

# Extremal Polygon Containment Problems

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## Abstract

Given a convex polygonal object  $P$  and an environment consisting of polygonal obstacles, we seek a placement for the largest copy of  $P$  that does not intersect any of the obstacles, allowing translation, rotation and scaling. We employ the parametric search technique of Megiddo [Me], and the fixed size polygon placement algorithms developed by Leven and Sharir [LS, LS1], to obtain an algorithm that runs in time  $O(k^2 n \lambda_4(kn) \log^3(kn) \log \log(kn))$ . We also present several other efficient algorithms for restricted variants of the extremal polygon containment problem, using the same ideas. These variants include: placement of the largest homothetic copies of one or two convex polygons in another convex polygon and placement of the largest similar copy of a triangle in a convex polygon.

## 1 Introduction

Let  $P$  be a convex polygon having  $k$  vertices and edges, and let  $Q$  be a closed two dimensional space bounded by a collection of polygonal obstacles (the “environment”) having altogether  $n$  corners. The main problem solved in this paper is to compute the largest possible placement of a similar copy of  $P$  that can be placed inside  $Q$ , that is, a placement in which the copy of  $P$  does not intersect any of the obstacles. We also give efficient algorithms that solve similar extremal polygon containment problems under more restrictive conditions, and an algorithm that computes largest disjoint placements of two polygons in a third.

Some papers study the *fixed-size* polygon containment problem, in which (the convex)  $P$  is only allowed to translate and rotate and we wish to determine

whether there is any placement of a copy of  $P$  inside  $Q$  [Ch, AB1].

Chazelle [Ch] studies the problem for the case where  $P$  and  $Q$  are arbitrary simple polygons and presents a naive algorithm that takes time  $O(k^3 n^3 (k+n) \log(k+n))$ . A more restricted case of the problem, in which both  $P$  and  $Q$  are convex is also studied by Chazelle [Ch], who solves this case in time  $O(kn^2)$ . Chazelle gives a simple solution to an even more restricted version in which  $P$  is a triangle; this version runs in time  $O(n^2)$ . Avnaim and Boissonnat [AB1] present an algorithm for the case where both  $P$  and  $Q$  are non-convex, possibly non-connected polygons, which runs in time  $O(k^3 n^3 \log(kn))$ . In another paper Avnaim and Boissonnat [AB] investigate the problem of simultaneous placement of two or three not necessarily convex polygons in a closed polygonal environment. For this problem they allow translations only.

Extremal polygon containment problems were also previously studied. Fortune [Fo], and Leven and Sharir [LS1] consider the following problem: find the largest homothetic copy of  $P$  inside  $Q$ . In other words, translation and scaling of  $P$  are allowed, but rotation is not. When  $P$  is convex and  $Q$  is an arbitrary polygonal environment, this problem is solved in time  $O(kn \log(kn))$  by constructing a generalized Voronoi diagram of  $Q$  under a convex distance function induced by  $P$ .

Chew and Kedem [CK] follow a related approach to solve a more difficult variant of the problem, in which  $P$  is also allowed to rotate, which is also the main problem studied in this paper. Instead of a Voronoi diagram, they compute the Delaunay triangulation of  $Q$  under the convex distance function induced by  $P$  at some arbitrary fixed orientation. By using a clever incremental technique for constructing all the topologically different triangulations obtained as the orientation of  $P$  varies, they solve the problem in time  $O(k^4 n \lambda_3(kn) \log n)$ , where  $\lambda_q(r)$  is the maximum length of an  $(r, q)$ -Davenport-Schinzel sequence (which is almost linear in  $r$  for any fixed  $q$ ) [ASS, HS].

In this paper we follow a different approach that applies the parametric search technique introduced by Megiddo [Me]. By exploiting efficient sequential and parallel algorithms for the fixed size contain-

ment problem, we solve the extremal problem in time  $O(k^2 n \lambda_4(kn) \log^3(kn) \log \log(kn))$ .

There are two advantages of our technique over the technique of [CK]. First, our solution is considerably faster than theirs when  $k$  is large — roughly two orders of magnitude faster. Second, the application of Megiddo’s technique to largest placement problems is so natural that it is surprising that no one has observed this connection before. Roughly speaking, a solution for the fixed-size problem allows us to determine whether any specified expansion ratio is too large or too small. This, plus an efficient parallel version of the fixed size containment algorithm, is all that is required for Megiddo’s technique to apply (see below for more details). We demonstrate the generality of our approach by considering several other extremal containment problems, and show that Megiddo’s technique applies to all of them. Specifically we consider the extremal versions of the following problems: placing a convex polygon in another convex polygon under translation, placing two convex polygons in a third convex polygon under translation, placing a triangle in a convex polygon under translation and rotation, and finally the general case of placing a convex polygon in a polygonal environment under translation and rotation. Except for the general case, these problems were never solved before. Some additional possible extensions of the technique are discussed at the end of the paper.

The paper is organized as follows. In the next subsection we give a brief review of Megiddo’s ingenious parametric search technique. In section 2 we investigate some simple versions of the extremal polygon containment problem, involving one and two polygons, and allowing the polygons only to translate. Section 3 is devoted to a simple version of the general case, in which we also allow rotation; we study the placement of a triangle in a convex polygon. This can be regarded as a warm-up exercise that sheds some light on the general and more complex algorithm. In section 4 we state some necessary definitions and results from [LS], and note that the combinatorial bound derived in that paper can be somewhat improved. In section 5 we describe a variant of the fixed size containment algorithm from [KS], which we use as a decision procedure, to decide whether a copy of  $P$  having some fixed expansion ratio can be placed in  $Q$ . In section 6 we give a parallel version of the fixed size containment algorithm. In section 7 we show how to combine the algorithms of section 5 and 6 to produce an algorithm for the largest placement problem. We conclude in section 8 with a discussion of our results and some open problems.

## 1.1 Parametric Search Technique

We describe below the idea behind Megiddo’s technique [Me]. The exposition is driven by our needs, so it is not the most general possible. Suppose we have a decision problem  $\mathcal{P}(I, \delta)$  that receives as input a collection  $I$  of  $n$  data items, and a real parameter  $\delta$ . Furthermore, assume that the decision is a monotone function of  $\delta$  for every fixed input data, that is, there exists a real number  $\delta^* = \delta^*(I)$  so that

$$\mathcal{P}(I, \delta) = \begin{cases} T & \delta \leq \delta^*(I) \\ F & \delta > \delta^*(I) \end{cases}$$

We want to find the value  $\delta^*$  for a given input  $I$ .

Assume that we have both an efficient sequential algorithm  $A_s$  for solving  $\mathcal{P}(I, \delta)$  at any given  $\delta$ , and a parallel algorithm  $A_p$ , assumed to run in Valiant’s model of computation [Va]. We will denote the running time of  $A_s$  by  $T_s$ , the running time of  $A_p$  by  $T_p$ , and the number of processors it uses by  $P$ . Assume moreover that the flow of execution of  $A_p$  depends only on comparisons, each of which is resolved by testing the sign of a low-degree polynomial in  $\delta$  and the input items. Megiddo’s technique then runs the algorithm  $A_p$  “generically”, without specifying the value of  $\delta$ , with the intention of simulating its execution at the unknown  $\delta^*$ .

At each step of  $A_p$ , where there are at most  $P$  comparisons to resolve, all the roots of all the associated polynomials are computed. If we knew between which two of them (in the natural ordering of the roots)  $\delta^*$  lies, we could resolve all the comparisons, because the sign of the polynomials is constant between consecutive roots, and continue to the next parallel step. So we search  $\delta^*$  in the list of  $O(P)$  roots, using binary search and utilizing  $A_s$  to decide if a given root  $p_i$  is below or above  $\delta^*$  (i.e., we solve  $\mathcal{P}(I, p_i)$ ). This search requires  $O(P + T_s \log P)$  time per parallel step, for a total of  $O(PT_p + T_s T_p \log P)$  time.

Through this execution of the generic  $A_p$ , we obtain a sequence of progressively smaller intervals, each known to contain  $\delta^*$ . In our applications, the value  $\delta^*$  must be a left endpoint of one of the intervals, and so at the end of the execution of  $A_p$ , the left endpoint of the resulting interval is  $\delta^*$ . A more detailed discussion is presented later.

## 2 Placement of Polygons Under Translation

In this section we investigate problems of placement of the largest homothetic copies of polygons inside another polygon (i.e. allowing translations only).

**Definitions:** We denote the set of translations of  $P$

that place it inside  $Q$  by  $\mathcal{C}(P, Q)$ , and the set of translations of  $P$  that make it intersect  $Q$  by  $\mathcal{O}(P, Q)$

We will assume that the polygons  $P$  and  $Q$  are given as an array of their vertices in counter clockwise direction,  $P = (p_1, \dots, p_k)$  and  $Q = (q_1, \dots, q_n)$ . We will consider  $Q$  as fixed and  $P$  as movable, and we will use the vertex  $p_1$  as a reference point for  $P$ .

**Proposition 2.1 ([Ch])** *If  $P$  and  $Q$  are convex, then  $\mathcal{C}(P, Q)$  is a convex polygon with at most  $n$  edges.*

**Proposition 2.2 ([GRS])** *If  $P$  and  $Q$  are convex, then  $\mathcal{O}(P, Q)$  is a convex polygon with at most  $n + k$  edges.*

## 2.1 Computation of $\mathcal{C}(P, Q)$

We assume that both  $P$  and  $Q$  are convex. The procedure given below for the computation of  $\mathcal{C}(P, Q)$  is taken from Chazelle [Ch].

1. For each edge  $q_i q_{i+1}$  of  $Q$ , we find the vertex  $p_i$  of  $P$  that is nearest to it, when  $P$  lies completely on the same side of the line  $q_i q_{i+1}$  as  $Q$  (there may be two such vertices, in which case we choose one of them arbitrarily).

This may be done in time  $O(n + k)$  by merging the normal diagrams of  $P$  and  $Q$ , i.e., merging the edges of  $P$  and  $Q$  to a single list sorted by slope, and finding for each edge of  $Q$  between which edges of  $P$  it lies in the merged list.

2. For every  $i = 1, 2, \dots, n$  we place  $P$  so that  $p_i$  lies on the edge  $q_i q_{i+1}$ , and  $P$  and  $Q$  lie on the same side of the line  $q_i q_{i+1}$ . Now we draw a line  $t_i$  parallel to  $q_i q_{i+1}$  and passing through the reference point  $p_1$  of  $P$ .

Let  $ht_i$  denote the half plane that lies below  $t_i$  if  $Q$  lies below the line  $q_i q_{i+1}$  and above  $t_i$  otherwise. The computation of all the  $ht_i$  takes  $O(n)$  time, a constant time for each half plane.

3. As shown in [Ch],  $\mathcal{C}(P, Q) = \bigcap_i ht_i$ , so what remains to do is to compute the intersection of the  $n$  half planes. We note that the half planes are given sorted by their slope. We compute the intersection by solving the dual convex hull problem, and the sorting of half planes by slope gives us a convex hull problem of  $n$  points sorted by their  $x$ -coordinate. This problem can be solved in  $O(n)$  time, using the beneath-beyond algorithm.

We conclude that the computation of  $\mathcal{C}(P, Q)$  can be done in  $O(n + k)$  time.

In order to apply the parametric search technique of Megiddo, we need a parallel version of this algorithm.

Step 1, the merging of the normal diagrams, could be performed in parallel in  $O(\log \log(\min(n, k)))$  parallel time using  $\sqrt{nk}$  processors, using Valiant's algorithm [Va]. However, the normal diagrams of  $P$  and  $Q$  are independent of the expansion ratio, so no comparison that this merge generates depends on  $\delta^*$ . We can thus implement this step sequentially, "outside" the generic scheme of Megiddo. Step 2 involves no comparisons, so it too can be performed sequentially. The coefficients of the  $t_i$ 's will be however functions of the expansion ratio of  $P$ . Step 3 is performed in parallel using the parallel algorithm for computing the convex hull of a plane point set, by Aggarwal et al. [ACGOY], that works in  $O(\log n)$  time and uses  $O(n)$  processors.

We now combine the sequential and parallel algorithms to obtain an algorithm that computes the largest homothetic copy of  $P$  that can be placed in  $Q$ . Note that the problem at hand satisfies the requirements of section 1.1, that is, when the expansion ratio  $\delta$  is smaller than some (unknown) value  $\delta^*$ , there is a placement of  $P$  inside  $Q$ , and when  $\delta > \delta^*$  there is no such placement. We run the generic parallel algorithm, without specifying  $\delta$ . We resolve comparisons needed by the algorithm by computing the set of real roots of the characteristic polynomials associated with the comparisons, and locating  $\delta^*$  in this (ordered) set by binary search. The decisions made during the binary search are based on the outcome of the fixed-size algorithm, applied to a copy of  $P$  with expansion ratio equal to the root  $\delta$  being compared. Note that the decision step only tells us whether  $\delta \geq \delta^*$  or  $\delta < \delta^*$ . In order not to get stuck, we interpret  $\delta \geq \delta^*$  as  $\delta > \delta^*$  and continue in this manner. When the entire algorithm terminates, it will have produced an interval  $I$  so that  $\delta^*$  is either its left endpoint or an interior point. However, the second case is impossible, because the output of the generic algorithm is the same for all  $\delta \in \text{int}(I)$ , but the output must change at  $\delta^*$ , by definition. Hence  $\delta^*$  is the left endpoint of  $I$ .

The running time of the algorithm is  $O(n + k)$ , for the initial step 1 performed just once, plus the cost of the parametric search itself, which, by Section 1.1, is  $O(n \log^2 n)$ . We thus obtain:

**Theorem 1** *Given a convex polygon  $P$  with  $k$  vertices and a convex polygon  $Q$  with  $n$  vertices, we can compute a placement of the largest homothetic copy of  $P$  inside  $Q$  in  $O(k + n \log^2 n)$  time.*

**Remark.** As noted by Chazelle [Ch], this will work even if  $P$  is not convex, because in this case we simply apply our algorithm to  $\text{conv}(P)$  instead of  $P$ .

## 2.2 Computation of $\mathcal{O}(P, Q)$

As shown by Guibas et al. [GRS], the calculation of  $\mathcal{O}(P, Q)$  in the case of two convex polygons  $P$  and  $Q$

amounts to the merging of the lists of their edges sorted by slope.

This takes time  $O(n+k)$  using a serial algorithm, or  $O(\log \log(\min(n,k)))$  parallel time using  $\sqrt{nk}$  processors, using Valiant's algorithm [Va].

### 2.3 Finding Largest Homothetic Placements of Two Convex Polygons Inside a Third

We now consider the following problem. Given two convex polygons  $P_1$  and  $P_2$  having  $k_1$  and  $k_2$  vertices respectively, and a third convex polygon  $Q$  having  $n$  vertices, find the largest expansion ratio  $\alpha$  such that  $\alpha P_1$  and  $\alpha P_2$  can be translated into  $Q$  without overlapping each other.

For the fixed-size containment problem we use the procedure given by Avnaim and Boissonnat [AB] and Guibas et al. [GRS]. The procedure computes the set  $U$  of all the valid translations  $T_r$  of  $P_1$  relative to  $P_2$ , for which there exists a translation that position both  $P_1$  and  $P_2$  in  $Q$  in their valid relative position without overlapping. This is done by several consecutive applications of the primitive operations  $\mathcal{C}$  and  $\mathcal{O}$ :

1. Compute  $C_1 = \mathcal{C}(P_1, Q)$ .
2. Compute  $C_2 = \mathcal{C}(P_2, Q)$ .
3. If  $C_1$  or  $C_2$  is empty then return  $\emptyset$ .
4. Compute  $I = \mathcal{O}(P_2, P_1)$ .
5. Compute  $S = \mathcal{O}(C_2, C_1)$ .
6. Compute  $S^*$ , the polygon symmetric to  $S$  with respect to the origin.
7. Return  $U = S^* \setminus I$ .

The correctness of this algorithm is proved in [AB]. From the propositions above and the descriptions of algorithms for the computations of  $\mathcal{C}$  and  $\mathcal{O}$ , it follows that the running time of this algorithm is  $O(n+k_1+k_2)$ .

The parallel version of the algorithms for computing  $\mathcal{C}$  and  $\mathcal{O}$  can be used for performing steps 1-5 of the algorithm above. Step 6 does not involve comparisons, so we need not perform it in parallel. Step 7 is more difficult to handle, but we exploit the fact that we are only interested in the existence of a translation in  $U$ , not in its full structure. So instead of computing  $U$ , we will only decide in step 7 whether  $U = S^* \setminus I$  is empty or not. As both  $S^*$  and  $I$  are convex polygons, the difference is not empty if and only if the convex hull of  $S^* \cup I$  is not simply  $I$ . So computing the convex hull of  $S^* \cup I$  is sufficient to decide on the non-emptiness of  $U$ , and this

computation can be performed in  $O(\log(n+k_1+k_2))$  parallel time using  $O(n+k_1+k_2)$  processors.

Applying the parametric search paradigm, we obtain

**Theorem 2** *Given two convex polygons  $P_1$  and  $P_2$  with  $k_1$  and  $k_2$  vertices respectively, and a convex polygon  $Q$  with  $n$  vertices we can compute disjoint placements of the largest homothetic copies of  $P_1$  and  $P_2$  inside  $Q$  (with the same expansion ratio), without intersecting each other in  $O((n+k_1+k_2)\log^2(n+k_1+k_2))$  time.*

The assumption that the expansion of  $P_1$  and  $P_2$  is the same is not necessary; we only have to assume that the expansion ratios of the two polygons are expressed by two monotone increasing functions of the same parameter,  $f_1(\alpha)P_1$ ,  $f_2(\alpha)P_2$ .

## 3 Placing a Triangle Under Translation and Rotation

Before we tackle the general problem of extremal containment of a convex polygon in a general polygonal environment, we consider a restricted version in which we compute the largest similar copy of a triangle  $T = ABC$  in a convex polygon  $Q = (q_1, \dots, q_n)$ .

This (fixed size) containment problem was studied by Chazelle [Ch]. He observed that there is a free placement of  $T$  in  $Q$  if and only if there is a placement of  $T$  in  $Q$  in which a vertex of  $T$  and a vertex of  $Q$  coincide. Thus in order to test if there exists a free placement of  $T$  in  $Q$ , we go over all the  $3n$  pairs of a vertex of  $T$  and a vertex of  $Q$  and for each pair test if there is a free placement such that the relevant vertices coincide.

When the vertices of such a pair, say  $A$  and  $q_1$ , coincide, we use the angle  $\theta$  of rotation around the fixed vertex  $A$  of  $T$  to describe the placement of  $T$ . The placement is free iff both edges  $AB$  and  $AC$  lie in the half-planes whose intersection is  $Q$ . As  $B$  and  $C$  can intersect the line defining the half-plane only twice when  $T$  rotates around  $A$ , we can generate an interval of placements (= angles) that are free relative to that half-plane. The intersection of all  $n$  intervals is the set of free placements. This intersection can be computed in time  $O(n)$  per pair, or  $O(n^2)$  overall.

The parallel version works by sorting all the endpoints of all free intervals for each vertex-vertex contact, and then computes their intersection using a standard "prefix-sum" technique. This takes  $O(\log n)$  time and uses  $O(n)$  processors per pair [Co], or  $O(\log n)$  time and  $O(n^2)$  processors overall. Thus we obtain:

**Theorem 3** *Given a triangle  $T$  and a convex polygon  $Q$  with  $n$  vertices, we can compute a placement of the largest possible similar copy of  $T$  inside  $Q$  in time  $O(n^2 \log^2 n)$ .*

The discussion above is similar in nature to the solution of the general case given below. The increased complexity caused by allowing rotations prevents us from computing the set of all possible free placements as we did when translation alone was allowed. Instead we restrict our attention to a distinguished subset of “critical” free placements that necessarily exist if any free placement exists. There is also an analogy between computing the intersection of relatively free intervals to find a free placement, and the use of lower envelopes below. The details of the general case, however, are much more complex.

## 4 The General Case — Finding Free Critical Orientations

We now begin the description of our solution to the general case. In this section we give a short exposition of the definitions and results in [LS]; this is needed in order to present the algorithms in subsequent sections. The material is taken almost verbatim from [LS].

Let  $P$  be a convex polygonal object having  $k$  vertices, free to translate and rotate (but not to change its size) in a closed two-dimensional space  $Q$  bounded by a collection of polygonal obstacles (“walls”) having altogether  $n$  corners.

A *free critical* placement of  $P$  is one at which it makes simultaneously three distinct contacts with the walls, and is fully contained in  $Q$ , so that it cannot penetrate any obstacle.

A (potential) *contact pair*  $O$  is a pair  $(W, S)$  such that either  $W$  is a (closed) wall edge and  $S$  is a corner of  $P$  or  $W$  is a wall corner and  $S$  is a (closed) side of  $P$ . The contact pair is said to be of type I in the first case, and of type II in the second case.

An actual *obstacle contact* is said to *involve* the contact pair  $O = (W, S)$  if this contact is of a point on  $S$  against a point on  $W$ , and, furthermore, if this contact is *locally free*, i.e., the inner angle of  $P$  at  $S$  lies entirely on the exterior side of  $W$  if  $S$  is a corner of  $P$ , and the entire angle within the wall region  $Q^c$  at  $W$  lies exterior to  $P$  if  $W$  is a wall corner.

The *tangent line*  $T$  of a contact pair  $O = (W, S)$  is either the line passing through  $W$  if  $W$  is a wall edge or the line passing through  $W$  and parallel to  $S$  if  $S$  is a side of  $B$  (in the second case  $T$  depends of course on the orientation of  $P$ ).

Let  $O_1, O_2$  two contact pairs. We say that  $O_2$  *bounds*  $O_1$  at the orientation  $\theta$  if the following conditions hold (see Figure 1):

1. There exists a (not necessarily free) placement  $Z = (X, \theta)$  of  $P$  at which it makes two simultaneous obstacle contacts involving  $O_1, O_2$ .

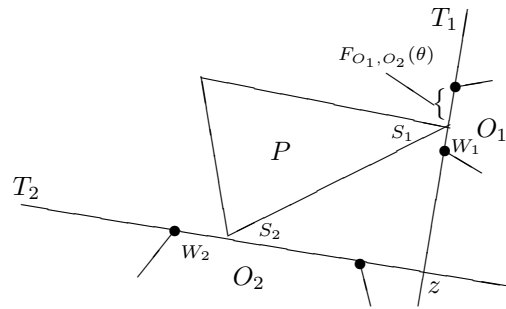


Figure 1: A bounding function.

2. As we move  $P$  from  $Z$  without changing the orientation  $\theta$ , along the tangent  $T_1$ , in the direction of the intersection  $z$  of the two tangents  $T_1$  and  $T_2$ , the subset  $P^* = \text{conv}(S_1 \cup S_2)$  of  $P$  intersects  $W_2$  until  $S_1$  touches  $W_1$ .

It is shown in [LS] that for any double obstacle contact, one of the contact pairs always bounds the other. Let  $O_1$  be any contact pair and consider all contact pairs that bound  $O_1$  (at any orientation  $\theta$ ). For each such pair  $O_2$  we define the *bounding function*  $F_{O_1, O_2}(\theta)$  over the domain  $\Pi = \Pi_{O_1, O_2}$  of orientations  $\theta$  of  $P$  in which  $O_2$  bounds  $O_1$ . For each  $\theta \in \Pi$ , we define  $F_{O_1, O_2}(\theta)$  to be the distance from the endpoint of the contact wall farthest from  $z$  (the intersection of the tangents) to the contact point of  $O_1$ , at the placement  $Z = (X, \theta)$  in which  $P$  simultaneously makes two obstacle contacts involving  $O_1, O_2$ ; (see Figure 1). Note that  $\Pi$  need not be connected, but it consists of at most five subintervals (this is proved in [LS], Lemma 2.2).

The dependence of the bounding function on a specific endpoint of the contact wall suggests that we group the bounding functions  $F_{O_1, O_2}$  of  $O_1$  into two classes,  $A_L$  and  $A_R$ , so that in each class the functions are related to the same endpoint of the contact wall of  $O_1$ .

With each class  $A_E$ ,  $E \in \{L, R\}$ , of each contact pair  $O_1$ , we associate a function

$$\Psi_{E; O_1}(\theta) = \min\{F_{O_1, O}(\theta) : F_{O_1, O} \in A_E\}.$$

This is the *lower envelope* of the functions in  $A_E$ . An intersection of two bounding functions of the same class,  $F_{O_1, O_2}$  and  $F_{O_1, O_3}$ , that lies on the lower envelope of that class, is called a *breakpoint* of the lower envelope.

Critical free orientations (i.e. orientations of critical free placements) can arise in three situations. The *first kind* of orientations is of critical placements at which two contact pairs simultaneously bound a third one, and both belong to the same class. Each such placement is represented as a breakpoint on some lower envelope. The *second kind* of orientations arise at critical

placements where two contact pairs bound a third one but belong to different classes. The *third kind* of orientations arise when no two contact pairs bound a third, but rather at critical placements involving three contact pairs  $O_1$ ,  $O_2$  and  $O_3$  so that  $O_1$  bounds  $O_2$ ,  $O_2$  bounds  $O_3$ , and  $O_3$  bounds  $O_1$ . see Figure 2 for an illustration.

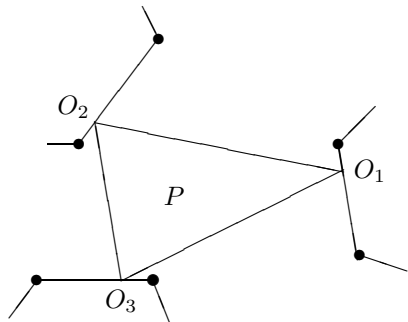


Figure 2: A critical contact of the third kind.

These are necessary conditions for a critical free placement of  $P$ , that is, one of the three situations must occur at a critical free placement. However, they are not sufficient, and while our algorithm will find every orientation of any of the three kinds, it must also be able to discard critical placements that are not free.

**Remark.** In [LS] it is proved that the number of breakpoints along one lower envelope is  $O(\lambda_s(kn))$  for some fixed  $s \leq 6$  (see the remark after Lemma 2.3 in [LS]). We give a simple argument that shows  $s \leq 4$ .

Our argument relies on the fact that we can partition the functions  $F_{O,O'}$  in a class  $A_E(O)$  into two subsets, one arising from contacts  $O'$  of type I and the other from contacts  $O'$  of type II. As shown in [LS], two functions from the same subset intersect at most twice, and functions from different subsets intersect at most four times. Hence the lower envelope of functions in the same subset has complexity  $O(\lambda_4(kn))$  (since they are only partial functions) and since the final envelope of the class is the envelope of these two sub-envelopes, it easily follows that it too has complexity  $O(\lambda_4(kn))$ .

**Remark.** The analysis of [LS], when turned into an algorithm, can produce a list of all these critical placements in time  $O(kn\lambda_4(kn)\log(kn))$ . However, detecting which of these placements is indeed free is not straightforward. In the context of motion planning, as studied in [KS], it is possible to sift out the critical orientations and obtain a subset of *free* critical placements that include all placements reachable from a given initial placement, and perhaps some other non-reachable but free placements. This can be done within the same time bound,  $O(kn\lambda_4(kn)\log(kn))$ , but cannot guarantee that *all* free placements are found, and is therefore unsuitable for our purpose. This issue is discussed in

the algorithms that we give below. In our solution, we do detect all free critical placements, at an extra cost of  $O(k\log n)$  per placement. Performing this faster still appears to be an open problem.

## 5 A Sequential Algorithm

In this section we present a sequential algorithm for the fixed size containment problem, that is to determine whether it is possible to place a similar copy of the convex polygonal object  $P$ , at some fixed expansion ratio to the original  $P$ , in the polygonal environment  $Q$ . To solve this decision problem we solve a related problem — finding all the *critical free orientations* of  $P$ . If the set of critical free orientations is empty, the solution to our decision problem is “no”, otherwise the answer is “yes”.

### 5.1 Generating All the Critical Placements

Below we give the algorithm that generates all the critical placements. Each one of them needs to be tested to decide whether it is free, using the algorithm of the next subsection.

The algorithm closely follows the first stages of the algorithm in [KS]. However, the data structures used are simpler, to ease the task of parallelizing the algorithm later. We do not consider critical contacts in which  $P$  has only one degree of freedom (a corner of  $P$  against a corner of  $Q$ , an edge of  $P$  against an edge of  $Q$ ), because they can be handled exactly like the triangle in section 3, that also has one degree of freedom in every critical placement.

*Step 1: Find all bounding functions.* For every two contact pairs  $O_i, O_j$ , find the range of orientations  $\Pi_{O_i, O_j}$  in which  $O_j$  bounds  $O_i$  toward a specific endpoint  $E$  of  $O_i$ . Split the resulting bounding functions  $F_{O_i, O_j}$  into (at most five) “subfunctions”, each defined over a connected interval, and add them to the appropriate collection  $A_L(O_i)$  or  $A_R(O_i)$ .

*Step 2: Calculate lower envelopes.* We describe the calculation of the lower envelope of  $A_L(O)$  which is denoted by  $\Psi_{L;O}$ ;  $\Psi_{R;O}$  is calculated similarly.

1. Fix a contact pair  $O$  and partition  $A_L(O)$  into two disjoint subsets  $A'_L$  and  $A''_L$  of roughly equal size.
2. Compute recursively the two lower envelopes

$$\Psi'(\theta) = \min\{F_{O,O_i}(\theta) : F_{O,O_i} \in A'_L\}$$

$$\Psi''(\theta) = \min\{F_{O,O_i}(\theta) : F_{O,O_i} \in A''_L\}.$$

Each of the recursive calculations produces a sequence of angular intervals, delimited by breakpoints, in each of which the corresponding partial lower envelope is attained by a single bounding function.

3. Merge these two sequences of intervals to obtain a refined sequence  $\Gamma$  of angular intervals. In the merging process mark every breakpoint in  $\Gamma$  as red if it was originally a breakpoint of  $\Psi'$  or as black if it was originally a breakpoint of  $\Psi''$ . In addition, maintain a pointer from each red node in  $\Gamma$  to the black interval it lies in (an interval of  $\Psi''$ ) and from each black node to the red interval it lies in. For each interval  $I \in \Gamma$  there exist two unique contact pairs  $O', O''$  with  $F_{O, O'} \in A'_L$ ,  $F_{O, O''} \in A''_L$  such that  $\Psi'(\theta) = F_{O, O'}(\theta)$ ,  $\Psi''(\theta) = F_{O, O''}(\theta)$  for each  $\theta \in I$ . By the analysis of [LS] the two functions  $F_{O, O'}, F_{O, O''}$  intersect in at most four points (some of which may not belong to  $I$ ), which can be calculated, as the roots of some quartic polynomial, in constant time. Each of these intersections which lies in  $I$  is clearly a breakpoint of  $\Psi = \Psi_{L, O}$ . Add these points to  $\Gamma$  and mark them as white nodes. Every breakpoint of  $\Psi$  is either of this kind (a white node) or is a breakpoint of  $\Psi'$  or of  $\Psi''$ , i.e. one of the red or black nodes. Now we need to eliminate from  $\Gamma$  the red and black nodes which do not lie on the lower envelope  $\Psi$ . For each red (black) node, we follow the pointer to the black (red) interval it lies in, and check which is higher — the red breakpoint in  $\Psi'$  or the bounding function on  $\Psi''$ . If the former is higher we prune it from the list  $\Gamma$ , otherwise the breakpoint remains in  $\Gamma$ . Thus at the end of the process  $\Gamma$  represents the breakpoints in  $\Psi$ .

Note that maintaining the red/black pointers can be done in time proportional to the length of the list (in one pass over the list), and the same time bound applies to the pruning of the redundant nodes.

The merging step can be done in time proportional to the length of  $\Gamma$ , which, by [LS1] and the comments in Section 2, is  $O(\lambda_4(kn))$ . Hence the calculation of the lower envelope  $\Psi_{L, O}$  takes  $O(\lambda_4(kn) \log kn)$  time, so all these envelopes can be computed in overall time  $O(kn\lambda_4(kn) \log kn)$ .

The collection of breakpoints is a superset of all the critical orientations of the first kind; every one of them will later be tested to decide whether it is free, in the manner described in the next subsection.

*Step 3: Calculate critical orientations of the second kind.* These are orientations at which  $P$  makes simultaneously, at some free placement, obstacle contacts involving three distinct contact pairs  $O_1, O_2, O_3$  such that two of them, say  $O_2, O_3$  bound  $O_1$  but with

$F_{O_1, O_2} \in A_L(O_1)$  while  $F_{O_1, O_3} \in A_R(O_1)$ . In this case we first reflect and translate one of the envelopes, so that they both measure the distance from the same endpoint of  $O_1$ . Then we merge the lists of breakpoints in  $\Psi_{L, O_1}$  and in  $\Psi_{R, O_1}$  and compute the intersections of the bounding functions from the two lower envelopes over each resulting interval in the same way as in the previous step. These orientations are added to the list of critical orientations.

Clearly, this step runs in  $O(kn\lambda_4(kn))$  time. Again, we will later discard non-free critical orientations found in this step.

*Step 4: Calculate critical orientations of the third kind.* Finally, we calculate the third and most complex kind of critical orientations. At each such orientation  $\theta$ ,  $P$  can make simultaneously a free triple contact involving three distinct contact pairs  $O_1, O_2, O_3$ , such that  $F_{O_1, O_2} \in A_{E_1}(O_1)$ ,  $F_{O_2, O_3} \in A_{E_2}(O_2)$ ,  $F_{O_3, O_1} \in A_{E_3}(O_3)$ , where  $E_i \in \{L, R\}$  for  $i = 1, 2, 3$ , and such that all three functions lie at  $\theta$  on the corresponding lower envelopes.

To find these orientations we first merge all breakpoint lists from all the lower envelopes calculated in step 2, to obtain a single sorted list  $\Phi$  consisting of  $O(kn\lambda_4(kn))$  refined noncritical intervals. Each interval  $I \in \Phi$  has the property that each lower envelope is attained over it by a single bounding function.

Next we find all the critical orientations of the third kind (not necessarily free). For each possible triple contact that the algorithm considers, we find its (at most four) critical orientations, and then test which of these orientations is indeed free.

We start by considering the first interval in  $\Phi$ , denoted  $I_0$ . For each contact pair  $O_1$  and each side  $E_1 \in \{L, R\}$ , find the unique contact pair  $O_2$  such that  $\Psi_{E_1, O_1} = F_{O_1, O_2}$  over  $I_0$ . For each  $E_2 \in \{L, R\}$ , find the unique contact pair  $O_3$  such that  $\Psi_{E_2, O_2} = F_{O_2, O_3}$  over  $I_0$ . For each  $E_3 \in \{L, R\}$  for which  $\Psi_{E_3, O_3} = F_{O_3, O_1}$  over  $I_0$  conclude that  $(O_1, O_2, O_3, E_1, E_2, E_3)$  is a critical contact, perhaps not free. Compute its critical orientations. Those that lie in  $I_0$  are tested to decide whether they are free, and if so they are reported as such (the algorithm can thus be halted right now with an affirmative answer to the decision problem). The other orientations do not lie in  $I_0$ , so we find the interval each of them lies in, by binary search over the sorted list  $\Phi$ , and test whether the corresponding critical placement is free. This takes  $O(kn \log(kn))$  time, excluding the tests for being free,  $O(\log(kn))$  time for each contact pair.

Each interval  $I \neq I_0$  in  $\Phi$  can induce new critical triplets but fortunately only a constant number of them. The interval  $I$  was formed because its left endpoint represents a break in one of the lower envelopes, say  $\Psi_{L, O_1}$ . So we need to repeat the process we did at  $I_0$ , but this

time starting with only one particular contact ( $O_1$ ) and one lower envelope ( $\Psi_{L;O_1}$ ). Thus every interval  $I$  induces only  $O(1)$  new candidates for the critical orientations that we seek. Finding the bounding functions on the lower relevant envelopes is now accomplished by a binary search over the list of breakpoints on each envelope. This takes  $O(kn\lambda_4(kn)\log(kn))$  time, as each of the  $O(kn\lambda_4(kn))$  intervals requires  $O(\log(kn))$  time for the binary searches.

## 5.2 Deciding whether a critical orientation represents a free placement

As mentioned above, the set of critical orientations computed so far may contain orientations that correspond to critical placements that are not free, so we need to test each critical placement whether it is indeed free.

To perform this test we use the following simple method. In a preliminary step, we prepare data structures that will enable us to perform this test, at any query placement of  $P$ , in time  $O(k\log n)$ . These data structures depend only on  $Q$ . As we are required to perform at most  $O(kn\lambda_4(kn))$  such tests, the total time they require is  $O(k^2n\lambda_4(kn)\log n)$ . As the environment  $Q$  is static during the execution of the (largest placement) algorithm, we need to build the data structures only once, “outside” the generic execution of the parallel version of the algorithm.

If a critical placement is not free, then either a vertex of  $Q$  lies inside  $P$ , or an edge of  $P$  intersects an edge of  $Q$ . To test whether the first situation occurs we insert the  $n$  vertices of  $Q$  into a data structure that supports fast counting of points inside a query polygon. We use the technique of [PY, Ed], which uses  $O(n^2)$  storage,  $O(n^2)$  preprocessing, and can answer a query in time  $O(k\log n)$  for a query polygon with  $k$  sides. Given a placement of  $P$ , we query the number of points inside it and declare the placement non-free if any such point is found. To test whether the second situation arises, we preprocess  $Q$  for segment intersection queries, that is, given a query segment, determine quickly whether it intersects an edge of  $Q$ . For this we use the technique of [Ch1]. Again, this can be implemented with  $O(n^2)$  storage,  $O(n^2)$  preprocessing, and can answer a segment intersection query in  $O(\log n)$  time. For each critical placement of  $P$ , we query this structure with each edge of  $P$ , and declare the placement as non-free if any such intersection is found. If none of these bad situations are detected, the placement is free.

We have thus shown:

**Theorem 4** *Given a convex polygon  $P$  with  $k$  sides, and a polygonal environment  $Q$  with  $n$  edges, we can compute all free critical placements of  $P$  inside  $Q$  in time  $O(k^2n\lambda_4(kn)\log(kn))$ .*

## 6 A Parallel Algorithm

We now present a parallel version of the algorithm, or rather comment on how to perform each step in parallel. Recall that we do not need a strong parallel computation model. All we seek is a scheme in which many independent comparisons are performed at each parallel step. Thus we ignore synchronization and other book-keeping problems, use Valiant’s weak model of parallel computation [Va], and perform tasks in a sequential manner when they do not involve comparisons but only manipulation of pointers.

Step 1 can clearly be carried out by  $O(k^2n^2)$  processors in  $O(1)$  parallel time, with each processor calculating one bounding function.

Step 2 is performed using a divide and conquer strategy. The divide phase and the recursive calls can be done in parallel. The merge phase can be done using Valiant’s merging algorithm [Va] which runs in  $O(\log\log(kn))$  parallel time using  $O(\lambda_4(kn))$  processors per envelope. Once the merge is done, maintenance of the red/black pointers can be done serially because this is a mere manipulation of pointers and involves no comparisons. The testing of red (black) breakpoints against their containing black (red) non-critical intervals can be done in  $O(1)$  parallel time using  $O(\lambda_4(kn))$  processors per envelope. The subsequent pruning of breakpoints can be done serially, because it involves no comparisons. The total time required to compute all the lower envelopes is thus  $O(\log(kn)\log\log(kn))$  using  $O(kn\lambda_4(kn))$  processors.

Step 3 also uses a merge, but only once for each contact pair, and the calculation of envelope intersections over all contact pairs can clearly be done in parallel. The total parallel time for this step is therefore  $O(\log\log(kn))$  using  $O(kn\lambda_4(kn))$  processors.

Step 4 first requires the merging of all  $O(kn)$  lower envelopes into one sorted list of breakpoints  $\Phi$ . This can be done recursively. We divide the lower envelopes into two collections of roughly equal size, compute the two merged lists of breakpoints  $\Phi'$ ,  $\Phi''$  and merge them. The merging step takes  $O(\log\log(kn))$  parallel time using  $O(kn\lambda_4(kn))$  processors, so the whole process takes  $O(\log(kn)\log\log(kn))$  time using  $O(kn\lambda_4(kn))$  processors.

The handling of the first interval  $I_0 \in \Phi$  can be done in parallel using  $O(kn)$  processors and  $O(\log(kn))$  time. All the other intervals are each assigned a single processor and the time it takes to find the new critical triple contacts is  $O(\log(kn))$ . The test to decide whether a critical orientation is free is performed in the same manner as in the sequential case. We use  $k$  processors to perform the  $O(k)$  queries in  $O(\log n)$  time. The data structures depend only on  $Q$  and so can be preprocessed just once, outside the generic parallel scheme.



We conclude that all critical free orientations can be calculated in parallel, under Valiant’s comparison model, in time  $O(\log(kn)\log\log(kn))$  using  $O(k^2n\lambda_4(kn))$  processors.

## 7 The Overall Algorithm

We now apply Megiddo’s technique to our problem, using the algorithms of sections 5 and 6. We run the parallel algorithm generically, without specifying the expansion ratio  $\delta$ . We resolve comparisons made by the parallel algorithm by using our sequential algorithm, in the manner explained in section 1.1.

The only fine point is to verify that comparisons involve only evaluations of signs of low degree polynomials in the unspecified  $\delta$ . Indeed, a free placement of  $P$  in  $Q$  has to satisfy a set of algebraic constraints (see [SS]). In our case these constraints are mainly algebraic inequalities that describe the disjointness of  $P$  and  $Q$ . Computing a critical triple contact amounts to setting three inequalities as tight constraints (i.e. equalities), and solving these three equations in three unknowns (the  $(x, y, \theta)$  coordinates of  $P$ ), discarding solutions which are not locally free. Computing a breakpoint thus reduces to computing the critical triple contact placement associated with the three contact pairs that define the breakpoint. Evaluating a bounding function at a given orientation amounts to setting the two constraints involved in the corresponding two contact pairs to be tight, and adding a third constraint that the slope of the line passing between two fixed points in  $P$  will be at the given orientation. This will give us the desired placement of  $P$ , and we can calculate the value of the bounding function which is simply an affine transformation of the placement. Using the standard transformation  $t = \tan(\theta/2)$ , all contact constraints, and thus all functions of  $\delta$  computed by the algorithm, become algebraic, and no trigonometric functions need be used.

Since the only place where dependence on  $\delta$  can arise is in the coefficients of the constraints, and since the functions of  $\delta$  are polynomials of the first or second degree, we are assured that all the equations in  $\delta$  we have to solve during the algorithm are algebraic equations of bounded degree. We assume that this kind of equations can be handled in constant time.

The running time of the algorithm can easily be deduced by “plugging in” the running time of the sequential and parallel algorithms into the analysis of section 1.1. The fact that  $\delta^*$  is the left endpoint of the final interval is justified as in section 2.1. This establishes our main result:

**Theorem 5** *Given a convex polygon  $P$  with  $k$  sides, and a polygonal environment  $Q$  with  $n$  edges, we can*

*compute a placement of the largest possible similar copy of  $P$  inside  $Q$  in time  $O(k^2n\lambda_4(kn)\log^3(kn)\log\log(kn))$ .*

## 8 Conclusion

In this paper we have applied Megiddo’s parametric search technique to a variety of extremal polygon placement problems. In addition, we presented a decision algorithm for the general fixed-size polygon containment problem, which improves upon results obtained in previous papers that studied related problems.

Our work raises a few open problems. One is to improve our algorithm by about an order of  $k$ , bringing its complexity close to the motion-planning algorithm of [KS]. We believe that Megiddo’s technique can be applied to many other extremal containment problems. As an example we mention the problem of finding the largest stick (line segment) that can be placed inside a simple polygon, and the problem of finding the largest stick that can be placed in a polyhedral environment in 3-space. Finally, can our technique be turned into a motion-planning algorithm, that finds a “highest-clearance” path among obstacles, as in [CK1].

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