Available online at www.sciencedirect.com
ScienceDirect
Advances in Mathematics 227 (2011) 730–744
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

Weighted covering numbers of convex sets ☆

S. Artstein-Avidan *, O. Raz

School of Mathematical Sciences, Tel Aviv University, Israel

Received 22 June 2010; accepted 15 February 2011

Available online 1 March 2011

Communicated by Erwin Lutwak

Abstract

In this paper we define the notions of weighted covering number and weighted separation number for convex sets, and compare them to the classical covering and separation numbers. This sheds new light on the equivalence of classical covering and separation. We also provide a formula for computing these numbers via a limit of classical covering numbers in higher dimensions.

Keywords: Covering numbers; Convex sets; Separation numbers; Weighted covering

1. Introduction

Covering numbers are a very useful tool in many mathematical fields, ranging from probability and combinatorics to analysis and geometry. Roughly speaking, for two convex bodies K and T (closed convex sets with non-empty interior) the covering number of K by tT as a function of $t > 0$ measures the complexity of the set K in terms of T, and is defined as the minimal number of translates of tT needed to cover K:

Definition 1. Let $K, T \subseteq R^d$. A sequence $\{x_i\}_{i=1}^N \subset K$ is called a *covering* of K by T if $K \subset \bigcup_{i=1}^N (T + x_i)$. The *covering number* of K by T is defined as the minimal N over all possible coverings of K by T, denoted by $N(K, T)$.

☆ Both authors were partially supported by ISF grant 865/07. This work is part of the M.Sc. thesis of the second named author.

* Corresponding author.
E-mail address: shiri@post.tau.ac.il (S. Artstein-Avidan).

0001-8708/$ – see front matter
doi:10.1016/j.aim.2011.02.009

Remark 2. In the case of classical covering numbers, it is often allowed to use translates of T with centers anywhere in $\mathbb{R}^d$, i.e. the centers x_i are not restricted to lie in K (denote this number by $N_0(K,T)$, though it will not be used outside this remark). We prefer to use the above definition (sometimes denoted $\bar{N}$) and note that there is a well-known connection between the two definitions given, for symmetric $T=-T$, by

$$N_0(K,T) \leqslant N(K,T) \leqslant N_0\left(K, \frac{1}{2}T\right).$$

[More generally, the right-hand side inequality for non-symmetric T reads $N(K, T-T) \leqslant N_0(K,T)$.]

The computation of these numbers is usually extremely difficult. However, they arise very naturally in the solution of many problems in analysis and in probability, for example in the study of Gaussian processes; understanding their behavior is an important goal (see e.g. [1,3–5, 7,8,11] and references therein).

A closely related notion to covering numbers is the notion of separation. Roughly speaking, for two convex bodies K and T, the separation number is the maximal number of non-intersecting copies of tT (as a function of t) one can put with centers all lying inside K.

Definition 3. A sequence $\{x_i\}_{i=1}^M \subset K$ is called a T-*separated set* in K if for every $i \neq j$ in $1,\dots,M$ we have $(T+x_i)\cap(T+x_j)=\emptyset$. The T-*separation number* in K is defined as the maximal M possible over all T-separated sets in K, denoted by $M(K,T)$.

Although there are known connections between the separation number of two bodies and their covering number, in general the two numbers can be very different. One such connection is given by the following inequalities, which hold for two symmetric convex bodies $K, T \subseteq \mathbb{R}^n$, which is well known and easy to prove:

$$N(K,2T) \leqslant M(K,T) \leqslant N(K,T). \tag{1}$$

In fact, the proof corresponds to the following well-known geometric riddle:

Riddle. On a triangular table, 100 identical coins are placed (non-intersecting) so that no room is left to insert another coin (without it intersecting one of the other coins). Show that the whole table can be covered using 400 such coins.

To solve the riddle, denote the table by T and a coin by C. We are trying to show that $N(T,C) \leqslant 400$ where we know $M(T,C)=100$. (Here is a slight inaccuracy: we know that there exists a maximal separated set of size 100. There could, in general, exist a larger separated set.) The key to the solution is to notice that the triangular table T can be seen as the union of four identical triangles (one of which is rotated 180°) of half the size, $T/2$. Then by Eq. (1) each of these four tables can be covered by 100 coins, and so T can be covered by 400 coins.

Thus, the main part in the solution of the riddle is to show why Eq. (1) holds. This is a simple matter of inflating the covering body by 2, and a similar argument is presented in the proof of Proposition 9 below. We write, in short, that $N \sim M$, where this stands for universal constant inside the argument, as in (1).

In general, equality need not hold in either side of this equation.

In combinatorics, already a couple of decades ago, the notion of a fractional covering of a hypergraph was introduced, where the covering sets can be given weights which are different than one, so long as the total weight on each point in the covered set is at least one. The weighted covering number is then the total weight distributed on the covering sets (precise definitions, in the context of convex bodies, are below). For reference see e.g. the book [10] and references therein. Our starting point was to ask whether a similar notion exists in the continuous case, and what can be said about it. Much to our surprise, the corresponding notion, which we call "weighted covering", not only exists but has much better properties than its combinatorial cousin.

Our main Theorem 6 asserts that usual covering numbers and our notion of weighted covering numbers are actually, up to some universal constants (independent of the dimension, and the sets of course), the same. Therefore, when these constants are not important, one can choose freely whether to use the classical or the weighted notion, being able to translate a result with one notion to the other. In combinatorics, the notion of fractional coverings plays a central role, their main advantage being that they allow to use linear programming over $\mathbb{R}$ rather than $\mathbb{Z}$. However, it is interesting to note that in combinatorics it was shown by L. Lovász in [6] that they are equivalent to usual covering numbers only up to a logarithmic factor, which cannot be eliminated (the difference from our case being that no natural homothety is allowed).

In fact, in the "weighted-world" covering and separation numbers for convex sets are not only equivalent but *equal* (whereas for the non-weighted notion there are cases where there is a difference). This is our Theorem 7. This fact by itself demonstrates another advantage the weighted notion has over the classical one, and one might find it easier to work with. We remark that an alternative proof of Theorem 6, with slightly worse constants, will be presented in [9].

To state the two main theorems precisely, let us give the definition of weighted covering and separation:

Definition 4. A sequence of pairs $S = \{(x_i, w_i)\colon\ x_i \in K,\ w_i \in \mathbb{R}^+\}_{i=1}^N$ with $N \in \mathbb{N}$ of points and weights will be called a *weighted covering* of K by T if for all $x \in K$

$$w(x) := \sum_{i=1}^{N} w_i \mathbb{1}_{T+x_i}(x) \geqslant 1.$$

Here $\mathbb{1}_A(x)$ denotes the indicator function of A, equal to 1 if $x \in A$ and 0 otherwise. We denote by $w(S) = \sum_{i=1}^N w_i$ the *total weight* of the weighted covering and define $N_w(K,T)$, the *weighted covering number* of K by T, to be the infimal total weight over all weighted coverings of K by T.

Definition 5. A sequence of pairs $S = \{(x_i, w_i)\colon\ x_i \in K,\ w_i \in \mathbb{R}^+\}_{i=1}^M$ with $M \in \mathbb{N}$ of points and weights will be called a *weighted T-separated set* in K if for all $x \in K$

$$w(x) := \sum_{i=1}^{M} w_i \mathbb{1}_{T+x_i}(x) \leqslant 1.$$

The *total weight* of S will be $w(S) = \sum_{i=1}^M w_i$, and we define $M_w(K,T)$ to be the supremal total weight over all weighted T-separated sets in K, which we will call the *weighted T-separation number* of K.

Our two main theorems are as follows:

Theorem 6. *Let K and T be two convex bodies in $\mathbb{R}^d$ and assume T is symmetric $T = -T$. Then,*

$$N(K, 2T) \leqslant N_w(K, T) \leqslant N(K, T). \tag{2}$$

Theorem 7. *Let K and T be two convex bodies in $\mathbb{R}^d$ and assume T is symmetric $T = -T$. Then,*

$$N_w(K, T) = M_w(K, T). \tag{3}$$

The third main result of this note can be viewed as an alternative definition to weighted covering number, as a limit of the classical covering numbers, given in Theorem 8. A main tool in the proof is again Theorem 7. The alternative definition allows us to characterize the cases where there is an actual equality between the weighted and the classical notions.

The representation is via a limit of classical covering numbers in increasing dimension. For $A \subset \mathbb{R}^d$ and $n \in \mathbb{N}$, denote $A^n = \{(x_1, \dots, x_n) \in (R^d)^n\colon\ x_i \in A\}$. We then have

Theorem 8. *Let K and T be convex bodies in $\mathbb{R}^d$ and assume T is symmetric $T = -T$. Then,*

$$N_w(K, T) = \lim_{n\to\infty} \sqrt[n]{N\big(K^n, T^n\big)}. \tag{4}$$

The paper in organized as follows. In Section 2 we show Theorem 6. In Section 3 we prove Theorem 7, which relies on linear programming duality. This requires several technical lemmas and continuity results. Finally, we give the proof of Theorem 8 in Section 4, which again requires some technical lemmas in addition to the study of how covering and weighted covering interact with direct products. Some of the proofs of the more technical lemmas are included in Appendix A.

2. Weighted covering is equivalent to the classical covering

Theorem 6 is a consequence of (the right-hand side of) the following inequality, which is proved in the second part of this section:

$$N_w(K, 2T) \leqslant M_w(K, T) \leqslant N_w(K, T). \tag{5}$$

This is the weighted version of inequality (1) above. This inequality is joined with the trivial inequalities:

$$N_w(K, T) \leqslant N(K, T), \tag{6}$$

$$M(K, T) \leqslant M_w(K, T). \tag{7}$$

Proof of Theorem 6. Indeed,

$$N(K, 2T) \leqslant M(K, T) \leqslant M_w(K, T) \leqslant N_w(K, T) \leqslant N(K, T). \qquad \square$$

We give the proof of (5) in the next two propositions.

Proposition 9. *Let K and T be two convex bodies in $\mathbb{R}^d$, T is symmetric, then*

$$N_w(K, 2T) \leqslant M_w(K, T). \tag{8}$$

More generally, for non-symmetric T

$$N_w(K, T - T) \leqslant M_w(K, T). \tag{9}$$

Proof. Fix $\varepsilon > 0$ and let $S = \{(x_i, w_i)\}_{i=1}^M$ be a weighted T-separated set in K with weight greater than $M_w(K, T) - \varepsilon$. We temporarily put $w_T(x)$ to denote x's weight in S when considered as a weighted T-separated set, and put $w_{T-T}(x)$ to denote x's weight in S when considered as a weighted covering with $T - T$. Then we claim that for every $x \in K$, $w_{T-T}(x) \geqslant 1 - \varepsilon$. Suppose otherwise that there is $x_0 \in K$ with $w_{T-T}(x_0) < 1 - \varepsilon$. This means that after adding $(x_0, 1 - w_{T-T}(x_0))$ to S, we still remain with a T-separated set in K. But since its total weight comes up to be greater than $M_w(K, T)$, we get a contradiction. It is possible to complete S to a weighted covering (of K by $T - T$) by adding to it the set $S_1 \times \{\varepsilon\}$, where S_1 is a classical covering of K by $T - T$. We therefore conclude that $N_w(K, T - T) \leqslant M_w(K, T) + C\varepsilon$ (with say $C = N(K, T - T)$). Since ε is arbitrary the result follows. □

For the proof of the right-hand side inequality, we need two simple definitions which will also be useful later on, of ε-covering and ε-separation numbers.

Definition 10. A sequence of pairs $\{(x_i, w_i)\}_{i=1}^N$ will be called an ε-*covering* if it is a weighted covering and $w_i = \varepsilon$ for all i. For a fixed ε, we denote the infimal (minimal) total weight over all ε-coverings of K by T by $N_\varepsilon(K, T)$.

A sequence of pairs $\{(x_i, w_i)\}_{i=1}^N$ will be called an ε-*separation* if it is a weighted T-separated set and $w_i = \varepsilon$ for all i. For a fixed ε, we denote the supremal (in fact, maximal) total weight over all ε-coverings of K by T by $N_\varepsilon(K, T)$.

In the proof we use the two following inequalities, which are simple, and are proved as Lemma 20 and Lemma 21 in Appendix A.

$$N_w(K, T) = \lim_{\varepsilon \to 0^+} N_\varepsilon(K, T), \tag{10}$$

$$M_w(K, T) = \lim_{\varepsilon \to 0^+} M_\varepsilon(K, T). \tag{11}$$

Proposition 11. *Let K, T be two convex bodies in $\mathbb{R}^d$, where $T = -T$ is symmetric. Then*

$$M_w(K, T) \leqslant N_w(K, T). \tag{12}$$

Proof. We will show that

$$N_{\varepsilon_1}(K, T) \geqslant M_{\varepsilon_2}(K, -T) \tag{13}$$

for any $\varepsilon_1 = 1/K_1, \varepsilon_2 = 1/K_2$ with $K_1, K_2 \in \mathbb{N}$. Then by letting K_1 and K_2 tend to ∞ on both sides, using Lemma 20 and Lemma 21 we will conclude $N_w(K, T) \geqslant M_w(K, -T)$ as needed.

Assume the contrary, namely that there exist an ε_1-covering of K by T,

$$S_1 = \{(x_i, \varepsilon_1)\}_{i=1}^{N}$$

and an ε_2-$(-T)$-separated set in K,

$$S_2 = \{(y_j, \varepsilon_2)\}_{j=1}^{M}$$

such that $N\varepsilon_1 < M\varepsilon_2$.

S_1 is covering K and $\{y_j\}_{j=1}^M \subset K$, therefore for each j there exist at least K_1 elements, $\{x_{i_k}\}_{k=1}^{K_1} \subset S_1$, such that $T + x_{i_k}$ contains y_j for each $k \in \{1, \ldots, K_1\}$.

We use the Pigeonhole Principle to show a contradiction. We have MK_1 points (K_1 points for each y_j in S_2) to put in N "buckets" (points of S_1). The assumption $N\varepsilon_1 < M\varepsilon_2$ implies $\frac{MK_1}{N} > K_2$, this means that there is a point $x_* \in S_1$ such that $T + x_*$ contains more than K_2 points of S_2, say $\{y_{j_k}\}_{k=1}^{K_2+1}$.

Equivalently, $x_* \in -T + y_{j_k}$ for each $k \in \{1, \ldots, K_2 + 1\}$. Then in the ε_2-$(-T)$-separated set S_2, $w(x_*) > 1$, which is a contradiction. We then conclude that $N_{\varepsilon_1}(K, T) \geqslant M_{\varepsilon_2}(K, -T)$ for any $\varepsilon_1, \varepsilon_2$ of the above form. $\square$

3. Duality: weighted covering is equal weighted separation

In this section we prove Theorem 7. The main tool for the proof of Theorem 7 is the well-known duality theorem of linear programming, which we next quote.

Theorem 12. *Let A be a real $m \times d$ matrix, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^d$. If at least one of the extreme* $\sup\{\langle c, x\rangle\colon\ Ax \leqslant b,\ x \geqslant 0\}$ *and* $\inf\{\langle y, b\rangle\colon\ yA \geqslant c,\ y \geqslant 0\}$ *is attained, then so is the other and*

$$\max\{\langle c, x\rangle\colon\ Ax \leqslant b,\ x \geqslant 0\} = \min\{\langle y, b\rangle\colon\ yA \geqslant c,\ y \geqslant 0\},$$

where $\langle\cdot,\cdot\rangle$ is the standard Euclidean inner product.

For reference see any standard Linear Programming book, for example [2]. To use the above theorem, we need to "discretize" our notions. Note that in the definition of weighted covering and separation, the covering and covered set can be any set, not necessarily convex. We thus may define, for $\Lambda \subset K$ (which will later be chosen to be some net)

$$N_w(K, T, \Lambda) := N_w(K \cap \Lambda, T),$$

$$M_w(K, T, \Lambda) := M_w(K \cap \Lambda, T).$$

We claim that for the discretized notion, linear programming duality translates to equality of covering and separation.

Theorem 13. *Let $K, T \subseteq \mathbb{R}^d$ with $T = -T$ and let $\Lambda \subset K$ be a finite subset of K. Then*

$$N_w(K, T, \Lambda) = M_w(K, T, \Lambda). \tag{14}$$

More generally, for non-symmetric T

$$N_w(K, T, \Lambda) = M_w(K, -T, \Lambda). \tag{15}$$

The proof will be a direct translation of the assumptions of the theorem to the language of vectors and matrices in which Theorem 12 can be applied.

Proof. Assume $\Lambda = \{x_i\}_{i=1}^n$. Let M be the $n \times n$ matrix given by

$$M_{i,j} = \begin{cases} 1, & x_i \in x_j + T, \\ 0, & \text{otherwise.} \end{cases}$$

Assigning weights $w = \{w_j\}_{j=1}^n$ to the columns, the condition of covering is translated to $(Mw)_i \geqslant 1$ for all i. The weighted cover $N(K, T, \Lambda)$ is thus the infimum over all such (non-negative) w of $\sum_j w_j$.

Using Theorem 12, we get that $N(K, T, \Lambda)$ is equal to the maximum of $\sum_i w_i$ under the condition $w_i \geqslant 0$ and $(w^T M)_j \leqslant 1$ for all j. This, in turn, means assigning weights w_j to the rows, and summing, for each column x_j, the weights of the rows x_i so that $x_i \in x_j + T$ which is the same as $x_j \in x_i - T$. All these summed weights must add up to at most 1. So we take the maximal $-T$ separated set in $K \cap \Lambda$. □

In order to use the discretized version, we need two simple lemmas regarding nets. A set $\Lambda(\delta) \subseteq K$ is called a δ-net of K if for every $x \in K$ there exists $x_i \in \Lambda$ such that $|x - x_i| \leqslant \delta$. In other words, if $K \subseteq \Lambda + \delta D$ where D is the Euclidean ball.

Lemma 14. *Let K and T be convex bodies in $\mathbb{R}^d$, $T = -T$, and let $\Lambda(\delta)$ be a δ-net for K then*

$$N_w(K, T + \delta D) \leqslant N_w\big(K, T, \Lambda(\delta)\big). \tag{16}$$

Indeed, using well-known inequalities for covering numbers (which translate directly to weighted covering) we see that

$$N_w\big(K, T, \Lambda(\delta)\big) = N_w\big(K \cap \Lambda(\delta), T\big) \geqslant N_w\big(K \cap \Lambda(\delta) + \delta D, T + \delta D\big) \geqslant N_w(K, T + \delta D).$$

Lemma 15. *Let K and T be two convex bodies in $\mathbb{R}^d$, and let $\Lambda(\delta)$ be a δ-net for K then*

$$M_w(K, T) \geqslant M_w\big(K, T + \delta D, \Lambda(\delta)\big). \tag{17}$$

Indeed, let S be a (weighted) separated set for $K \cap \Lambda$ by $T + \delta D$. Then it is easy to check that S also forms a weighted T-separated set in K, for if there is too much weight on a point $x \in K$, there would have been too much weight on the closest net-point.

Proof of Theorem 7. We use the two above lemmas, together with Theorem 13, as follows: Let $\Lambda(\delta_n)$ be a sequence of δ_n-nets for K with $\delta_n \to 0^+$. For each n we have

$$\begin{aligned} M_w(K, T) &\geqslant M_w\big(K, T + \delta_n D, \Lambda(\delta_n)\big) \\ &= N_w\big(K, T + \delta_n D, \Lambda(\delta_n)\big) \\ &\geqslant N_w(K, T + 2\delta_n D). \end{aligned} \tag{18}$$

Thus

$$M_w(K, T) \geqslant \lim_{n\to\infty} N_w(K, T + 2\delta_n D). \tag{19}$$

To complete the proof of Theorem 7 we need the following continuity result, which is proved in Proposition 22 in Appendix A.

$$\lim_{\delta\to 0^+} N_w(K, T + \delta D) = N_w(K, T). \tag{20}$$

Taking Eqs. (19) and (20) into account, together with Proposition 11, the proof of Theorem 7 is thus complete. □

4. An alternative definition to weighted covering

In this section we prove Theorem 8. For the proof we use the next proposition.

Proposition 16. *Let $K_1, T_1 \subset \mathbb{R}^{d_1}$, $K_2, T_2 \subset \mathbb{R}^{d_2}$ be convex bodies. Then we have*

$$N_w(K_1 \times K_2, T_1 \times T_2) = N_w(K_1, T_1) N_w(K_2, T_2), \tag{21}$$

$$M_w(K_1 \times K_2, T_1 \times T_2) = M_w(K_1, T_1) M_w(K_2, T_2). \tag{22}$$

Proof. We first show the two inequalities

$$N_w(K_1 \times K_2, T_1 \times T_2) \leqslant N_w(K_1, T_1) N_w(K_2, T_2), \tag{23}$$

$$M_w(K_1, T_1) M_w(K_2, T_2) \leqslant M_w(K_1 \times K_2, T_1 \times T_2). \tag{24}$$

The two simply follow from the fact that the direct product of two covers is a cover, and the direct product of two separated sets is a separated set. Indeed, given $S_1 = \{(x_i, w_i)\}_{i=1}^{N_1}$ which a weighted covering of K_1 by T_1 and $S_2 = \{(y_j, u_j)\}_{j=1}^{N_2}$ a weighted covering of K_2 by T_2, it is easy to check that the set

$$S = \big\{\big((x_i, y_j), w_i u_j\big)\big\}_{i,j}$$

is a weighted covering of $K_1 \times K_2$ by $T_1 \times T_2$, with total weight $w(S) = w(S_1) w(S_2)$. Similarly for separated sets. This proves both inequalities. Finally, using Theorem 7, we get

$$\begin{aligned} N_w(K_1 \times K_2, T_1 \times T_2) &\leqslant N_w(K_1, T_1) N_w(K_2, T_2) \\ &= M_w(K_1, -T_1) M_w(K_2, -T_2) \\ &\leqslant M_w\big(K_1 \times K_2, (-T_1) \times (-T_2)\big) \\ &= N_w\big(K_1 \times K_2, -\big((-T_1) \times (-T_2)\big)\big) \\ &= N_w(K_1 \times K_2, T_1 \times T_2). \end{aligned} \tag{25}$$

Implying that all inequalities in (25) are equalities. The statement then follows. □

We will also use the next lemma which is proved (in a more general setting) in Appendix A (Corollary 25 there).

Lemma 17. *Let S be a weighted covering of K by T. Then there exists a finite set $K' \subset K$, such that for every $x \in K^n$ there exists $y(x) \in (K')^n$ such that for any sub-covering $\bar{S} \subset S^n$, x is covered by $\bar{S}$ if and only if $y(x)$ is.*

Proof of Theorem 8. We first note that the limit exists and is equal to the infimum over n. Indeed, it is easy to see that $N(K^{n+m}, T^{n+m}) \leqslant N^n(K, T) N^m(K, T)$ (as in the first part of the proof of Proposition 16, with the weights now all being 1), and using the property of sub-multiplicative sequences, the limit $\lim_{n\to\infty} \sqrt[n]{N(K^n, T^n)}$ exists and is equal to the infimum of the sequence.

The fact that this limit is at least the weighted covering number is easy. Indeed, Proposition 16 yields that for every n,

$$N_w(K, T) = \sqrt[n]{N_w\big(K^n, T^n\big)} \leqslant \sqrt[n]{N\big(K^n, T^n\big)}.$$

For the opposite inequality, fix $\varepsilon > 0$ and let S be an ε-covering of K by T. Fix $\delta > 0$ (arbitrarily small) and choose a set $M \subset S^n$ of cardinality

$$L = \ulcorner \big(w(S)(1+\delta)\big)^n \urcorner$$

by selecting elements from S^n, randomly and independently, according to the uniform distribution (assigning probability $[\varepsilon/(w(S))]^n$ to each element in S^n).

Denote by $K' \subset K$ the finite set given by Lemma 17. We next show that with positive probability, M is a (classical) covering of $(K')^n$ by T^n (and thus, by Lemma 17, also a covering of K^n).

Let $x = (x_1, \ldots, x_n) \in (K')^n$ and $\bar{s} = (s_1, \ldots, s_n) \in M$. We have for every i, as x_i is covered by at least $1/\varepsilon$ elements in S, that

$$\Pr(x_i \in T + s_i) \geqslant \frac{1}{\varepsilon} \frac{\varepsilon}{w(S)} = \frac{1}{w(S)}$$

and therefore

$$\Pr\big(x \in T^n + \bar{s}\big) \geqslant \left(\frac{1}{w(S)}\right)^n.$$

For a fixed x, the probability that no point in M gives a translation that covers x is then bounded by

$$\left[1-\left(\frac{1}{w(S)}\right)^n\right]^L \leqslant \left(\frac{1}{2}\right)^{(1+\delta)^n}$$

where the last inequality is true for sufficiently large n (because $\lim_{t\to 0}(1-t)^{1/t} = 1/e < 1/2$).

Since we have $(|K'|)^n$ elements to cover, the probability that M doesn't form a covering of K'^n is less than

$$\frac{(|K'|)^n}{2^{(1+\delta)^n}}$$

which is less than 1 (and actually goes to 0) for n large enough (and depending on δ and $|K'|$). For such an n the probability that M is a covering of $(K')^n$ is positive and hence at least one such a covering exists. Lemma 17 gives us that M is then a covering of K^n. We thus found, for sufficiently large n, a classical covering of K^n by T^n with L elements, and hence

$$\lim \sqrt[n]{N\big(K^n, T^n\big)} \leqslant w(S)(1+\delta)$$

with $\delta > 0$ arbitrary small. This gives us

$$\lim_{n\to\infty} \sqrt[n]{N\big(K^n, T^n\big)} \leqslant w(S)$$

for every ε-covering S of K by T, with any $\varepsilon > 0$.

Finally, the continuity-type result Lemma 20 from Appendix A implies that, taking the limit as $\varepsilon \to 0^+$, we get

$$\lim_{n\to\infty} \sqrt[n]{N\big(K^n, T^n\big)} \leqslant N_w(K, T)$$

as needed. □

To end this section, we give a corollary of the above reasoning, giving a criterion for when there is equality of covering and fractional covering. It turns out that in such cases, the classical covering numbers have a multiplicativity (rather than just sub-multiplicativity) property.

Corollary 18. *Let K and T be convex bodies in $\mathbb{R}^d$, T symmetric $T = -T$. Then*

$$N_w(K, T) = N(K, T) \tag{26}$$

if and only if, for any $m \in \mathbb{N}$ and any two convex bodies G_1 and G_2 in $\mathbb{R}^m$ with $G_2 = -G_2$ symmetric,

$$N(K \times G_1, T \times G_2) = N(K, T)N(G_1, G_2). \tag{27}$$

The proof of this inequality follows from the following inequality, which is in itself quite interesting, as it is of a super-multiplicativity type (usual covering number exhibit only sub-multiplicativity result).

Proposition 19. *Let $K_1, T_1 \subset \mathbb{R}^{d_1}$, $K_2, T_2 \subset \mathbb{R}^{d_2}$ be convex bodies. Then*

$$N(K_1 \times K_2, T_1 \times T_2) \geqslant N_w(K_1, T_1) N(K_2, T_2). \tag{28}$$

Proof. Let

$$S_1 = \{(x_i, y_i)\}_{i=1}^{N}$$

be a (classical) covering of $K_1 \times K_2$ by $T_1 \times T_2$. We claim that

$$S = \{(x_i, w_i): N(K_2, T_2) w_i = |\{y: (x_i, y) \in S_1\}|\}$$

(no multiple elements here) forms a weighted covering of K_1 by T_1. Indeed, let $x \in K_1$, then $\{x\} \times K_2$ is covered, say by $S_x = \{(x_{i_k}, y_{i_k})\}_{k=1}^{|S_x|} \subset S_1$. In particular $\{y_{i_k}\}_{k=1}^{|S_x|}$ forms a covering of K_2 by T_2. Hence, $|S_x| \geqslant N(K_2, T_2)$ and $w(x) \geqslant 1$. This means that S is a covering and its total weight is

$$w(S) = \frac{|S_1|}{N(K_2, T_2)} \leqslant \frac{N(K_1 \times K_2, T_1 \times T_2)}{N(K_2, T_2)}. \qquad \square$$

Proof of Corollary 18. We first prove that (27) implies (26). Indeed, let $G_1 = K^n$, $G_2 = T^n$, then by (27) we have

$$N(K \times K^n, T \times T^n) = N(K, T) N(K^n, T^n)$$

with arbitrary $n \in \mathbb{N}$. This implies that for any n

$$N(K^n, T^n) = N^n(K, T).$$

Together with Theorem 8 we get (26). For the other direction let G_1 and G_2 be any two convex bodies with $G_2 = -G_2$. By Proposition 19 and (26) we have

$$N(K \times G_1, T \times G_2) \geqslant N_w(K, T) N(G_1, G_2) = N(K, T) N(G_1, G_2).$$

The opposite inequality

$$N(K \times G_1, T \times G_2) \leqslant N(K, T) N(G_1, G_2)$$

follows by the simple fact that for S_1, S_2 coverings of K by T and of G_1 by G_2 respectively, $S_1 \times S_2$ is a covering of $K \times G_1$ by $T \times G_2$ (with $|S_1||S_2|$ elements). $\square$

Acknowledgment

The first named author would like to thank Prof. N. Linial for suggesting the question of studying the analogues of combinatorial fractional covering in convexity.

Appendix A

The first two propositions deal with a continuity type result, stating that the limit of ε-covering or separation numbers are weighted covering or separation numbers.

Lemma 20. *Let K, T be two convex bodies in $\mathbb{R}^d$. Then for every $1 > r > 0$, $0 < \varepsilon < r/2N_w(K, T)$*

$$N_w(K,T) \leqslant N_\varepsilon(K,T) \leqslant N_w(K,T) + 2r. \tag{29}$$

In particular,

$$N_w(K,T) = \lim_{\varepsilon \to 0^+} N_\varepsilon(K,T) \tag{30}$$

and the limit exists.

Lemma 21. *Let K, T be two convex bodies in $\mathbb{R}^d$, where $T = -T$ is symmetric. Then*

$$M_w(K,T) = \lim_{\varepsilon \to 0^+} M_\varepsilon(K,T) \tag{31}$$

and in particular the limit exists.

Proof of Lemma 20. Clearly,

$$N_w(K,T) \leqslant N_\varepsilon(K,T) \tag{32}$$

because an ε-covering is in particular a weighted covering. Let $r > 0$. We take $S = \{(x_i, w_i)\}_{i=1}^N$ a weighted covering of K by T with $w(S) \leqslant N_w(K,T) + r$. Then for every $\varepsilon < r/N =: \varepsilon_0$ we may take $\{\bar{w}_i\}_{i=1}^N$ where $\bar{w}_i = m_i \varepsilon$ for some $m_i \in \mathbb{N}$ and $0 \leqslant \bar{w}_i - w_i \leqslant r/N$ (for every $x \in \mathbb{R}$ and $\varepsilon > 0$, one of the intervals $[0, \varepsilon] + m\varepsilon$ with $m \in \mathbb{N}$ contains x, and therefore exists $m_{x,\varepsilon} \in \mathbb{N}$ such that $m_{x,\varepsilon}\varepsilon$ is close to x, say from the right, up to ε). Then the sequence

$$S_1 = \{(x_i, \bar{w}_i)\}_{i=1}^N$$

is an ε-covering of K by T, because we only enlarged some of the weights in S, and also all the weights are multiples of ε. In addition, $w(S_1) \leqslant w(S) + r$ because $\sum_{i=1}^N (\bar{w}_i - w_i) \leqslant r$. Thus for every $r > 0$ we found an $\varepsilon_0 > 0$ such that $0 < \varepsilon < \varepsilon_0$ implies

$$N_w(K,T) \leqslant N_\varepsilon(K,T) \leqslant N_w(K,T) + 2r. \qquad \square$$

Proof of Lemma 21. The proof is similar to the one of Lemma 20. Clearly, $M_w(K,T) \geqslant M_\varepsilon(K,T)$ because T-ε-separated set is in particular a weighted T-ε-separated set. Let $r > 0$. We take $S = \{(x_i, w_i)\}_{i=1}^M$ to be a weighted T-separated set in K with $M_w(K,T) - r < w(S)$. Then for every $\varepsilon < r/M =: \varepsilon_0$ we may take $\{\bar{w}_i\}_{i=1}^M$ where $\bar{w}_i = m_i\varepsilon$ for some $m_i \in \mathbb{N}$ and $0 \leqslant w_i - \bar{w}_i \leqslant r/M$. Then the sequence

$$S_1 = \{(x_i, \bar{w}_i)\}_{i=1}^M$$

is a T-ε-separated set in K, because we only reduced some of the weights in S and also all the weights are multiples of ε. In addition, $w(S) - r \leqslant w(S_1)$ because $\sum_{i=1}^M (w_i - \bar{w}_i) \leqslant r$. Then for every $r > 0$ we found an $\varepsilon_0 > 0$ such that $0 < \varepsilon < \varepsilon_0$ implies

$$M_w(K,T) - 2r \leqslant M_\varepsilon(K,T) \leqslant M_w(K,T).$$

That is, by definition,

$$\lim_{\varepsilon \to 0^+} M_\varepsilon(K,T) = M_w(K,T). \qquad \square$$

Proposition 22. *Let K, T be two convex bodies in $\mathbb{R}^d$, and let D denote, as usual, the Euclidean unit ball in $\mathbb{R}^d$. Then*

$$\lim_{\delta \to 0^+} N_w(K, T + \delta D) = N_w(K,T). \tag{33}$$

For the proof we need the following lemma, which is a version of Proposition 22 for ε-covering.

Proposition 23. *Let K, T be two convex bodies in $\mathbb{R}^d$, and let D denote the usual Euclidean unit ball in $\mathbb{R}^d$. Then*

$$\lim_{\delta \to 0^+} N_\varepsilon(K, T + \delta D) = N_\varepsilon(K,T). \tag{34}$$

Proof. Let $\delta_n \to 0$. For each n, let us choose an ε-covering of K by $T + \delta_n D$ of weight $N_\varepsilon(K, T + \delta_n D)$. This simply means choosing $N_\varepsilon(K, T + \delta_n D)/\varepsilon$ centers of covering. Note that this number is bounded by $N = N_\varepsilon(K,T)/\varepsilon$. It is easy to see that we may choose a subsequence of this sequence of coverings which converges in the sense that the location of the kth center converges, for $k = 1, \ldots, N$.

We claim that the limiting points in K form an ε-covering of K by T. Indeed, suppose that $x \in K$ is not covered, then there exists a positive distance, say $r > 0$, between x and all the translations of T (since T is closed). By choosing N_0 large enough, we get that the distance between x and the cover of K, using the same set of centers (the limiting points) by $T + \delta_n D$ is also bounded from below, say by $r/2$, for all $n \geqslant N_0$. However, using the fact that the centers are the limits of the centers of covering in the subsequence, we have that for n large enough, the distance between the actual centers of the covering and these limit-centers in less than $r/2$, which means that x is also not covered by $T + \delta_n D$ with the original centers, and this is, of course, a contradiction. $\square$

Proof of Proposition 22. We immediately have

$$\lim_{\delta\to 0^+} N_w(K, T+\delta D) \leqslant N_w(K,T)$$

since $N_w(K, T+\delta D) \leqslant N_w(K,T)$ for every positive δ. For the opposite inequality, suppose that there is a constant $c>0$ such that

$$\lim_{\delta\to 0} N_w(K, T+\delta D) \leqslant N_w(K,T) - c. \tag{35}$$

Then, by (29), for every $r>0$, $0<\delta<1$ and every $\varepsilon \leqslant \frac{r}{N_w(K,T)}$

$$N_\varepsilon(K, T+\delta D) \leqslant N_w(K, T+\delta D) + r.$$

By choosing $r = c/2$, we get by (35) that for all $0<\delta<1$

$$N_\varepsilon(K, T+\delta D) \leqslant N_w(K,T) - c + c/2,$$

taking δ to 0 using Proposition 23, we get

$$N_\varepsilon(K,T) \leqslant N_w(K,T) - c/2,$$

for any $\varepsilon \leqslant \frac{c/2}{N_w(K,T)}$ which is a contradiction. □

Lemma 24. *Let S be a covering of K by T. Then there exists a finite subset $K' \subset K$ (depending on S) such that for every $x \in K$ there exists $y(x) \in K'$ such that for a translated copy of T in S, say $s+T$, we have that*

$$x \in T+s \quad \textit{if and only if} \quad y(x) \in T+s.$$

Proof. Let N be the number of elements in S. For every $x \in K$ we match a vector $v_x \in \{0,1\}^N$ with ith coordinate 1 if T translated to the ith element in S intersects x, and 0 otherwise. We define an equivalence relation for $x, y \in K$ by $x \sim y$ if $v_x = v_y$. Since v_x has only $2^N - 1$ possibilities (every element is covered, so that no x is mapped to the all-0 vector), the quotient set of K by $\sim$ is finite. Now take K' to be a set containing exactly one element from each non-empty equivalence class. By the definition of $\sim$, for a given translation $T+s$ in S, $x \in T+s$ implies $y \in T+s$ for every $y \in [x]$, and the proof is complete. □

Corollary 25. *Let S_i be a covering of K_i by T_i where $i = 1,\dots,n$. Then there exist finite sets $K_i' \subset K_i$, $i=1,\dots,n$, depending only on S_i and K_i, such that for every $x \in K_1 \times \cdots \times K_n$ there exists $y(x) \in K_1' \times \cdots \times K_n'$ such that for any subcover $\bar{S} \subset S_1 \times \cdots \times S_n$ we have that x is covered by $\bar{S}$ if and only if $y(x)$ is covered by $\bar{S}$.*

Proof. We choose K_i' by Lemma 24. We then define for $x = (x_1,\dots,x_n) \in K_1 \times \cdots \times K_n$ the point $y(x) = (y_1(x_1),\dots,y_n(x_n)) \in K_1' \times \cdots \times K_n'$ where y_i is the function from K_i to K_i' given in Lemma 24. Assume that $x \in T_1 \times \cdots \times T_n + \bar{s}$ with $\bar{s} = (s_1,\dots,s_n) \in S_1 \times \cdots \times S_n$ then for every i we have $x_i \in T_i + s_i$. By Lemma 24 this happens if and only if $y_i(x_i) \in T_i + s_i$, which implies that $y(x) = (y_1(x_1), \ldots, y_n(x_n)) \in \bar{s} + T_1 \times \cdots \times T_n$. Thus, x is covered by $\bar{S}$ if and only $y(x)$ is. $\square$

References

[1] S. Artstein, V. Milman, S.J. Szarek, Duality of metric entropy, Ann. Math. 159 (2004) 1–16.
[2] George B. Dantzig, Mukund N. Thapa, Linear Programming. 1. Introduction, Springer Ser. Oper. Res., ISBN 0-387-94833-3, Springer-Verlag, New York, 1997.
[3] R.M. Dudley, The sizes of compact subsets of Hilbert space and continuity of Gaussian processes, J. Funct. Anal. 1 (1967) 290–330.
[4] J. Kuelbs, W.V. Li, Metric entropy and the small ball problem for Gaussian measures, J. Funct. Anal. 116 (1993) 133–157.
[5] M. Ledoux, Isoperimetry and Gaussian analysis, in: Lectures on Probability Theory and Statistics, Saint-Flour, 1994, in: Lecture Notes in Math., vol. 1648, Springer-Verlag, New York, 1996, pp. 165–294.
[6] L. Lovász, On the ratio of optimal integral and fractional covers, Discrete Math. 13 (4) (1975) 383–390.
[7] A. Pietsch, Theorie der Operatorenideale (Zusammenfassung), Friedrich-Schiller-Universität Jena, 1972.
[8] G. Pisier, The Volume of Convex Bodies and Banach Space Geometry, Cambridge Tracts in Math., vol. 94, Cambridge Univ. Press, Cambridge, 1989.
[9] O. Raz, M.Sc. Thesis, Tel Aviv University, 2011.
[10] Edward R. Scheinerman, Daniel H. Ulman, Fractional Graph Theory, John Wiley & Sons, 2008.
[11] C. Schütt, Entropy numbers of diagonal operators between symmetric Banach spaces, J. Approx. Theory 40 (1984) 121–128.