\sigma^2) dt$.

$$\|X\|^2 = \sum_{n=0}^{\infty} \int \dots \int_{0<s_1<\dots<s_n<\infty} \sum_{k_1,\dots,k_n \in \{1,\dots,m\}} |f_{n,k_1,\dots,k_n}(s_1,\dots,s_n)|^2 \, ds_1 \dots ds_n \,.$$

All formulas obtained before hold for each B_k, but also for $B_1 \cos\alpha + B_2 \sin\alpha$ for any α. Especially, the formula $\iint_{0<r<s<t} dB(r)dB(s) = \frac{1}{2}(B^2(t) - t)$ gives us

$$\iint_{0<r<s<t} \big(\cos\alpha \cdot dB_1(r) + \sin\alpha \cdot dB_2(r)\big)\big(\cos\alpha \cdot dB_1(s) + \sin\alpha \cdot dB_2(s)\big) = \\ = \frac{1}{2}\big((\cos\alpha \cdot B_1(t) + \sin\alpha \cdot B_2(t))^2 - t\big) \,,$$

therefore

(3c2) $$\iint_{0<r<s<t} \big(dB_1(r)dB_2(s) + dB_2(r)dB_1(s)\big) = B_1(t)B_2(t) \,.$$

In the spirit of (2c3) we may write[14]

$$(dB_1(t))(dB_2(t)) = 0 \,.$$

How to generalize (3a10)? We may equip points of M with indices,

$$Y_t = \int_{\Delta_{0,t}} \Bigg(\sum_{k \in \{1,\dots,m\}^M} f_t(M,k) \prod_{s \in M} dB_{k(s)}(s) \Bigg) dM \,,$$

or alternatively, we may introduce m finite sets $M_1, \dots, M_m$ and write

$$Y_t = \int_{\Delta_{0,t}^m} f_t(M_1,\dots,M_m) \Bigg(\prod_{k=1}^m \prod_{s \in M_k} dB_k(s) \Bigg) dM_1 \dots dM_m \,;$$

it is only a matter of convenience. In any case, the argument of f_t may be split according to $0 < s < t$, and (3a10) may be generalized. Similarly to (3a12),

$$f_t(\emptyset,\dots,\emptyset) = \exp(at) \quad \text{or} \quad f_t(\emptyset, \text{nothing}) = \exp(at) \,,$$

further,

$$f_t(\{s\},\emptyset,\dots,\emptyset) = \exp(as)\sigma_1 \exp(a(t-s)), \; \dots, \; f_t(\emptyset,\dots,\emptyset,\{s\}) = \exp(as)\sigma_m \exp(a(t-s)) \,,$$

that is,

$$f_t(\{s\},k) = \exp(as)\sigma_k \exp(a(t-s)) \,,$$

and the counterpart of (3b4) is evident:

$$f_t(\{s_1,\dots,s_n\},k) = \exp(s_1 a)\sigma_{k(s_1)} \exp((s_2 - s_1)a)\sigma_{k(s_2)} \dots \exp((t-s_n)a) \,.$$

[14] $\big(\frac{dB_1(t) \pm dB_2(t)}{\sqrt{2}}\big)^2 = dt$, thus $dB_1(t)dB_2(t) = 0$.

Similarly to (3b),

$$Y_t = 1 + \sigma_1 B_1(t) + \dots + \sigma_m B_m(t) + O(t)\,;$$
$$1 = Y_t Y_t^* = 1 + (\sigma_1 + \sigma_1^*) B_1(t) + \dots + (\sigma_m + \sigma_m^*) B_m(t) + o(\sqrt{t})\,;$$
$$\sigma_1^* = -\sigma_1, \dots, \sigma_m^* = -\sigma_m\,.$$

The quadratic term (taking for simplicity $m = 2$) is

$$\iint_{0<r<s<t} \exp(ra)(\sigma_1 dB_1(r) + \sigma_2 dB_2(r)) \exp((s-r)a)(\sigma_1 dB_1(s) + \sigma_2 dB_2(s)) \exp((t-s)a) =$$
$$= \iint_{0<r<s<t} (1+O(t))(\sigma_1 dB_1(r) + \sigma_2 dB_2(r))(1+O(t))(\sigma_1 dB_1(s) + \sigma_2 dB_2(s))(1+O(t)) =$$
$$= \iint_{0<r<s<t} (\sigma_1 dB_1(r) + \sigma_2 dB_2(r))(\sigma_1 dB_1(s) + \sigma_2 dB_2(s)) + O(t^2) =$$
$$= \sigma_1^2 \cdot \frac{1}{2}(B_1^2(t) - t) + \sigma_1\sigma_2 \cdot B_1(t)B_2(t) + \sigma_2^2 \cdot \frac12 (B_2^2(t) - t) + O(t^2)$$

and we get

$$Y_t = 1 + \sum_{k=1}^m \sigma_k B_k(t) + ta + \frac12 \sum_{k=1}^m \sigma_k^2 (B_k^2(t) - t) + \sum_{k<l} \sigma_k \sigma_l B_k(t) B_l(t) + O(t^{3/2})\,;$$
$$Y_t Y_t^* = 1 + \sum_{k=1}^m \underbrace{(\sigma_k + \sigma_k^*)}_{=0} B_k(t) + t(a + a^*) + \frac12 \sum_{k=1}^m \big(\sigma_k^2 + (\sigma_k^2)^*\big)(B_k^2(t) - t) +$$
$$+ \sum_{k<l} \big(\sigma_k\sigma_l + (\sigma_k\sigma_l)^*\big) B_k(t) B_l(t) + \Big(\sum_{k=1}^m \sigma_k B_k(t)\Big)\Big(\sum_{k=1}^m \sigma_k B_k(t)\Big)^* + O(t^{3/2})\,;$$
$$t(a+a^*) + \sum_k \sigma_k^2 (B_k^2(t) - t) + \sum_{k<l} (\sigma_k\sigma_l + \sigma_l\sigma_k) B_k(t) B_l(t) - \sum_{k,l} \sigma_k \sigma_l B_k(t) B_l(t) = o(t)\,;$$
$$t(a+a^*) - t\sum_k \sigma_k^2 = o(t)\,;$$
$$a + a^* = \sigma_1^2 + \dots + \sigma_m^2\,.$$

Similarly to (3b7) we conclude that $Y_t \in \mathrm{SO}(n)$ if and only if $\sigma_1^* = -\sigma_1, \dots, \sigma_m^* = -\sigma_m$ and $a + a^* = \sigma_1^2 + \dots + \sigma_m^2$. Replacing for convenience $\sigma_1, \dots, \sigma_m$ and $a - \frac12(\sigma_1^2 + \dots + \sigma_m^2)$ with $i\sigma_1, \dots, i\sigma_m$ and iv we get $\sigma_1^* = \sigma_1, \dots, \sigma_m^* = \sigma_m$, $v^* = v$ and similarly to (3b9), $Y_t = \exp(i\sigma_1 B_1(t) + \dots + i\sigma_m B_m(t) + ivt) + O(t^{3/2})$. Finally,

$$Y_t = \operatorname{Texp}\Big(\int_0^t (i\sigma_1 dB_1(s) + \dots + i\sigma_m dB_m(s) + iv\,ds)\Big) = \operatorname{Texp}\Big(i\int_0^t dX_s\Big)\,,$$
$$X_t = \sigma_1 B_1(t) + \dots + \sigma_m B_m(t) + vt\,.$$

3c3 Example. In SO(3), let $i\sigma_1, i\sigma_2, i\sigma_3$ be the generators of the three rotation groups (3b1),

$$i\sigma_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\;;\quad i\sigma_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\;;\quad i\sigma_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,.$$

Then

$$a = iv - \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)\,;$$

$$\sigma_1^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma_2^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma_3^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Taking $v = 0$ we get

$$a = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = -\mathbf{1}\,;$$

here, a commutes with everyting, and we may combine $\exp(s_1 a), \dots, \exp((t-s_n)a)$ into $\exp(ta)$ (as simple as in 3a);

$$Y_t = \mathrm{Texp}\Big(i \int_0^t \big(\sigma_1 dB_1(s) + \sigma_2 dB_2(s) + \sigma_3 dB_3(s)\big) \Big) =$$

$$= e^{-t} + e^{-t} i \int_0^t \big(\sigma_1 dB_1(s) + \sigma_2 dB_2(s) + \sigma_3 dB_3(s)\big) +$$

$$+ e^{-t} i^2 \iint_{0<r<s<t} \big(\sigma_1 dB_1(r) + \sigma_2 dB_2(r) + \sigma_3 dB_3(r)\big)\big(\sigma_1 dB_1(s) + \sigma_2 dB_2(s) + \sigma_3 dB_3(s)\big) + \dots$$

The linear term of $e^t Y_t$ is simply i times

$$\sigma_1 B_1(t) + \sigma_2 B_2(t) + \sigma_3 B_3(t)\,.$$

The quadratic term contains $i^2 = -1$ times

$$\frac{1}{2}\sigma_1^2\big(B_1^2(t) - t\big) + \frac{1}{2}\sigma_2^2\big(B_2^2(t) - t\big) + \frac{1}{2}\sigma_3^2\big(B_3^2(t) - t\big)$$

but also 6 terms like $\sigma_1\sigma_2 \iint_{0<r<s<t} dB_1(r) dB_2(s)$. We may write $\sigma_1\sigma_2 = \frac{1}{2}(\sigma_1\sigma_2 + \sigma_2\sigma_1) + \frac{1}{2}(\sigma_1\sigma_2 - \sigma_2\sigma_1) = \sigma_1 \circ \sigma_2 + \frac{1}{2}[\sigma_1, \sigma_2]$;

$$\sigma_1 \circ \sigma_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad [\sigma_1, \sigma_2] = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Using (3c2) we may write the symmetric terms as

$$\sigma_1 \circ \sigma_2 B_1(t) B_2(t) + \sigma_1 \circ \sigma_3 B_1(t) B_3(t) + \sigma_2 \circ \sigma_3 B_2(t) B_3(t)$$

and the symmetric part of the quadratic term as

$$\frac{1}{2}\big(\sigma_1 B_1(t) + \sigma_2 B_2(t) + \sigma_3 B_3(t)\big)^2 - t$$

(compare it with the linear term). However, there is also an asymmetric part,

$$\frac{1}{2}[\sigma_1, \sigma_2] \iint_{0<r<s<t} \big(dB_1(r) dB_2(s) - dB_2(r) dB_1(s)\big) + \text{two more terms.}$$

Unlike the commutative case of 3a, here $Y(t)$ is not a function of $B_1(t), B_2(t), B_3(t)$, it depends on the past.