An example of public goods

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Consider an economy with two identical agents, A and B, who consume one public good G, and one private good y. The preferences of the two agents are given by the following quasi-linear utility function:

$$U(G, y_i) = \left(ZG - \frac{G^2}{2}\right) + y_i, \qquad i = A, B, \tag{1}$$

where Z is a positive parameter. Note that since the two agents are identical we do not need to index the utility function. As will become clear later, the assumption that the utility function of the two agents is quasi-linear in good y will simplify the analysis by allowing us to focus on the public good G without worrying about the private good y.

Suppose that agents A and B have initial endowments of good y, $\bar{y}_A$ and $\bar{y}_B$, respectively, that they can either consume directly, or convert to a public good on a one-to-one basis. That is, each agents can take g units of his initial endowment of good y and convert it to g units of the public good. Therefore, if agents A converts g_A units of y to a public good and agent B converts g_B units then the aggregate level of the public good is g_A+g_B and agent A is left with $\bar{y}_A-g_A$ units of good y while agent B is left with $\bar{y}_B-g_B$ of the public good. By definition, both agents consume the aggregate level of the public good so g_A+g_B can be viewed as a common pool that the two agents share. Given g_A and g_B, the utilities of the two agents are given by:

$$\begin{aligned} U(G, y_A) &= \left(Z(g_A + g_B) - \frac{(g_A + g_B)^2}{2}\right) + \left(\bar{y}_A - g_A\right), \\ U(G, y_B) &= \left(Z(g_A + g_B) - \frac{(g_A + g_B)^2}{2}\right) + \left(\bar{y}_B - g_B\right). \end{aligned} \tag{2}$$

Now suppose that the two agents decide simultaneously how much to contribute to the common pool of the public good. This can be viewed as a two-players "voluntary contribution" normal form game in which the strategy of each player is his/hers contribution to the common pool and the payoffs are the resulting utilities. A Nash equilibrium of this game is a pair (g_A*, g_B*), defined by the intersection of the following two best response functions:

$$\frac{\partial U(G,y_A)}{\partial g_A} = \left(Z - (g_A + g_B)\right) - 1 = 0,$$
$$\frac{\partial U(G,y_B)}{\partial g_B} = \left(Z - (g_A + g_B)\right) - 1 = 0. \qquad \textbf{(3)}$$

Equation (3) shows that at a Nash equilibrium of the voluntary contribution game, each agents contributes to the common pool up to the point where the marginal utility from G, given by Z-(g_A+g_B) is just equal to 1 which is the marginal cost of the public good since in order to produce one unit of G it is necessary to give up one unit of the private good.

At a symmetric Nash equilibrium, g_A* = g_B*, so equation (3) implies that

$$g_A^* = g_B^* = \frac{Z-1}{2}. \qquad \textbf{(4)}$$

Hence, in the voluntary contribution game, the level of public good will be

$$G^* = g_A^* + g_B^* = Z - 1. \qquad \textbf{(5)}$$

Is G* Pareto efficient? To answer this question we first need to characterize the set of Pareto efficient allocations. To this end, note that G is produced by converting units of private good to public good on a one-to-one basis. Hence the transformation curve in this economy that describes the feasible allocations of private and public goods is given by:

$$T(G,y_A,y_B) = G + y_A + y_B - \bar{y} = 0, \qquad \textbf{(6)}$$

where $\bar{y} \equiv \bar{y}_A + \bar{y}_B$.

Given $T(G,y_A,y_B)$, the set of Pareto efficient allocations is determined by the following maximization problem:

$$\begin{aligned} \underset{G,y_A,y_B}{Max} \quad & U(G,y_A) \\ s.t. \quad & U(G,y_B) = \bar{U}^B \\ & T(G,y_A,y_B) = 0. \end{aligned} \qquad \textbf{(7)}$$

Substituting for y_B from the last constraint into the second constraint, and using equation (1), the problem can be written as:

$$\begin{aligned} \underset{G,y_A}{Max} \quad & \left(Z\,G - \frac{G^2}{2}\right) + y_A, \\ s.t. \quad & \left(Z\,G - \frac{G^2}{2}\right) + (\bar{y} - y_A - G) = \bar{U}^B. \end{aligned} \qquad \textbf{(8)}$$

Now, substituting for y_A from the constraint into the objective function, the maximization problem becomes:

$$\underset{G}{Max} \quad \left(Z\,G - \frac{G^2}{2}\right) + \left(Z\,G - \frac{G^2}{2}\right) + (\bar{y} - G - \bar{U}^B). \qquad \textbf{(9)}$$

The first order condition for this problem is given by:

$$2\,(Z - G) - 1 = 0. \qquad \textbf{(10)}$$

This condition is just the Samuelson condition since it shows that the sum of the marginal utilities from consuming the public good (recall that the marginal utility of each agent from consiming the public good is Z-G) should be equal to the marginal cost of producing the public good.

Solving equation (10) for G reveals that the Pareto efficient level of public good is

$$G^{**} = Z - \frac{1}{2} > Z - 1 > G^{*}. \qquad \textbf{(11)}$$

Thus we just found that the voluntary contribution game leads to "too little" provision of public good relative to the Pareto efficient level. Intuitively, the reason is that while the agents share the benefits from consuming the public good, each agent alone bears the cost of providing the good. As a result, the agent have an incentive to "free ride" by waiting for the other agent to provide the public good.

Corrective taxation

One way to correct the market failure and restore Pareto efficiency is to give the two agents a subsidy, s, for each unit of public good that they provide in order to encourage them to increase their contributions. To compute the required subsidy, note that given s, the agents' utility functions become:

$$\begin{aligned} U(G,y_A) &= \left(Z(g_A + g_B) - \frac{(g_A + g_B)^2}{2} \right) + \left(\bar{y}_A - g_A + s g_A \right), \\ U(G,y_B) &= \left(Z(g_A + g_B) - \frac{(g_A + g_B)^2}{2} \right) + \left(\bar{y}_B - g_B + s g_B \right). \end{aligned} \qquad \textbf{(12)}$$

The best response functions of the two agents in the voluntary contribution game are now given by:

$$\begin{aligned} \frac{\partial U(G,y_A)}{\partial g_A} &= \left(Z - (g_A + g_B) \right) - 1 + s = 0, \\ \frac{\partial U(G,y_B)}{\partial g_B} &= \left(Z - (g_A + g_B) \right) - 1 + s = 0. \end{aligned} \qquad \textbf{(13)}$$

At a symmetric Nash equilibrium, $g_A{}^* = g_B{}^*$, so

$$g_A^* = g_B^* = \frac{Z - 1 + s}{2}. \tag{14}$$

Hence, in the voluntary contribution game, the level of public good will be

$$G^* = g_A^* + g_B^* = Z - 1 + s. \tag{15}$$

Thus, if s = 1/2, then the level of the public good in a Nash equilibrium is Z-1/2 which is also the Pareto efficient level.