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A characterization of the support map ☆

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Abstract

In this short note we give a characterization of the support map from classical convexity. We show it is the unique additive transformation from the class of closed convex sets in R^n which include 0 to the class of positive 1-homogeneous functions on R^n. This will be a consequence of a theorem about transforms from the class of convex sets to itself which preserve Minkowski addition.

Keywords: Characterization; Support

1. Introduction

In this short note we give a characterization of the support map from classical convexity.

To state the results, we need first some notation. We denote the class of all closed convex sets in $\mathbb{R}^n$ by $\mathcal{K}^n$. The subclass of the sets in $\mathcal{K}^n$ which contain 0 (possibly at the boundary) will be denoted by $\mathcal{K}_0^n$. For this subclass, the operation of duality is well defined and involutive, namely if $K \in \mathcal{K}_0^n$ then after fixing some scalar product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^n$, its dual set

$$K^\circ = \{x \in \mathbb{R}^n\colon\ \langle x, y\rangle \leqslant 1 \text{ for all } y \in K\}$$

is also in $\mathcal{K}_0^n$, and moreover, $(K^\circ)^\circ = K$.

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In this note we mainly restrict our attention to the above two classes, however there are several other natural classes which are frequently used, the class $\mathcal{K}^n_{0,b}$ consisting of bounded (thus compact) sets in $\mathcal{K}^n$ which include 0, and the class $\mathcal{K}^n_{(0),b}$ which consists of sets in $\mathcal{K}^n_{0,b}$ which include 0 in their interior (again, this is a class which is invariant under the operation of duality).

For a body $K \in \mathcal{K}^n$, the support function $h_K : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is defined by

$$h_K(u) = \sup_{x \in K} \langle x, u \rangle.$$

(In fact, if we allow the empty set to belong to $\mathcal{K}^n$ then formally it makes sense to let $h_\phi \equiv -\infty$, but we omit the empty set from $\mathcal{K}^n$ and do not have to handle this small technical detail.)

For the subclass $\mathcal{K}^n_0$, we see that h_K is a convex non-negative function, and it is also 1-homogeneous. It is easy to see that in fact $K^\circ = \{u\colon\ h_K(u) \leqslant 1\}$ and h_K is the norm associated with the dual body of K. Note that although usually the term "body" is reserved for compact convex sets, and "norm" is reserved for even convex 1-homogeneous functions which do not assume the value $+\infty$ and assume the value 0 only at 0, here we use the more general (and less standard) definition, allowing the (convex) function to be only positively homogeneous, to be 0 on a large set, so that the body which is its unit ball can be unbounded, and to have 0 on the boundary of the "unit ball" so that the associated "norm" can assume the value $+\infty$.

The well-known operation of Minkowski addition is defined on the class $\mathcal{K}^n$ as follows

$$K_1 + K_2 = \{x_1 + x_2\colon\ x_1 \in K_2,\ x_2 \in K_2\},$$

and the class $\mathcal{K}^n_0$ is closed under this operation, namely the Minkowski sum of two sets in $\mathcal{K}^n_0$ is again in $\mathcal{K}^n_0$.

It is easy to verify that the support map is additive with respect to the operation of Minkowski addition, namely for $K_1, K_2 \in \mathcal{K}^n$

$$h_{K_1+K_2}(u) = h_{K_1}(u) + h_{K_2}(u).$$

In this note, we show that up to a linear transformation, this property characterizes the support function. We need one more notation before stating our main theorem. Let $\mathcal{H}^n$ stand for support functions of bodies in $\mathcal{K}^n_0$, namely

$$\mathcal{H}^n = \bigl\{\text{positively homogeneous convex functions } h : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0} \cup \{+\infty\},\ h(0) = 0\bigr\}.$$

We prove the following theorem.

Theorem 1. *Let $n \geqslant 2$. Assume we are given a bijective transform $\mathcal{T} : \mathcal{K}^n_0 \to \mathcal{H}^n$ satisfying for every $K_1, K_2 \in \mathcal{K}^n_0$ that*

$$\mathcal{T}(K_1 + K_2) = \mathcal{T}K_1 + \mathcal{T}K_2.$$

Then $\mathcal{T}$ is essentially the standard support function, namely there exists a linear transformation $B \in GL_n$ such that for all $K \in \mathcal{K}^n_0$

$$\mathcal{T}K = h_{BK}.$$

For dimension $n = 1$, the assumption implies that there are two constants $c > 0$ and $d > 0$ such that either for all $x, y \geqslant 0$ we have $\mathcal{T}([-y, x]) = h_{[-cy,dx]}$ or for all $x, y \geqslant 0$ we have $\mathcal{T}([-y, x]) = h_{[-cx,dy]}$.

Theorem 1 is a direct corollary of the following theorem.

Theorem 2. *Let $n \geqslant 2$. Assume we are given a bijective transform $\mathcal{T} : \mathcal{K}_0^n \to \mathcal{K}_0^n$ satisfying for every $K_1, K_2 \in \mathcal{K}_0^n$ that*

$$\mathcal{T}K_1 + \mathcal{T}K_2 = \mathcal{T}(K_1 + K_2).$$

Then, there exists a linear transformation $B \in GL_n$ such that for all K

$$\mathcal{T}K = BK.$$

For dimension $n = 1$, the assumption implies that there are two constants $c > 0$ and $d > 0$ such that either for all $x, y \geqslant 0$ we have $\mathcal{T}([-y, x]) = [-cy, dx]$ or for all $x, y \geqslant 0$ we have $\mathcal{T}([-y, x]) = [-cx, dy]$.

The paper is organized as follows. First, we examine the background for Theorem 1. We show how it fits in the more general framework of the theory of convex functions, where we have proved a similar theorem. We then recall a different characterization of the support function in terms of order preserving transforms, which gives rise to the same transform. We then devote Section 3 to the proof of Theorem 2, and Theorem 1, and in Section 4 remark on a few other variants of it. Finally, in Section 5 we sketch the proof of a similar theorem for convex functions and give some final remarks.

2. Background

There are two natural contexts in which one should examine Theorem 1. The first is how it fits in the framework of convex functions. A lot has been written on the topic of extending results from convex bodies to log-concave functions (which are $e^{-\phi}$ for convex ϕ), see for example [1,7] and references therein. Here we have an opposite situation, where the corresponding theorem for convex functions was proven first. We recall this theorem and its relatives in Section 2.1. A second context in which one may view these results is in the frameworks of different characterizations of the support function. A different family of such results was obtained recently by means of order preserving transformations, and we present these theorems in Section 2.2 below.

2.1. *Convex functions*

Let us recall our earlier results about convex functions (which can be also stated in the language of log-concave functions) from [1]. We denote the class of lower-semi-continuous convex functions $\phi : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ by $Cvx(\mathbb{R}^n)$. (There is a slight ambiguity about how convexity fits with functions assuming infinite values; it is reasonable to let the only function actually assuming the value $-\infty$ be the constant $-\infty$ function.) The definition of the inf-convolution of two functions $f, g \in Cvx(\mathbb{R}^n)$ is given by

$$(f \,\square\, g)(z) = \inf_{x+y=z} \big(f(x) + g(y)\big).$$

We need some convention to decide what to do with $-\infty + \infty$. We omit the justification for the following reasonable agreement: $(-\infty)\Box f \equiv -\infty$ for all f, and $(-\infty) + f \equiv -\infty$ for all $f \not\equiv +\infty$, and, as functions, $-\infty + \infty \equiv +\infty$.

The following theorem was proven in the paper [1].

Theorem 3. *Assume we are given a transform $\mathcal{T} : Cvx(\mathbb{R}^n) \to Cvx(\mathbb{R}^n)$ (defined on the whole domain) satisfying for all $f, g \in Cvx(\mathbb{R}^n)$:*

1. $\mathcal{T}\mathcal{T} f = f$,
2. $\mathcal{T} f + \mathcal{T} g = \mathcal{T}(f \,\Box\, g)$.

Then, there exists a symmetric linear transformation $B \in GL_n$ such that for all f

$$(\mathcal{T} f)(x) = \sup_{y \in \mathbb{R}^n} \big(\langle Bx, y\rangle - f(y)\big).$$

Note that the transform $\mathcal{T}$ which comes up is, up to a linear change induced by B (which can be thought of as a change of coordinates) no other than the well-known Legendre transform, defined by

$$(\mathcal{L} f)(x) = \sup_{y \in \mathbb{R}^n} \big(\langle x, y\rangle - f(y)\big).$$

Note that B need not be positive definite, which corresponds to the fact that one may choose different sets of coordinates in the base and in the target space.

In fact, we were able to prove the more general theorem below. We sketch its proof, for the reader's convenience, in Section 5.

Theorem 4. *Assume we are given a bijective transform $\mathcal{T} : Cvx(\mathbb{R}^n) \to Cvx(\mathbb{R}^n)$ (defined on the whole domain) satisfying for all $f, g \in Cvx(\mathbb{R}^n)$*

$$\mathcal{T} f + \mathcal{T} g = \mathcal{T}(f + g).$$

Then, there exists a constant C_0, a vector v_0 and a linear transformation $B_0 \in GL_n$ such that for all f

$$(\mathcal{T} f)(x) = C_0 f(B_0 x + v_0).$$

As a corollary one then gets the slightly stronger theorem below.

Theorem 5. *Assume we are given a bijective transform $\mathcal{T} : Cvx(\mathbb{R}^n) \to Cvx(\mathbb{R}^n)$ (defined on the whole domain) satisfying for all $f, g \in Cvx(\mathbb{R}^n)$*

$$\mathcal{T} f + \mathcal{T} g = \mathcal{T}(f \,\Box\, g).$$

Then, there exists a constant C_0, a vector v_0 and a linear transformation $B_0 \in GL_n$ such that for all f

$$(\mathcal{T} f)(x) = C_0 (\mathcal{L} f)(B_0 x + v_0).$$

Similarly, if one knows that for all $f, g \in Cvx(\mathbb{R}^n)$

$$\mathcal{T}(f+g) = \mathcal{T}f \,\Box\, \mathcal{T}g$$

then there exists a constant C_1*, a vector* v_1 *and a linear transformation* $B_1 \in GL_n$ *such that for all* f

$$(\mathcal{T}f)(x) = \langle x, v_1\rangle + C_1(\mathcal{L}f)(B_1x).$$

Indeed, to see that Theorem 4 implies Theorem 5 simply consider the transform $\mathcal{T}\mathcal{L}$, which satisfies

$$\mathcal{T}\mathcal{L}(f+g) = \mathcal{T}(\mathcal{L}f \,\Box\, \mathcal{L}g) = \mathcal{T}\mathcal{L}f + \mathcal{T}\mathcal{L}g$$

and apply Theorem 4 to get that $(\mathcal{T}\mathcal{L}f)(x) = C_0 f(B_0x + v_0)$ and thus (using $\mathcal{L}\mathcal{L}f = f$) we have $(\mathcal{T}f)(x) = C_0(\mathcal{L}f)(B_0x + v_0)$. For the second case consider $\mathcal{L}\mathcal{T}$, then

$$\mathcal{L}\mathcal{T}(f+g) = \mathcal{L}(\mathcal{T}f \,\Box\, \mathcal{T}g) = \mathcal{L}\mathcal{T}f + \mathcal{L}\mathcal{T}g$$

so that by Theorem 4 we have that $(\mathcal{L}\mathcal{T}f)(x) = C_0 f(B_0x + v_0)$ and applying $\mathcal{L}$ on both sides we get,

$$(\mathcal{T}f)(x) = \sup_y\big(\langle x, y\rangle - C_0 f(B_0y + v_0)\big) = \langle x, v_1\rangle + C_0\mathcal{L}f(B_1x).$$

Note also that Theorem 5 is stronger than Theorem 3, since in the conditions of Theorems 3 (the involution assumption) $\mathcal{T}$ must satisfy both assumptions $\mathcal{T}f + \mathcal{T}g = \mathcal{T}(f \,\Box\, g)$ and $\mathcal{T}(f + g) = \mathcal{T}f \,\Box\, \mathcal{T}g$. Thus it must be of both forms given in the conclusion of Theorem 5, meaning that both v_0 and v_1 must be 0, and we are left with $f \to C_0\mathcal{L}f \circ B$. Applying it twice and using the involution assumption, we also see that B must be symmetric and that $C_0 = 1$.

Let us restate some of the above theorems in terms of the support function. To this end, we should introduce the class $LC(\mathbb{R}^n)$ of log-concave functions on $\mathbb{R}^n$. This class, which is simply the class of functions of the form $e^{-\phi}$ for $\phi \in Cvx(\mathbb{R}^n)$, are usually seen as a generalization of the class of convex bodies, which can be naturally embedded in this class by taking $K \to 1_K$, its characteristic function, which equals 1 on K and 0 elsewhere (other natural embeddings can be given as well).

Consider the transform $S : LC(\mathbb{R}^n) \to Cvx(\mathbb{R}^n)$ given by $S(f) = \mathcal{L}(-\log f)$. It is easy to check that on the subclass of $LC(\mathbb{R}^n)$ consisting of indicators of convex bodies, we have that $S(1_K) = h_K$. Thus, in this very weak sense, S is an extension of the support function from the class of convex bodies to the class $LC(\mathbb{R}^n)$. In the next section it will be explained why S is the correct extension in terms of order preservation. However, Theorem 5 shows that it is the right extension in terms of additivity. To explain this, one has to recognize what is the meaning of Minkowski addition for log-concave functions. This was thoroughly discussed in [4], where we showed that if one defines the sum of two log-concave functions $f, g \in LC(\mathbb{R}^n)$ by the Asplund product

$$(f \star g)(z) = \sup_{\{x+y=z\}} f(x)g(y)$$

then one gets in particular that $1_{K_1} \star 1_{K_2} = 1_{K_1+K_2}$, so that it is a generalization of Minkowski addition, and more importantly, the Prékopa Leindler inequality (see e.g. [8]) implies a Brunn–Minkowski inequality for the integrals (to state it one has to define also the homothety of a log-concave function, again, see [4] for details).

Our Theorem 5 above implies that S is essentially the only transform $S : LC(\mathbb{R}^n) \to Cvx(\mathbb{R}^n)$ which preserves this additivity.

Theorem 6. *Assume we are given a transform $\mathcal{T} : LC(\mathbb{R}^n) \to Cvx(\mathbb{R}^n)$ (defined on the whole domain) satisfying for all $f, g \in LC(\mathbb{R}^n)$*

$$\mathcal{T}(f \star g) = \mathcal{T} f + \mathcal{T} g.$$

Then, there exists a constant C_0, a vector v_0 and a linear transformation $B_0 \in GL_n$ such that for all f

$$(\mathcal{T} f)(x) = C_0\big(\mathcal{L}(-\log f)\big)(B_0 x + v_0).$$

Indeed, define $\mathcal{T}' : Cvx(\mathbb{R}^n) \to Cvx(\mathbb{R}^n)$ by $\mathcal{T}'\varphi = \mathcal{T}(e^{-\phi})$. Then, since for $f = e^{-\varphi}$ and $g = e^{-\psi}$ we have $f \star g = e^{-(\varphi \Box \psi)}$, we get that the condition on $\mathcal{T}$ implies that

$$\mathcal{T}'(\varphi \Box \psi) = \mathcal{T}\big(e^{-\varphi} \star e^{-\psi}\big) = \mathcal{T} e^{-\varphi} + \mathcal{T} e^{-\psi} = \mathcal{T}'\varphi + \mathcal{T}'\psi$$

so that $\mathcal{T}'$ satisfies the conditions of Theorem 5 and so

$$(\mathcal{T}'\varphi)(x) = C_0(\mathcal{L}\varphi)(B_0 x + v_0).$$

This in turn implies that

$$\mathcal{T} f = C_0\big(\mathcal{L}(-\log f)\big)(B_0 x + v_0).$$

2.2. *Another characterization of the support map*

It is easy to see that the support map, mapping a convex body K to its support function h_K, is an order preserving transformation (with respect to the partial order of inclusion for sets and the partial order of point-wise inequality on functions). In [3] we state the following theorem, which is an easy corollary of a simple variant of Theorem 11 from [2], see also [10] for a full proof and a more general theorem.

Theorem 7. *Let $n \geqslant 2$. Assume a bijection $h : \mathcal{K}_0^n \to \mathcal{H}^n$ is given, which satisfies that for all $K_1, K_2 \in \mathcal{K}_0^n$*

$$K_1 \subseteq K_2 \quad \Leftrightarrow \quad h(K_1) \leqslant h(K_2).$$

Then h is essentially the standard support function, namely there exists a linear $B \in GL_n$ such that for every $K \in \mathcal{K}_0^n$

$$h(K) = h_{BK}.$$

Note that here the assumption on the dimension is crucial, since in dimension 1 any increasing function will do.

A similar theorem holds for the class of compact convex sets which include 0 in the interior, and this is a consequence of a theorem of Böröczky and Schneider from [5]. Also, a similar theorem holds for the Minkowski functional $K \to \|\cdot\|_K$ where order preservation is replaced by order reversion.

Moreover, in [3] we prove that also in the case of log-concave functions, order preservation is a characterization of the generalized support function. More precisely, as before, one extends the operation of support function from the class $\mathcal{K}_0^n$ to the class $LC(\mathbb{R}^n)$ (where the embedding $\mathcal{K}_0^n \subset LC(\mathbb{R}^n)$ is simply $K \to 1_K$). The extension is given by

$$S(f) = \mathcal{L}(-\log f).$$

Again this is an essentially unique order preserving mapping, this time between $LC(\mathbb{R}^n)$ and $Cvx(\mathbb{R}^n)$. More precisely we have shown in [3].

Theorem 8. *Let $n \geqslant 1$. Any bijective mapping $S : LC(\mathbb{R}^n) \to Cvx(\mathbb{R}^n)$ which preserves the (partial) order must be, up to a linear change, the 'support' map S defined above.*

For more information, a discussion and proofs, regarding order preserving transforms, see [3].

3. Proof of Theorem 2

The case $n = 1$ is easy. Since $\{0\} + K = K$ for all K, clearly $\mathcal{T}(\{0\}) = \{0\}$. Similarly, $\mathcal{T}\mathbb{R} = \mathbb{R}$ since $\mathbb{R} + K = \mathbb{R}$ for all K. Next, as $\mathcal{T}\mathbb{R}_{\geqslant 0} + \mathcal{T}\mathbb{R}_{\geqslant 0} = \mathcal{T}\mathbb{R}_{\geqslant 0}$, and since this is true for $\mathbb{R}_{\geqslant 0}$ and also for $\mathbb{R}_{\leqslant 0}$, and apart from $\{0\}$ and $\mathbb{R}$ they are the only sets satisfying $K + K = K$, we have that either $\mathcal{T}\mathbb{R}_{\geqslant 0} = \mathbb{R}_{\geqslant 0}$ and $\mathcal{T}\mathbb{R}_{\leqslant 0} = \mathbb{R}_{\leqslant 0}$ or the other way around. We may assume w.l.o.g. that we are the first case, since we may always compose $\mathcal{T}$ with $K \to -K$. Next, as $[0, x] + \mathbb{R}_{\geqslant 0} = \mathbb{R}_{\geqslant 0}$ for $x \geqslant 0$, we see that $\mathcal{T}[0, x] + \mathbb{R}_{\geqslant 0} = \mathbb{R}_{\geqslant 0}$ and so $\mathcal{T}([0, x]) = [0, a(x)]$ for some function $a : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$, and since

$$\mathcal{T}\big([0, x] + [0, y]\big) = \mathcal{T}\big([0, x + y]\big)$$

we see that $a(x)$ is additive. Also, $a(x)$ must be monotone since for $x > y$ we have that

$$[0, x] = [0, y] + [0, x - y]$$

so that

$$\big[0, a(x)\big] = \mathcal{T}\big([0, x]\big) = \mathcal{T}\big([0, y]\big) + \mathcal{T}\big([0, y - x]\big) \supset \mathcal{T}\big([0, y]\big) = \big[0, a(y)\big].$$

Therefore, $a(x)$ must be linear, as so we have that for some positive constant $a > 0$ we have $\mathcal{T}([0, x]) = [0, ax]$. Doing the same thing for $\mathcal{T}[-x, 0]$, we see that there is another positive constant $b > 0$ for which we have $\mathcal{T}([-x, 0]) = [-bx, 0]$. Thus, for all $x, y \geqslant 0$ we have $\mathcal{T}([-y, x]) = [-by, ax]$. If we composed $\mathcal{T}$ with $K \to -K$, we get that for all $x, y \geqslant 0$ we have $\mathcal{T}([-y, x]) = [-bx, ay]$, as claimed.

We now concentrate on the case $n \geqslant 2$. The proof is divided into several simple steps.

3.1. The inverse map

Note that if $\mathcal{T}$ is additive and bijective then so is $\mathcal{T}^{-1}$, since

$$\mathcal{T}^{-1}(K_1+K_2) = \mathcal{T}^{-1}\big(\mathcal{T}\big(\mathcal{T}^{-1}K_1\big)+\mathcal{T}\big(\mathcal{T}^{-1}K_2\big)\big)$$
$$= \mathcal{T}^{-1}\big(\mathcal{T}\big(\mathcal{T}^{-1}(K_1)+\mathcal{T}^{-1}(K_2)\big)\big) = \mathcal{T}^{-1}(K_1)+\mathcal{T}^{-1}(K_2).$$

3.2. Cones

A set K which satisfies $K+K=K$ is mapped to a set which satisfies $\mathcal{T}K+\mathcal{T}K=\mathcal{T}K$, so we know that cones are mapped to cones, bijectively (because of $\mathcal{T}^{-1}$, which also maps cones to cones). Here by a "cone" we simply mean a closed convex subset C of $\mathbb{R}^n$ which satisfies $C+C=C$, with no non-degeneracy assumption. Moreover, since for two cones C_1 and C_2 we have that $C_1 \subseteq C_2$ if and only if $C_1+C_2=C_2$, so that

$$C_1 \subseteq C_2 \quad \Leftrightarrow \quad C_1+C_2=C_2 \quad \Leftrightarrow \quad \mathcal{T}C_1+\mathcal{T}C_2=\mathcal{T}C_2 \quad \Leftrightarrow \quad \mathcal{T}C_1 \subseteq \mathcal{T}C_2,$$

and since $\mathcal{T}$ restricted to cones is bijective, we have that on the class of cones $\mathcal{T}$ is an order preserving bijection which also implies that it preserves the lattice of cones.

Thus, in dimension $\geqslant 3$ we could use R. Schneider's Theorem from [9] to say that the restriction of $\mathcal{T}$ to cones is linear. However, our theorem also holds in dimension 2, and so we will proceed in a slightly different manner.

3.3. Subsets of cones

For a cone C and any $K \in \mathcal{K}_0^n$ we have that $K \subseteq C$ if and only if $K+C=C$, a relation which is preserved after application of $\mathcal{T}$. Thus, we see that the restriction of $\mathcal{T}$ to any fixed cone C is still a bijection, from all the subsets of the cone C to all the subsets of the cone $\mathcal{T}C$ (again, we use the same for $\mathcal{T}^{-1}$).

3.4. Rays

A ray is a particular case of a cone. We can take an intersection of two cones which is some given ray. It is mapped again to a non-empty intersection of two cones, which also contains a ray. Clearly, no two different rays can be mapped to the same ray since this would contradict injectivity — the image of a ray being a cone it contains at least one ray. The image of a ray cannot contain two rays, because then they would both have to be mapped by $\mathcal{T}^{-1}$ to the same ray so we see that in particular rays are mapped to rays — bijectively.

3.5. Subspaces

Let us start with a subspace L of dimension 1. It can be written at the sum of two opposite rays, $L=R-R$, and so its image is $\mathcal{T}L=\mathcal{T}R+\mathcal{T}(-R)$. Since L does not include any other ray R' (with $R'+L=L$), neither can $\mathcal{T}L$ include any other ray (since it would then be $\mathcal{T}R'$ for some R') and so we see that $\mathcal{T}(-R)=-\mathcal{T}R$, and that $\mathcal{T}L$ is a one-dimensional subspace. We continue by induction. Assume we know that the image under $\mathcal{T}$ of any k-dimensional subspace is a k-dimensional subspace. Let L be a $(k+1)$-dimensional subspace. Then $L = L_1 + L_2$ where L_1 is k-dimensional and L_2 is one-dimensional, and L_2 is not contained in L_1 (so that $L_1 + L_2 \neq L_1$). We then have that $\mathcal{T}L = \mathcal{T}L_1 + \mathcal{T}L_2$ so it is the sum of a k-dimensional subspace and a one-dimensional subspace, so that it is at most $(k+1)$-dimensional. Further, we know that $\mathcal{T}L_1 + \mathcal{T}L_2 \neq \mathcal{T}L_1$ (by injectivity) so that $\mathcal{T}L_2 \not\subset \mathcal{T}L_1$ and thus the dimension of the sum is indeed $k+1$ (we could have used, instead, the induction hypothesis for $\mathcal{T}^{-1}$). Thus we have shown that the image under $\mathcal{T}$ of a k-dimensional subspace is k-dimensional.

3.6. *Intervals*

Any interval $[0, x]$ is a subset of a ray $\mathbb{R}_{\geqslant 0}x$ which is mapped to some ray $\mathbb{R}_{\geqslant 0}y$, and so the interval must be mapped to some interval $[0, z]$ with $z = ay$ for some $a \in \mathbb{R}^+$. We thus have a mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ defined by $\mathcal{T}([0, x]) = [0, F(x)]$. Moreover, we know that the mapping maps rays to rays: for every $a > 0$ and $x \in \mathbb{R}^n$ we have that $F(ax) = bF(x)$ for some $b = b(x, a)$.

3.7. *Monotonicity of F on rays*

As in the one-dimensional case, since for $a > 1$ we have that $[0, ax] = [0, x] + [0, (a-1)x]$, we know that $\mathcal{T}([0, ax]) = \mathcal{T}([0, x]) + \mathcal{T}([0, (a-1)x])$ and in particular (since all these sets are on the same ray $\mathbb{R}^+y$) we see that $\mathcal{T}([0, x]) \subset \mathcal{T}([0, ax])$ when $a > 1$.

3.8. *Linearity of F on rays*

Fix $x \in \mathbb{R}^n$ and let $y = F(x)$. Let us denote $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ the function given by $F(ax) = \varphi(a)y$ (so, formally, φ depends on x). As in the one-dimensional case, we get that φ is an additive function. Since by the previous paragraph φ is monotone, we get that it must be linear. Since we postulated $F(ax) = \varphi(a)y$ we see that $\varphi(1) = 1$ so that $\varphi(a) = a$. Thus we have proven that for every $x \in \mathbb{R}^n$ and $a \in \mathbb{R}^+$, $F(ax) = aF(x)$.

3.9. *Linearity of F*

Consider linearly independent x and y in $\mathbb{R}^n$ and let $z = x + y$. Consider any line (one-dimensional subspace) L which lies in the same 2-dimensional subspace as x and y and such that both x and y lie on the same side of L. It is mapped to a line $\mathcal{T}L = L'$. Since $[0, x] + [0, y] + L = [0, z] + L$ we have that $[0, F(x)] + [0, F(y)] + L' = [0, F(z)] + L'$. This means that $F(z) - (F(x) + F(y))$ is a point in the line L'. However, since x and y are independent, we have more than one possible choice for L (and thus also for L' — by injectivity) and so we must have that $F(z) = F(x) + F(y)$. Together with the fact that $F(ax) = aF(x)$, we get that F is a linear mapping, thus there is some $B \in GL_n$ such that $F(x) = Bx$ and so $\mathcal{T}([0, x]) = [0, Bx]$.

3.10. *Rays and double intervals*

Note that since for rays we have that $[0, x] + \mathbb{R}_{\geqslant 0}x = \mathbb{R}_{\geqslant 0}x$, we get that $[0, Bx] + \mathcal{T}(\mathbb{R}_{\geqslant 0}x) = \mathcal{T}(\mathbb{R}_{\geqslant 0}x)$, and in particular we see that $\mathcal{T}(\mathbb{R}_{\geqslant 0}x) = \mathbb{R}_{\geqslant 0}Bx$, that is, rays are mapped into rays by the same linear transformation B.

Similarly, since an interval of the form $I = [-ax, x]$ can be written as the sum $[-ax, 0] + [0, x]$, we see that $\mathcal{T}I = [-aBx, Bx] = BI$.

3.11. Slabs

We call an L-slab (or simply a slab) a set of the form $S = I + L$ where I is an interval of the form $[-ax, x]$ for some $x \in \mathbb{R}^n \setminus L$ and $a \in \mathbb{R}^+$ (we also allow $a = 0$) and L is an $(n-1)$-dimensional subspace, or, I can be a ray $\{tx\colon t \geqslant -a\}$ which is infinite in one direction (again we allow $a = 0$). Clearly by the above arguments, a slab S is mapped into a slab $S' = BS$. We also call the set $\mathbb{R}^n$ a slab, for technical reasons, and clearly $\mathcal{T}(\mathbb{R}^n) = \mathbb{R}^n$ since $\mathbb{R}^n + K = \mathbb{R}^n$ for all K, in particular for $K = \mathcal{T}^{-1}\mathbb{R}^n$.

Fix a convex $K \in \mathcal{K}_0^n$. For every $(n-1)$-dimensional subspace L, there exists a unique slab $I + L$ such that $K + L = I + L$: this is the smallest slab containing L which contains K — we call it the "supporting L-slab for K" (note: I is not unique, since we do not ask for orthogonality of I and L, but the slab $L + I$ is the unique L-slab supporting K. Also — the supporting L-slab can be all of $\mathbb{R}^n$ for some L's, if, for instance, K is a half-space). We see that when $I + L$ is the supporting L-slab, $\mathcal{T}K + \mathcal{T}L = \mathcal{T}I + \mathcal{T}L$ which means that $\mathcal{T}I + \mathcal{T}L$ is the smallest BL-slab which includes $\mathcal{T}K$. Since we now know the form of $\mathcal{T}$ on both rays and intervals, we get that if K is contained in the slab $I + L$ and it is its supporting L-slab, then $\mathcal{T}K$ is contained in $BI + BL$ and it is its supporting BL-slab. We thus know the form of the supporting slabs for $\mathcal{T}K$: they are exactly BS where S are the supporting slabs for K. Since a convex set is determined by its supporting slabs, and $\mathcal{T}K$ has the same supporting slabs as BK, we get that $\mathcal{T}K = BK$, and the theorem is proven.

The proof that Theorem 2 implies Theorem 1 is standard, namely we take $\mathcal{T} : \mathcal{K}_0^n \to \mathcal{H}^n$ and compose it with the inverse of the support map, by this obtaining a mapping $\mathcal{T}' : \mathcal{K}_0^n \to \mathcal{K}_0^n$, which, by the properties of the support map, satisfies the assumptions of Theorem 2, and so must be induced by a linear transformation B, which implies that the original $\mathcal{T}$ was of the form $K \to h_{BK}$.

4. Variations

4.1. The class $\mathcal{K}^n$

Since our general interest in the area arose from duality, we usually consider, as in Theorem 2, the class $\mathcal{K}_0^n$, on which there is a natural duality transform. However, in the context of Theorem 2, it seems even more natural to consider the class of all closed convex sets in $\mathbb{R}^n$, with Minkowski addition. Indeed, by very small variations on the proof of Theorem 2, we can prove the following theorem.

Theorem 9. *Let $n \geqslant 1$. Assume we are given a bijective transform $\mathcal{T} : \mathcal{K}^n \to \mathcal{K}^n$ satisfying for every $K_1, K_2 \in \mathcal{K}^n$ that*

$$\mathcal{T}K_1 + \mathcal{T}K_2 = \mathcal{T}(K_1 + K_2).$$

Then, there exists a linear transformation $B \in GL_n$ such that for all K

$$\mathcal{T}K = BK.$$

Note that here there is actually no difference between $n = 1$ and $n \geqslant 2$, as opposed to the case of $\mathcal{K}_0^n$.

A sketch of the proof. The case $n = 1$ is very easy, but has many of the ingredients for the higher-dimensional case. Because $\{0\} + K = K$ for all K, we must have that $\mathcal{T}(\{0\}) = \{0\}$. Similarly, $\mathcal{T}\mathbb{R} = \mathbb{R}$ since $\mathbb{R} + K = \mathbb{R}$ for all K. The ray $\{x \geqslant 0\}$ is mapped either to itself or to the ray $\{x \leqslant 0\}$ since these are the only (other) two sets which satisfy $K + K = K$. Let us assume w.l.o.g. that we are in the first case (else, combine $\mathcal{T}$ with $K \to -K$). Next, as $[0, r] + \{x \geqslant 0\} = \{x \geqslant 0\}$ for all $r > 0$, we see that $\mathcal{T}[0, r] = [0, s]$ for some $s(r) > 0$, and $s(r)$ is additive and monotone, hence for some $c_1 > 0$ we have $s(r) = c_1 r$, and similarly $\mathcal{T}[-r, 0] = [-c_2 r, 0]$ for some $c_2 > 0$. Finally, we must determine where $\{x\}$ is mapped to. Since $\{x\} + \{-x\} = \{0\}$, we must have $\mathcal{T}(\{x\}) = \{u(x)\}$, and u is additive. Finally, since

$$\{x\} + [-2x, 0] = [-x, x] = \{-x\} + [0, 2x]$$

we see that

$$\big\{u(x)\big\} + [-2c_2 x, 0] = \big\{u(-x)\big\} + [0, 2c_1 x]$$

so that $u(x) - 2c_2 x = u(-x)$ which implies that $u(x) = c_2 x$, and similarly $u(x) = c_1 x$, so that $c_1 = c_2$. We thus see that for a general interval $I = [x, y]$ we have that $\mathcal{T}I = \mathcal{T}(\{x\} + [0, y - x]) = c_1 x + \mathcal{T}[0, y - x] = [c_1 x, c_1 y]$.

For $n \geqslant 1$ we continue in the same way. Clearly, as in 3.1, $\mathcal{T}^{-1}$ is additive as well. First, note that $\mathcal{T}\{0\} = \{0\}$ since $\{0\}$ is the only set K_0 in $\mathcal{K}^n$ satisfying for every $K \in \mathcal{K}^n$ that $K_0 + K = K$. Similarly, $\mathcal{T}(\mathbb{R}^n) = \mathbb{R}^n$.

Secondly, since $\{x\} + \{-x\} = \{0\}$, we have that $\mathcal{T}\{x\} + \mathcal{T}\{-x\} = \{0\}$ which implies that $\mathcal{T}\{x\} = \{y\}$ for some $y = U(x)$ and that $U(-x) = -U(x)$. Further, by additivity of $\mathcal{T}$ we get that U is additive as a function $U : \mathbb{R}^n \to \mathbb{R}^n$ (we will later show that it must be linear).

As in 3.2 a set K which satisfies $K + K = K$ is mapped to a set which satisfies $\mathcal{T}K + \mathcal{T}K = \mathcal{T}K$, and these sets are exactly cones. Thus we see that, as above, cones are mapped to cones, bijectively (because of $\mathcal{T}^{-1}$, which also maps cones to cones). By the same reason as in 3.2 we see that $\mathcal{T}$ preserves the lattice of cones. The first difference appears in 3.3, since $K \subset C$ for $K \in \mathcal{K}^n$ and a cone C does not imply that $K + C = C$ (but we do not use this fact).

Section 3.4 remains the same, since rays are again mapped bijectively to rays, similarly subspaces of dimension k are mapped to subspaces of dimension k as in 3.5. An interval $[0, x]$ is mapped to an interval $[0, F(x)]$ since $[0, x] + \mathbb{R}_{\geqslant 0}x = \mathbb{R}_{\geqslant 0}x$ and this is a characterization of the intervals $[0, \lambda x]$ for $\lambda \geqslant 0$. This defines a mapping $F : \mathbb{R}^n \to \mathbb{R}^n$, bijective, and which is monotone on rays. In fact, by the same argument as in Sections 3.8 and 3.9, $F(ax) = aF(x)$ and $F(x + y) = F(x) + F(y)$ so that F is a linear mapping, thus there is some $B \in GL_n$ such that $F(x) = Bx$ and so $\mathcal{T}([0, x]) = [0, Bx]$. Similarly, $\mathcal{T}(\mathbb{R}_{\geqslant 0}x) = \mathbb{R}_{\geqslant 0}Bx$, that is, rays are mapped into rays by the same linear transformation B and for an interval of the form $I = [-ax, x]$ we have $\mathcal{T}I = [-aBx, Bx] = BI$.

Now we are ready to get back to U and show that it is not only additive, but linear (with the same B as in F). To this end, fix $x \in \mathbb{R}^n$. Since

$$\{x\} + [-2x, 0] = [-x, x] = \{-x\} + [0, 2x]$$

we see by applying $\mathcal{T}$ and using its additivity that

$$\big\{U(x)\big\} + [-2Bx, 0] = \big\{U(-x)\big\} + [0, 2Bx]$$

so that $U(x) - 2Bx = U(-x)$ which implies that $U(x) = Bx$, as claimed. We thus see that for a general interval $I = [x, y]$ we have that $\mathcal{T}I = \mathcal{T}(\{x\} + [0, y-x]) = Bx + \mathcal{T}[0, y-x] = B[x, y]$. Similarly, the interval can be infinite in one direction and the same result holds.

Going back to the original proof, we consider general slabs $I + L$ where L is an $(n-1)$-dimensional subspace, mapped by T to the subspace BL, and so a slab S is mapped into a slab $S' = BS$. Using the same argument as above for general slabs, a convex $K \in \mathcal{K}^n$ has a unique "supporting L-slab" $K + L = I + L$, and since $\mathcal{T}K + \mathcal{T}L = \mathcal{T}I + \mathcal{T}L$ we have that $\mathcal{T}I + \mathcal{T}L = BI + BL$ is a supporting BL-slab for $\mathcal{T}K$. We thus get that $\mathcal{T}K = BK$, and the theorem is proven. □

5. Sketch of a proof of Theorem 4

For the reader's convenience, we sketch the proof of Theorem 4. First, it is easy to check that $\mathcal{T}(+\infty) = +\infty$ and $\mathcal{T}(-\infty) = -\infty$ since these are the only functions which when summed with any other function are the same, and $+\infty$ has a representation as a sum of two other functions, which $-\infty$ does not. Similarly, as $0 + f = f$ for all f characterizes 0, we see $\mathcal{T}(0) = 0$. Turning to functions satisfying $f + f = f$, these are exactly functions of the form D_A for some convex closed A, where $D_A(x) = 0$ for $x \in A$ and $+\infty$ otherwise. So we get a mapping $B : \mathcal{K}^n \to \mathcal{K}^n$ such that $\mathcal{T}(D_A) = D_{B(A)}$. Clearly, as $A \subseteq A'$ if and only if $D_A + D_{A'} = D_{A'}$, we get that $A \subseteq A'$ if and only if $B(A) \subseteq B(A')$. Also, by bijectivity of $\mathcal{T}$, the mapping B is bijective as well. This implies easily that B preserves the lattice of closed convex sets in $\mathbb{R}^n$, and by [10] (a result which is in the spirit of [6] and [5]) it must be induced by a linear map. That is, for some $B \in GL_n$ we have that $B(K) = BK + v_0$.

To see what can be $\mathcal{T}(c)$, note first that since $\mathcal{T}(c) + \mathcal{T}(-c) = \mathcal{T}(0) = 0$, we must have that $\mathcal{T}(c)$ is an affine linear function for every c (since for two convex functions to add up to a constant, they must both have 0 as their second derivative). Let us denote this function by $\phi_c = \langle u_c, \cdot\rangle + b(c)$. From additivity of $\mathcal{T}$ we have that $u_{c_1+c_2} = u_{c_1} + u_{c_2}$ and $b(c_1 + c_2) = b(c_1) + b(c_2)$. Thus, at least for rational c we can state that there exist $v_1 \in \mathbb{R}^n$ and $a_0 \in \mathbb{R}$ such that for $c \in \mathbb{Q}$

$$\mathcal{T}(c)(y) = c\big(\langle v_1, y\rangle + a_0\big).$$

Consider a function $f \in Cvx(\mathbb{R}^n)$. To compute $(\mathcal{T}f)(x)$ we use that

$$(\mathcal{T}f)(x) = (\mathcal{T}f + D_x)(x) = \mathcal{T}(f + D_{B^{-1}(x-v_0)})(x) = \mathcal{T}(c + D_{B^{-1}(x-v_0)})(x)$$

where $c = f(B^{-1}(x - v_0))$. This in turn equals to $(\mathcal{T}c)(x)$, and so when $f(B^{-1}(x - v_0))$ is rational, we get that

$$(\mathcal{T}f)(x) = f\big(B^{-1}(x - v_0)\big)\big(\langle v_1, x\rangle + a_0\big).$$

Next, it is clear that we must have $v_1 = 0$, otherwise the transform will not preserve convexity even for very simple functions (recall that it is invertible). Finally, the values of $\mathcal{T}(c)$ for $c \in \mathbb{Q}$ determine the values for all c because one may take, say, a linear function, and its rational values determine all the others by convexity.

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