Mean Variance Utility

In this note I show how exponential utility function and normally distributed consumption give rise to a mean variance utility function where the agent's expected utility is a linear function of his mean income and the variance of his income. The analysis is taken from p. 154-155 in T. Sargent, *Macroeconomic Theory*, 2nd. edition.

Suppose that the utility function from consumption, C, is exponential and given by

$$U(C) = -e^{-\lambda C}, \qquad \lambda > 0. \tag{1}$$

This utility function is increasing and concave since

$$U'(C) = \lambda e^{-\lambda C} > 0, \qquad U''(C) = -\lambda^2 e^{-\lambda C} < 0. \tag{2}$$

Since the utility function is concave, it reflects risk aversion. Moreover note that the Arrow-Pratt index of absolute risk aversion is given by

$$-\frac{U''(C)}{U'(C)} = \lambda. \tag{3}$$

This means that the larger λ is, the more risk averse the agent is.

Next, suppose that C is distributed normally with mean, μ, and standard deviation, σ. Then the density of C is given by:

$$f(C) = \frac{e^{-\frac{(C-\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}. \tag{4}$$

Therefore, expected utility is given by:

$$\begin{aligned} EU(C) &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} -e^{-\lambda C} e^{-\frac{(C-\mu)^2}{2\sigma^2}} dC \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} -e^{-\left(\lambda C + \frac{(C-\mu)^2}{2\sigma^2}\right)} dC. \end{aligned} \tag{5}$$

It is no useful to rewrite the exponent so as to group terms that depend on C and terms that do not depend on C. To this end note that

$$\lambda C + \frac{(C-\mu)^2}{2\sigma^2} = \frac{\left(C-\mu+\lambda\sigma^2\right)^2}{2\sigma^2} + \lambda\left(\mu - \frac{\lambda\sigma^2}{2}\right). \tag{6}$$

Substituting in EU(C), gives

$$EU(C) = -\frac{e^{-\lambda\left(\mu - \frac{\lambda\sigma^2}{2}\right)}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\left(C-\mu+\lambda\sigma^2\right)^2}{2\sigma^2}}\, dC. \tag{7}$$

Now, for all μ',

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(C-\mu')^2}{2\sigma^2}}\, dC = 1, \tag{8}$$

because the left hand side is just the area under the density function over the entire support when the mean is μ' and the standard deviation is γ. Since this is so for any μ' including $\mu' = \mu - \lambda\gamma^2$, it follows that

$$EU(C) = -e^{-\lambda\left(\mu - \frac{\lambda\sigma^2}{2}\right)}. \tag{9}$$

Hence, the objective of the agent is to maximize the expression

$$\mu - \frac{\lambda\sigma^2}{2}. \tag{10}$$

That is, the agent is interested in maximizing his mean consumption minus the variance multiplied by a constant. As we saw before, the constant λ measures the degree of risk aversion: the larger λ is, the more risk averse the agent is. Hence the utility of the agent is increasing with the mean of his consumption and decreases with the variance. The rate of decrease with the variance is larger the more risk averse the agent is.