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Order isomorphisms in cones and a characterization of duality for ellipsoids

Shiri Artstein-Avidan · Boaz A. Slomka

Abstract We study order isomorphisms in finite-dimensional ordered vector spaces. We generalize theorems of Alexandrov, Zeeman, and Rothaus (valid for "non-angular" cones) to wide classes of cones, including in particular polyhedral cones, using a different and novel geometric method. We arrive at the following result: whenever the cone has more than n generic extremal vectors, an order isomorphism must be affine. In the remaining case, of precisely n extremal rays, the transform has a restricted diagonal form. To this end, we prove and use a new version of the well-known Fundamental theorem of affine geometry. We then apply our results to the cone of positive semi-definite matrices and get a characterization of its order isomorphisms. As a consequence, the polarity mapping is, up to a linear map, the only order-reversing isomorphism for ellipsoids.

Keywords Cones · Order isomorphisms · Ordered linear spaces · Polarity

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S. Artstein-Avidan (✉) · B. A. Slomka
School of Mathematical Science, Tel Aviv University,
Ramat Aviv, 69978 Tel Aviv, Israel
e-mail: shiri@post.tau.ac.il

B. A. Slomka
e-mail: boazslom@post.tau.ac.il

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1 Introduction

1.1 Definitions and notation

To state our main results, we need some standard definitions and notations. The impatient reader may skip this section and go directly to Sect. 1.3 where the main results are stated.

A non-empty subset K of a vector space is said to be a cone if $K + K \subset K, \alpha K \subset K$ for all $\alpha \geq 0$, and K does not contain any one-dimensional subspaces, namely $K \cap (-K) = \{0\}$. Clearly, every cone is a convex set. Throughout this note, we will only deal with closed cones in normed linear spaces and mainly in $\mathbb{R}^n$.

Recall that a set A is said to be partially ordered if it is equipped with a reflexive, antisymmetric, and transitive relation $\leq$ or, in short, an order relation. A vector space X is called a partially ordered vector space or an ordered vector space if it is equipped with an order relation that is also compatible with the algebraic structure of X. Namely if $x \geq y$, then $x + z \geq y + z$ for each $z \in X$ and $\alpha x \geq \alpha y$ for all $\alpha \geq 0$. It is easy to check that in an ordered vector space X, the set $X_+ = \{x \in X : x \geq 0\}$ is a cone and that every cone K of a vector space X induces a vector ordering on X by letting $x \geq y$ whenever $x - y \in K$, in which case $X_+ = K$. In order to avoid ambiguity, we shall denote the vector ordering induced by a cone K by $\leq_K$ and the standard ordering of $\mathbb{R}$ by $\leq$. For further reading about cones, we refer the reader to [5].

We are interested in the following objects.

Definition 1.1 Let $(X, \leq_1)$, $(Y, \leq_2)$ be two ordered vector spaces. Let $A \subset X$ and $B \subset Y$. A bijection $T : A \to B$ is said to be an **order isomorphism** if it preserves order in both directions. That is,

$$x \leq_1 y \iff T(x) \leq_2 T(y). \tag{1}$$

Note that the injectivity of T is automatically implied by Eq. (1).

In this note, we shall restrict ourselves to three special cases of order isomorphisms:

- $T : (K_1, \leq_{K_1}) \to (K_2, \leq_{K_2})$,
- $T : (int(K_1), \leq_{K_1}) \to (int(K_2), \leq_{K_2})$
- $T : (\mathbb{R}^n, \leq_{K_1}) \to (\mathbb{R}^n, \leq_{K_2})$.

That is, we only deal with the cases in which the mapping is defined either on the entire space $\mathbb{R}^n$, on the order-inducing cone K or on its interior $int(K)$.

1.2 Some history

The first results in this topic are those of Alexandrov from the 1950s onwards, beginning with [1], where he proved that in 4-dimensional space "constancy of light velocity alone implies Lorentz group with translations and dilatations." This means, mathematically, that a mapping on affine space mapping any right circular cone

$$C = \{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = (t - t_0)^2\}$$

onto a cone of the same kind, is affine. Note that in our setting, this is not a cone but rather the boundary of a cone. The case of "solid" cones, namely when one considers not only the boundary, which is more in the spirit of our setting, was published in [2].

The results of Alexandrov seem not to have been noticed in certain circles, and some have been reproved. For example, Zeeman [34] has shown that for a right circular cone C in $\mathbb{R}^4$, any order isomorphism $T : (\mathbb{R}^4, \leq_C) \to (\mathbb{R}^4, \leq_C)$ must be affine, with a Lorentz transformation as its linear part, and such results appear in the papers of Alexandrov. Zeeman's result was then generalized by Rothaus [26]. To state Rothaus's results recall that a cone K is said to have a compact base B if there exists a linear functional h satisfying that $h(x) > 0$ for all $x \in K\setminus\{0\}$, and for which $B = h^{-1}(1) \cap K$ is compact. A cone is called non-angular if it has a compact base with no isolated extreme points. Note that the correspondence between two bases of a cone is given by a projective linear transformation so that if one base has the aforementioned property, so do all others. Rothaus proved the following.

Theorem 1.2 *Let $n \geq 3$. Let $K \subset \mathbb{R}^n$ be a non-angular cone and let A denote either $\mathbb{R}^n$ or $int(K)$. Assume $T : (A, \leq_K) \to (A, \leq_K)$ is an order isomorphism. Then, there exist $v_0 \in \mathbb{R}^n$ and $B \in GL_n$ so that $T(x) = Bx + v_0$, with $BK = K$. In case $A = int(K)$, it follows that $v_0 = 0$.*

Let us mention at this point that non-angularity is a relatively strong requirement on the cone, in particular no polytope-based cone is such, and this is a disadvantage. We also mention that Alexandrov in [4] generalized his previous results for strictly convex cones (namely whose base is a strictly convex set). This, however, is not exactly the setting of Rothaus's results as non-angular cones that are not strictly convex do exist.

The results of Alexandrov, Zeeman, and Rothaus have found various applications in the literature, some of which are described in [29], where the physical significance of order isomorphisms of cones with respect to mathematical relativity theory is discussed. This field has been named, as Alexandrov [3] notes, by A. D. Fokker "Chronogeometry" and has applications in the study of dynamics generated by hyperbolic partial differential equations, where the case of an *arbitrary* non-trivial closed convex cone appears and is physically natural (the convexity assumption is, however, crucial). We remark that some of their results have also been extended to an infinite dimensional setting, e.g., [22,23] and [31].

1.3 Main results

Our aim in this note is to extend Rothaus's results to fit the case of angular cones as well as non-angular cones and to shed light on the general form of order isomorphisms associated with any given cone.

Before we proceed with stating our main results, we need to recall a few standard definitions. A nonzero vector $e \in K$ is said to be an extremal vector of the cone K if $0 \leq_K x \leq_K e$ implies that $x = \lambda e$ for some $\lambda \geq 0$. In this case, the ray $R_e = \{\lambda e : \lambda \geq 0\}$ is called an extremal ray of K and the one-dimensional subspace spanned by e will be called an extremal line of K. Alternatively, extremal rays of a cone are exactly the rays that intersect any base of the cone at an extremal point. We will say that a set of $n+1$ vectors in $\mathbb{R}^n$ is n-independent if any n vectors among them form a basis of $\mathbb{R}^n$.

The following theorem demonstrates that there is a quite natural, large family of cones, different from the family of non-angular cones described in Theorem 1.2, but for which the same conclusion is drawn, that is, all associated order isomorphisms must be affine linear as well. Namely, this family is all cones that have $n+1$ n-independent extremal vectors. We prove the following.

Theorem 1.3 *Let $n > 2$. Let $K \subset \mathbb{R}^n$ be a closed cone and let A stand for either $\mathbb{R}^n$, K or $int(K)$. Assume K has at least $n+1$ extremal vectors that are n-independent. Let $T : (A, \leq_K) \to (A, \leq_K)$ be an order isomorphism. Then, T is an affine transformation, i.e., $\exists v_0 \in \mathbb{R}^n$ $\exists B \in GL_n$ such that*

$$T(x) = v_0 + Bx \quad and \quad BK = K.$$

In the case that A is either K or $int(K)$, we must have $v_0 = 0$.

Remark 1.4 Although the conditions on the cone in Theorem1.3 are rather mild, we point out that they do not formally generalize Theorem 1.2. Indeed, one may construct a non-angular cone that does not have $n+1$ extremal rays that are n-independent. For example, take any non-angular cone K in $\mathbb{R}^3$ and construct a cone K' in $\mathbb{R}^6 = \mathbb{R}^3 \oplus \mathbb{R}^3$ by taking the convex hull two copies of K, one in each copy of $\mathbb{R}^3$. It is not hard to check that such a cone is non-angular, but does not have $n+1$ extremal vectors that are n-independent. However, in this case, one may use Theorem 1.7 given below and Theorem 1.3 to deduce that all order isomorphisms associated with K' must be affine linear.

Remark 1.5 The condition, in Theorem 1.3, that there are $n+1$ generic extremal rays cannot be relaxed; for example, consider the cone $K \in \mathbb{R}^4$ whose extremal rays are generated by the standard basis $e_1, \ldots, e_4$ and the vector $(0, 1, 1, -1)^T$. The transformation defined by $(x, y, z, w)^T \mapsto (x^3, y, z, w)^T$ preserves $\leq_K$ but is certainly not linear. Its special form is, however, not accidental, see Theorem 1.7 below.

It is easy to verify that any three nonzero extremal vectors of a cone in a vector space that generate three distinct extremal rays are linearly independent. Thus, for $n = 3$, Theorem 1.3 can be restated as follows.

Corollary 1.6 *Let $K \subset \mathbb{R}^3$ be a closed cone with at least four distinct extremal rays. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be an order isomorphism. Then, T is an affine transformation, i.e., there exist $v_0 \in \mathbb{R}^3$ and a linear transformation $B \in GL_3$ such that for every $x \in \mathbb{R}^3$, $T(x) = v_0 + Bx$ and $BK = K$. Similarly, if $T : K \to K$ or $T : int(K) \to int(K)$, then $Tx = Bx$ for some $B \in GL_3$ with $BK = K$.*

When dealing with a cone that does not satisfy the condition of Theorem 1.3, we do not know linearity (and as in Remark 1.5, it may be non-linear). However, we do know that even in the most general case that is for any order-inducing cone, all order isomorphisms have quite a restricted "diagonal" form. This is specified in the following theorem.

Theorem 1.7 *Let $n \geq 2$. Let $A \subset \mathbb{R}^n$ stand for either K, $int(K)$ or $\mathbb{R}^n$ and let $K \subset \mathbb{R}^n$ be a closed non-degenerate cone. Assume $T : (A, \leq_K) \to (A, \leq_K)$ is an order isomorphism. Then, there exist bijective increasing functions $f_1, \dots, f_n : \mathbb{R} \to \mathbb{R}$, linearly independent vectors $v_1, \dots, v_n \in K$ and another set of linearly independent vectors $w_1, w_2, \dots, w_n \in K$ such that*

$$T(\sum_{i=1}^{n} \alpha_i v_i) = \sum_{i=1}^{n} f_i(\alpha_i) w_i$$

for $\alpha_1 v_1 + \cdots + \alpha_n v_n \in A$. In particular, T is continuous. In case $A = \mathbb{R}^n$ or $A = K$, we have that $w_i = T(v_i)$.

Remark 1.8 Rothaus [26] has shown that an order isomorphism defined on any open set must be a continuous map (even for infinite dimensional normed spaces). In Theorem 1.7, the continuity follows by different arguments and holds also for the case where the map is defined on the closed order-inducing cone, in $\mathbb{R}^n$.

Remark 1.9 Theorems 1.3 and 1.7 are stated for order isomorphisms in which the order-inducing cone in the domain is the same as in the image. However, this is only for the reader's (and writers') convenience. The same results are valid if we take two different order-inducing cones, and the proofs are literally the same. Moreover, in such a case of, say, $T : (K_1, \leq_{K_1}) \to (K_2, \leq_{K_2})$, the assumption of the existence of $(n+1)$ extremal rays that are n-independent need to be assumed only on the cone K_1.

Note that in each of the theorems, there are three cases to consider, namely $\mathbb{R}^n$, K, and $int(K)$. We usually provide the proof of one of the three, and remark on the changes to be made for the other two cases.

1.4 An application

In recent years, many results concerning a characterization of order isomorphisms and of duality for various classes of convex bodies and functions, as well as characterizations of endomorphisms of lattices of convex bodies have been established. To name a few of these results: Böröczky and Schneider [14] characterized duality for convex bodies (compact convex sets with the origin in their interior) in, a similar result for general closed convex sets that include the origin appears in [9]. Order isomorphisms for convex functions where characterized in [7] together with the fact that up to linear terms the only duality (order-reversing involution) on convex functions is the Legendre transform. Duality for cones was characterized in [28]. In this section, we will be interested in the class of all centered non-degenerate ellipsoids in $\mathbb{R}^n$, which we denote by $\mathcal{E}_c^n$. We will also have need of the class of all generalized ellipsoids, which we denote by $\mathcal{E}_g^n$, and the class of bounded generalized ellipsoids, which can be thought of as all linear images of the Euclidean ball, which we denote by $\mathcal{E}_b^n$. More formally, denoting the Euclidean unit ball $B_2^n = \{x : \sum_{i=1}^n x_I^2 \leq 1\}$, we have

- $\mathcal{E}_c^n = \{AB_2^n : A \in GL_n\}$

- $\mathcal{E}_b^n = \{AB_2^n : A \in L(\mathbb{R}^n, \mathbb{R}^n)\}$.
- $\mathcal{E}_g^n = \{\{x : Ax \in B_2^n\} : A \in L(\mathbb{R}^n, \mathbb{R}^n)\} \cup \mathcal{E}_b^n$

where $L(\mathbb{R}^n, \mathbb{R}^n)$ denotes all linear transformations on $\mathbb{R}^n$ and GL_n stands for all the invertible linear transformations.

Recall the well-known polarity transformation for convex sets given by

$$K^\circ = \left\{x \in \mathbb{R}^n : \langle x, y\rangle \leq 1, \quad \forall y \in K\right\}.$$

It is not hard to check that this operation is an involution on $\mathcal{E}_c^n$ and also on $\mathcal{E}_g^n$.

As an application of Theorem 1.3, we will prove a characterization of order isomorphisms and of duality for classes of ellipsoids. For order isomorphisms, we have

Theorem 1.10 *Let $n \geq 2$. Let $T : \mathcal{E}_c^n \to \mathcal{E}_c^n$ be an order isomorphism (with respect to inclusion). Then, T is induced by a linear point map on $\mathbb{R}^n$, namely for some $A \in GL_n$, we have $TE = AE$ for every $E \in \mathcal{E}_c^n$. The same holds when one replaces the class $\mathcal{E}_c^n$ by $\mathcal{E}_g^n$ or by $\mathcal{E}_b^n$.*

Since we know one order-reversing involution on the classes $\mathcal{E}_c^n$ and $\mathcal{E}_g^n$, namely the aforementioned duality, we get an immediate consequence of Theorem 1.10:

Corollary 1.11 *Let $n \geq 2$. Let $T : \mathcal{E}_c^n \to \mathcal{E}_c^n$ be an order-reversing isomorphism. Then, there exists a linear transformation $A \in GL_n$ such that $T(E) = AE^\circ$, for all $E \in \mathcal{E}_c^n$. The same is true for the class $\mathcal{E}_g^n$.*

Remark 1.12 It is worth mentioning that one may approach the proof of Theorem 1.10 by standard methods, as used in [14,33] and [7–9]. However, via that approach, one encounters difficulties which do not seem very easy to overcome. For example, the set $\mathcal{E}_c^n$, while ordered by inclusion, does not satisfy lattice requirements, namely there is no maximal object that is the "smallest" greater than some two given ellipsoids (and similarly minimum does not exist), a fact that is extensively used in the proof of similar results.

The paper is organized as follows. The proof of the main theorems relies on propositions regarding maps that map parallel lines in certain directions to parallel lines. These propositions, together with some history on line preserving maps, are presented in Sect. 2. In Sect. 3, we investigate the behavior of (translates of) extremal vectors and extremal lines of an order-inducing cone under order isomorphisms. We then apply the conclusion together with the results in Sect. 2 to our setting and get proofs of our main theorems, in Sect. 4. In Sect. 5, we give the application to order isomorphisms and duality on ellipsoids, and some further comments.

2 Maps preserving lines in some directions

It is a classical fact, sometimes called the fundamental theorem of affine geometry, that a bijective map which maps any line in $\mathbb{R}^n$ onto a line must be affine linear (see e.g., [24]). This fact has been extensively generalized to many settings, see e.g., [6,15,21].

For our purposes, it will become clear in Sect. 3 that we need a corresponding fact when the assumption is given only for lines that are in certain directions, namely the directions of the extremal vectors. It is part of an ongoing project of the authors (which will be published elsewhere) to determine the general form of a bijection that preserves lines in a fixed set of directions. Some results in this direction are known, for example [13,19,25] and [30], but none of them suffice for our purposes.

For the purpose of cone order isomorphisms, as we will see in Sect. 3, we may add an extra assumption on the mapping, namely that in the fixed directions in concern, it maps parallel lines to parallel lines. Note that in $\mathbb{R}^2$, any two non-intersecting lines must be parallel, so that a bijective map mapping two parallel lines to lines must automatically map them to *parallel lines*. This is far from true in $\mathbb{R}^n$ for larger n, and one may easily generate a map $F : \mathbb{R}^3 \to \mathbb{R}^3$ mapping lines in three directions to lines, but not parallel, say $F(x, y, z) = (x + yz, y, z)$.

In this section, we state and prove our results concerning this variation of the fundamental theorem of affine geometry. We then explain some structural fact about so-called doubly ruled surfaces that will enable us to get the parallelism assumption for our map in the proof of the main theorems for cones, and thus to apply the former.

2.1 Maps preserving parallel lines in some directions

In this section, we discuss maps mapping parallel lines in n linearly independent directions to parallel lines and then discuss the case where an additional $(n+1)^{th}$ direction is added. The domain of the map shall be either all of $\mathbb{R}^n$, or just some fixed closed cone, or its interior. The parallelism assumption allows one to consider "coordinates" in the space, and then in this system of coordinates, the map is of a very special diagonal form.

It is worth mentioning that the fundamental theorem of affine geometry, which holds in $\mathbb{R}^n$, does not hold on subsets of $\mathbb{R}^n$, and in the case of subsets, there is a much bigger family of maps mapping lines to lines. These are the so-called fractional linear maps, see [10] and [11]. They may even be chosen so that the domain and the range of the map are a cone. However, these will not map parallel lines in n independent directions to parallel lines (at most $n - 1$ directions), unless they are truly linear.

The following is our main result about maps preserving parallel lines in n linearly independent directions. For technical reasons, we state the theorem twice, once in the case of the closed cone and again, slightly reformulated, in the case of the interior of the cone.

Theorem 2.1 *Let $n \geq 2$. Let D_1 and D_2 be the interiors of two closed non-degenerate cones K_1 and K_2 in $\mathbb{R}^n$ and let $v_1, \ldots, v_n \in K_1$ be linearly independent vectors. Assume $F : D_1 \to D_2$ is an injection that maps parallel lines in directions $v_1, \ldots, v_n \in K_1$ to parallel lines in directions $w_1, \ldots, w_n \in K_2$. More precisely, for every $x \in K_1$ and any $i \in \{1, \ldots, n\}$, we have that $F((x + \mathrm{sp}\, v_i) \cap D_1) = (F(x) + \mathrm{sp}\, w_i) \cap D_2$. Then, there exist bijective functions $f_1, f_2, \ldots, f_n : \mathbb{R} \to \mathbb{R}$, so that whenever $a_1 v_i + \cdots + a_n v_n \in D_1$, we have that*

$$F(x) = \sum_{i=1}^{n} f_i(a_i) w_i.$$

Theorem 2.2 *Let $n \geq 2$. Let K_1 and K_2 be two closed non-degenerate cones in $\mathbb{R}^n$ and let $v_1, \dots, v_n \in K_1$ be linearly independent vectors. Assume $F : K_1 \to K_2$ is an injection that maps parallel lines in directions $v_1, \dots, v_n \in K_1$ to parallel lines in directions $F(v_1), \dots, F(v_n) \in K_2$. More precisely, for every $x \in K_1$ and any $i \in \{1, \dots, n\}$, we have that $F((x + \mathrm{sp}\, v_i) \cap K_1) = (F(x) + \mathrm{sp}\, F(v_i)) \cap K_2$. Then, there exist bijective functions $f_1, f_2, \dots, f_n : \mathbb{R} \to \mathbb{R}$, so that whenever $a_1 v_i + \cdots + a_n v_n \in K_1$, we have that*

$$F(x) = \sum_{i=1}^{n} f_i(a_i) F(v_i).$$

Remark 2.3 In Theorems 2.1 and 2.2, one could choose a scalar product such that $v_1, \dots, v_n$ are orthogonal and then we would have that $F(x) = \sum_{i=1}^{n} f_i(\langle x, v_i \rangle) F(v_i)$ and $F(x) = \sum_{i=1}^{n} f_i(\langle x, v_i \rangle) w_i$ for all $x \in K_1$ and for all $x \in D_1$, respectively.

Remark 2.4 The same theorem holds when the cones K_i are replaced with $\mathbb{R}^n$, the same proof works and is actually even slightly easier since one may work with the full lines and not intersect with the respective cone each time.

For the proof of Theorem 2.1 and 2.2, we will need the following lemma that explains what a map preserving parallel lines in n linearly independent directions must do to (translates of) linear subspaces spanned by these vectors. Again, for technical reasons, we state the lemma twice.

Lemma 2.5 *Let $n \geq 2$. Let D_1 and D_2 be the interiors of two closed non-degenerate cones K_1 and K_2 in $\mathbb{R}^n$ and let $v_1, \dots, v_n \in K_1$ be linearly independent vectors. Assume $F : D_1 \to D_2$ is an injection that maps parallel lines in directions $v_1, \dots, v_n \in K_1$ to parallel lines in directions $w_1, \dots, w_n \in K_2$. More precisely, for every $x \in D_1$ and any $i \in \{1, \dots, n\}$, we assume that $F((x + \mathrm{sp}\, v_i) \cap D_1) = (F(x) + \mathrm{sp}\, w_i) \cap D_2$. Then, for every $1 \leq k \leq n$, the following holds.*

(a) *$w_1, \dots, w_k$ are linearly independent.*
(b) *For any $x \in D_1$, $F((x + \mathrm{sp}\{v_1, \dots, v_k\}) \cap D_1) = (F(x) + \mathrm{sp}\{w_1, \dots, w_k\}) \cap D_2$.*

Proof of Lemma 2.5 We will use induction on k. For $k = 1$, the claim holds by assumption. Assume that the claim holds for $k - 1$. First, we prove that (a) holds. Indeed, if $w_k \in \mathrm{sp}\{w_1, \dots, w_{k-1}\}$, then, by the induction hypothesis, (b) implies that whenever for some $x \in D_1$, we have $F(x) + w_k \in D_2$,

$$F(x) + w_k \in F((x + \mathrm{sp}\{v_1, \dots, v_{k-1}\}) \cap D_1).$$

However, by assumption if $F(x) + w_k \in D_2$, then it may be written as $F(x + \lambda v_k)$. We then get by injectivity that $x + \lambda v_k = x + \sum_{i=1}^{k-1} \alpha_i v_i$, which is a contradiction. Note that we may always find some $x \in D_1$ with $F(x) + w_k \in D_2$ because D_2 is a cone. In fact, note that if a vector x is in the interior of a cone K, then $x + y$ is in its interior as well, for any $y \in K$. We will also use this fact, next, to prove that (b) holds.

We now turn to prove (b) for k. First, we show the inclusion

$$F((x+\mathrm{sp}\{v_1,\dots,v_k\})\cap K_1)\subseteq(F(x)+\mathrm{sp}\{w_1,\dots,w_k\})\cap K_2.$$

From this point on, one needs to bear in mind the fact that if a vector x is in the interior of a cone K, then $x+y$ is in its interior as well, for any $y\in K$.

Let $x\in D_1$ and let $\alpha_1,\dots,\alpha_k\in\mathbb{R}$ so that $x+\alpha_1v_1+\cdots+\alpha_kv_k\in D_1$. If $\alpha_k=0$, then we are done, by the induction hypothesis. Assume $\alpha_k>0$. Then, $y=x+\alpha_kv_k\in D_1$ and so, by the induction hypothesis, (b) implies that

$$F(y+\alpha_1v_1+\cdots+\alpha_{k-1}v_{k-1})=F(y)+\beta_1w_1+\cdots+\beta_{k-1}w_{k-1}$$

for some $\beta_1,\dots,\beta_k\in\mathbb{R}$, and that the above vector belongs to D_2. Moreover, by assumption, $F(y)=F(x+\alpha_kv_k)=F(x)+\beta_kw_k$ for some $\beta_k\in\mathbb{R}$. Thus,

$$F(x+\alpha_1v_1+\cdots+\alpha_kv_k)=F(x)+\beta_1w_1+\cdots+\beta_kw_k.$$

Assume $\alpha_k<0$. Then, since $-\alpha_kv_k\in D_1$, we have that also $y=x+\alpha_1v_1+\cdots+\alpha_{k-1}v_{k-1}\in D_1$ and we may apply the induction hypothesis, (b) to get that

$$F(x+\alpha_1v_1+\cdots+\alpha_{k-1}v_{k-1})=F(x)+\beta_1w_1+\cdots+\beta_{k-1}w_{k-1}$$

for some $\beta_1,\dots,\beta_{k-1}\in\mathbb{R}$, and is in D_1. Moreover, by assumption, again since $y+\alpha_kv_k\in D_1$, there exists $\beta_k\in\mathbb{R}$ for which $F(y+\alpha_kv_k)=F(y)+\beta_kw_k$ which means that

$$F(x+\alpha_1v_1+\cdots+\alpha_kv_k)=F(x)+\beta_1w_1+\cdots+\beta_kw_k.$$

The inclusion in the opposite direction is proven similarly. Let $x\in D_1$ and let $\beta_1,\dots,\beta_k\in\mathbb{R}$ so that $F(x)+\beta_1w_1+\cdots+\beta_kw_k\in D_2$. If $\beta_k=0$, we are done due to the induction hypothesis. Assume $\beta_k>0$. Then, $F(x)+\beta_kw_k\in D_2$. By assumption, there exists $\alpha_k\in\mathbb{R}$ for which $y=x+\alpha_kv_k\in D_1$ and $F(x+\alpha_kv_k)=F(x)+\beta_kw_k$. Next, by the induction hypothesis, (b) implies that there exist $\alpha_1,\dots,\alpha_{k-1}\in\mathbb{R}$ so that $y+\alpha_1v_1+\cdots+\alpha_{k-1}v_{k-1}\in D_1$ and

$$F(y+\alpha_1v_1+\cdots+\alpha_{k-1}v_{k-1})=F(y)+\beta_1w_1+\cdots+\beta_{k-1}w_{k-1}.$$

That is,

$$F(x+\alpha_1v_1+\cdots+\alpha_kv_k)=F(x)+\beta_1w_1+\cdots+\beta_kw_k.$$

In the case of $\beta_k<0$, since $-\beta_kw_k\in D_2$, we see that $F(x)+\beta_1w_1+\cdots+\beta_{k-1}w_{k-1}\in D_2$ and so, the induction hypothesis, (b) implies that there exist $\alpha_1,\dots,\alpha_{k-1}\in\mathbb{R}$ so that $y=x+\alpha_1v_1+\cdots+\alpha_{k-1}v_{k-1}\in D_1$ and

$$F(x+\alpha_1v_1+\cdots+\alpha_{k-1}v_{k-1})=F(x)+\beta_1w_1+\cdots+\beta_{k-1}w_{k-1}.$$

Again $F(y) + \beta_k w_k \in D_2$ so by assumption, there exists $\alpha_k \in \mathbb{R}$ for which $F(y + \alpha_k v_k) = F(y) + \beta_k w_k$, which together gives

$$F(x + \alpha_1 v_1 + \cdots + \alpha_k v_k) = F(x) + \beta_1 w_1 + \cdots + \beta_k w_k.$$

This completes the induction, and thus the proof. □

Lemma 2.6 *Let $n \geq 2$. Let K_1 and K_2 be two closed non-degenerate cones in $\mathbb{R}^n$ and let $v_1, \ldots, v_n \in K_1$ be linearly independent vectors. Assume $F : K_1 \to K_2$ is an injection that maps parallel lines in directions $v_1, \ldots, v_n \in K_1$ to parallel lines in directions $F(v_1), \ldots, F(v_n)$. More precisely, for every $x \in K_1$ and any $i \in \{1, \ldots, n\}$, we assume that $F((x + \mathrm{sp}\, v_i) \cap K_1) = (F(x) + \mathrm{sp}\, F(v_i)) \cap K_2$. Then, for every $1 \leq k \leq n$, the following holds.*

(a') $F(v_1), \ldots, F(v_k)$ *are linearly independent .*
(b') *For any* $x \in K_1$, $F((x + \mathrm{sp}\{v_1, \ldots, v_k\}) \cap K_1) = (F(x) + \mathrm{sp}\{F(v_1), \ldots, F(v_k)\}) \cap K_2$.

Remark 2.7 Again, the lemma holds and the same proof works for $F : \mathbb{R}^n \to \mathbb{R}^n$.

Proof of Lemma 2.6 The proof is literally the same as the proof of Lemma 2.5. In fact, showing that $F(v_1), \ldots, F(v_k)$ are linearly independent is actually slightly simpler since F is defined on the entire cone K_1, and in particular on $v_1, \ldots, v_k$. □

Remark 2.8 Notice that the lemmas above imply that the functions f_i in the statement of Theorems 2.1 and 2.2 satisfy that for all i, the image of the restriction of f_i to $\mathbb{R}^+$, satisfies $f_i(\mathbb{R}^+) = \mathbb{R}^+$, and that in Theorem 2.2 also $f_i(0) = 0$ is satisfied.

We are now ready to prove the diagonal form of a map preserving parallelism in n independent directions.

Proof of Theorem 2.1 First, choose a scalar product $\langle \cdot, \cdot \rangle$ such that $v_1, \ldots v_n$ are orthogonal, and a scalar product which we also denote $\langle \cdot, \cdot \rangle$ such that $w_1, \cdots w_n$ are orthogonal (by Lemma 2.5, $\{w_1, \ldots, w_n\}$ are linearly independent and so this is possible).

By Lemma 2.5, if $x, y \in D_1$ satisfy that $x - y \in \mathrm{sp}\{v_j : j \neq i\}$, then $F(x) - F(y) \in \mathrm{sp}\{w_j : j \neq i\}$. In other words, if $\langle x, v_i \rangle = \langle y, v_i \rangle$, then $\langle F(x), w_i \rangle = \langle F(y), w_i \rangle$. Hence, we can define the following functions. For $t \in \mathbb{R}$ for which there exists $x \in K_1$ with $\langle x, v_i \rangle = t$ define $f_i(t) = \langle F(x), w_i \rangle$. By the above, this is well defined. Note that for all i either for all $t < 0$, there exists $x \in D_1$ with $\langle x, v_i \rangle = t$ or there does not exist any $x \in D_1$ for which $\langle x, v_i \rangle < 0$, due to the fact that K_1 is a cone. In the latter case, we set $f_i(t) = -f_i(-t)$ and clearly in such a case $f_i : \mathbb{R}^+ \to \mathbb{R}^+$. Also, in the general case, by Lemma 2.5 applied to the subspace $sp\{v_i\}$, $f_i(\mathbb{R}^+) = \mathbb{R}^+$. Note that if there exists $x \in D_1$ such that $\langle x, v_i \rangle = 0$, then Lemma 2.5 implies that $f_i(0) = 0$. In the case that there does not exist such a vector, we set $f_i(0) = 0$.

Having defined f_i, the claim readily follows: By Lemma 2.5, $w_1, \ldots, w_n$ are a basis of $\mathbb{R}^n$ and so for every $x = t_1 v_1 + \cdots t_n v_n \in D_1$, we have $F(x) = \beta_1 w_1 + \cdots \beta_n w_n$ for a unique choice of $\beta_1, \ldots, \beta_n \in \mathbb{R}$. By the argument above, it follows that $\beta_i = f_i(t_i)$. Since F is injective, it follows that the f_i's are bijective. □

The proof of Theorem 2.2 is literally the same as the proof of Theorem 2.1, only one should replace Lemma 2.5 with Lemma 2.6.

We proceed to a theorem concerning maps that preserve parallel lines in $n+1$ directions that are further assumed to be n-independent. In such a case, we show that in the form given in the statement of Theorem 2.2, the functions f_i are in fact additive, and correspondingly the whole function F is additive. It is well known that "well behaved" additive maps on $\mathbb{R}^n$ are linear (for one of the most general such theorems see [12] and [32]), which is a fact we shall use in the sequel (the sense in which these maps are "well-behaved" is that their coefficient functions are monotone, as explained in the proof of Theorem 1.3 in Sect. 4).

Theorem 2.9 *Let $n \geq 2$. Let K_1 and K_2 be two closed non-degenerate cones in $\mathbb{R}^n$ and let $v_1, \dots, v_n, v_{n+1} \in K_1$ be $(n+1)n$-independent vectors. Assume $F : K_1 \to K_2$ is an injection that maps parallel lines in directions $v_1, \dots, v_{n+1}$ to parallel lines in directions $F(v_1), \dots, F(v_{n+1})$. More precisely, we assume that for every $x \in K_1$ and any $i \in \{1, \dots, n+1\}$, we have that $F((x + \mathrm{sp}\, v_i) \cap K_1) = (F(x) + \mathrm{sp}\, F(v_i)) \cap K_2$. Then, F is an additive mapping. More precisely, if we choose a scalar product such that $\{v_i\}_{i=1}^n$ form an orthogonal basis, then there exist bijective additive functions $f_1, f_2, \dots, f_n : \mathbb{R} \to \mathbb{R}$, so that for every $x \in K_1$,*

$$F(x) = \sum_{i=1}^{n} f_i(\langle x, v_i \rangle) F(v_i)$$

Remark 2.10 As in Remark 2.8, the f_i's in Theorem 2.9 satisfy that $f_i(\mathbb{R}^+) = \mathbb{R}^+$ and $f_i(0) = 0$. In fact, up to scaling factors, all the f_i' s in Theorem 2.9 are actually the same function. To be more precise, for any i and j, there exist $c_j, d_j \in \mathbb{R}$ such that $f_i(x) = d_j f_j(c_j x)$. Indeed, write $v_{n+1} = a_1 v_1 + \cdots + a_n v_n$ and assume that lines in direction v_{n+1} are mapped to lines in direction $F(v_{n+1}) = b_1 F(v_1) + \cdots + b_n F(v_n)$. Since F maps all points of the form $t v_{n+1}$ to $\sum_{i=1}^n f_i(t a_i) F(v_i)$, which is parallel to $F(v_{n+1})$, we get that $f_i(t a_i) b_j = b_i f_j(t a_j)$, as required.

Remark 2.11 One may reformulate Theorem 2.9 for mappings between the interiors of two cones (as was done with Theorems 2.1 and 2.2) or mappings on $\mathbb{R}^n$ in order to prove Theorem 1.3 for the different cases ($A = K$, $A = \mathrm{int} K$ and $A = \mathbb{R}^n$). However, by using Rothaus's argument [26, Proposition 2], a proof of Theorem 1.3 for the case $A = K$, where the mapping is defined on the cone, will imply the same result in the two remaining cases, as we explain in the proof of Theorem 1.3 below.

Proof of Theorem 2.9 By Theorem 2.2, there exist bijections $f_1, \dots, f_n : \mathbb{R} \to \mathbb{R}$, so that for all $x \in K_1$,

$$F(x) = \sum_{i=1}^{n} f_i(\langle x, v_i \rangle) F(v_i). \tag{2}$$

Next, we will show that the f_i's are additive. By Lemma 2.6, $F(v_1), \dots, F(v_{n+1})$ are n-independent and so we may write $F(v_{n+1}) = b_1 F(v_1) + \cdots + b_n F(v_n)$ where

$b_i \neq 0$ for all i. Similarly, by assumption, $v_{n+1} = a_1 v_1 + \cdots + a_n v_n$ for $a_i \neq 0$. Notice that $a_i > 0$ if and only if $b_i > 0$. Indeed, by Eq. (2), $b_i = f_i(a_i)$ and now one should recall Remark 2.8.

Let us show that the f_i's are additive.

Assume that $a_i > 0$. We show that in this case $f_i(x+y) = f_i(x) + f_i(y)$. Let $x, y \geq 0$. Then, on the one hand, we have that

$$F\left(xv_i + ya_i^{-1}v_{n+1}\right) \in f_i(x)F(v_i) + \mathrm{sp}F(v_{n+1})$$

and so $F(xv_i + ya_i^{-1}v_{n+1}) - f_i(x)F(v_i) \in \mathrm{sp}F(v_{n+1})$.On the other hand, we have that

$$\begin{aligned} F\left(xv_i + ya_i^{-1}v_{n+1}\right) &= F\left(xv_i + ya_i^{-1}\sum_{j\neq i} a_j v_j\right) \\ &= f_i(x+y)F(v_i) + \sum_{j\neq i} f_j\left(ya_i^{-1}a_j\right)F(v_j). \end{aligned}$$

Thus,

$$[f_i(x+y) - f_i(x)]F(v_i) + \sum_{j\neq i} f_j\left(ya_i^{-1}a_j\right)F(v_j) \in \mathrm{sp}F(v_{n+1}). \tag{3}$$

Since $ya_i^{-1}(a_1v_1 + \cdots + a_nv_n) \in \mathrm{sp}\ v_{n+1}$, it follows that

$$F\left(ya_i^{-1}(a_1v_1 + \cdots + a_nv_n)\right) = f_i(y)F(v_i) + \sum_{j\neq i} f_i\left(ya_i^{-1}a_j\right)F(v_j) \in \mathrm{sp}F(v_{n+1})$$

and so, by comparing to Eq. (3), we conclude that $f_i(x+y) - f_i(x) = f_i(y)$.

Assume that $a_i < 0$. Next, we show that in this case, $f_i(x-y) = f_i(x) + f_i(-y)$, for all $x, y \geq 0$. On the one hand, we have that

$$F\left(xv_i + y|a_i^{-1}|v_{n+1}\right) \in f_i(x)F(v_i) + \mathrm{sp}F(v_{n+1})$$

and so, $F(xv_i + y|a_i^{-1}|v_{n+1}) - f_i(x)F(v_i) \in \mathrm{sp}\ F(v_{n+1})$. On the other hand, we have that

$$F\left(xv_i + y|a_i^{-1}|v_{n+1}\right) = f_i(x-y)F(v_i) + \sum_{j\neq i} f_j\left(y|a_i^{-1}|a_j\right)F(v_j).$$

Thus,

$$[f_i(x-y) - f_i(x)]F(v_i) + \sum_{j\neq i} f_j\left(y|a_i^{-1}|a_j\right)F(v_j) \in \mathrm{sp}F(v_{n+1}). \tag{4}$$

Since $y|a_i^{-1}|(a_1v_1+\dots a_nv_n)\in \mathrm{sp}v_{n+1}$, it follows that

$$F\left(y|a_i^{-1}|(a_1v_1+\dots a_nv_n)\right)=f_i(-y)F(v_i)+\sum_{j\neq i}f_j\left(y|a_i^{-1}|a_j\right)F(v_j)\in \mathrm{sp}F(v_{n+1})$$

and so, by comparing to Eq. (4), we conclude that $f_i(x-y)-f_i(x)=f_i(-y)$.

Next, we check that $f_i(x)=-f_i(-x)$ for all i and every x (>0). Notice that if $a_i<0$, then we already established that $f_i(x)+f_i(-y)=g_i(x-y)$ for $x,y\geq 0$ and so plugging $x=y$ implies that $f_i(-x)=-f_i(x)$ (recall that $f_i(0)=0$). So, we assume that $a_i>0$. If there does not exist a vector $z\in \mathrm{sp}\{v_j\}_{j\neq i}$ such that $-xv_i+z\in K_1$, then $f_i(-x)=-f_i(x)$ by the definition of f_i (here, one should recall the definition of f_i in the proof of Theorem 2.2). Assume there exists a vector $z\in \mathrm{sp}\{v_j\}_{j\neq i}$ such that $-xv_i+z\in K_1$. Without loss of generality, $z=\sum_{j\neq i}^n z_i$ with $z_i>0$, as for every $p>0$, adding $pv_j\in K_1$ to $-xv_1+z$ remains in K_1. Now, on the one hand, we have that for some $c\in\mathbb{R}$,

$$\begin{aligned}F\left(-xv_i+z+xa_i^{-1}v_{n+1}\right)&=f_i(-x)F(v_i)+F(z)+cF(v_{n+1})\\&\in F(-xv_i+z)+\mathrm{sp}F(v_{n+1}).\end{aligned}\tag{5}$$

Since $F(-xv_1+z+xa_i^{-1}v_{n+1})\in \mathrm{sp}\{F(v_j)\}_{j\neq i}$, it follows that $c=-f_i(-x)b_i^{-1}$. On the other hand, we have that

$$F\left(-xv_i+z+xa_i^{-1}v_{n+1}\right)=\sum_{j\neq i}f_j\left(z_2+xa_i^{-1}a_j\right)F(v_j).$$

By the previous arguments, we know that $f_i(z_i+xa_i)=f_i(z_i)+f_i(xa_i)$ (recall that $a_i>0$) and so the above equality becomes

$$\begin{aligned}F\left(-xv_i+z+xa_i^{-1}v_{n+1})\right)&=F(z)+\sum_{j\neq i}f_j\left(z_j+xa_i^{-1}a_j\right)F(v_j)\\&=-f_i(x)F(v_i)+F(z)+f_i(x)F(v_i)+\sum_{j\neq i}f_j\left(z_j+xa_i^{-1}a_j\right)F(v_j)\\&=-f_1(x)F(v_1)+F(z)+F\left(xa_i^{-1}v_{n+1}\right).\end{aligned}$$

Since $F(-xv_1+z+xa_i^{-1}v_{n+1})\in \mathrm{sp}\{F(v_j)\}_{j\neq i}$, it follows that $F(xa_i^{-1}v_{n+1})=f_i(x)b_i^{-1}F(v_{n+1})$ (recall that by assumption $F(cv_{n+1})\in \mathrm{sp}F(v_{n+1})$). So, the above equation becomes

$$F\left(-xv_i+z+xa_i^{-1}v_{n+1})\right)=-f_i(x)F(v_i)+F(z)+f_i(x)b_i^{-1}F(v_{n+1})$$

Comparing the coefficient of, say, $F(v_2)$ above with its coefficient in Eq. (5), we get that $f_i(x)=-f_i(-x)$.

Now, we are ready to conclude that $f_i(x+y) = f_i(x) + f_i(y)$ for all $x, y \in \mathbb{R}$. Indeed, assume that $a_i > 0$. Then, for all $x, y \geq 0$, we have already checked that $f_i(x+y) = f_i(x) + f_i(y)$. If $x \geq y$, then $f_i(x) = f_i(x-y+y) = f_i(x-y) + f_i(y)$ and so, $f_i(x-y) = f_i(x) - f_i(y) = f_i(x) + f_i(-y)$. If $y > x$, then

$$f_i(x-y) = -f_i(y-x) = -(f_i(y) - f_i(x)) = f_i(x) - f_i(y) = f_i(x) + f_i(-y).$$

If $a_i < 0$, then we already showed that $f_i(x-y) = f_i(x) + f_i(-y)$ and now we have that $f_i(y) = f_i(x+y) + f_i(-x) = f_i(x+y) - f_i(x)$ and so $f_i(x+y) = f_i(x) + f_i(y)$. Thus, the f_i's are additive, and in particular F is an additive map. □

2.2 Doubly ruled mappings

Here, we prove the following proposition which will help us later on in showing that cone order isomorphisms map parallel extremal lines to parallel extremal lines.

Proposition 2.12 *Let $n \geq 3$. Let $F : \mathbb{R}^2_+ \to \mathbb{R}^n$ be an injective mapping. Assume that F maps each ray parallel either to the X-axis or the Y-axis onto a ray, and that the endpoint is mapped to the endpoint. That is, for any $x_0 \in \mathbb{R}_+$, $F(x_0, \mathbb{R}_+)$ is a ray emanating from $F(x_0, 0)$ and for any $y_0 \in \mathbb{R}_+$, $F(\mathbb{R}_+, y_0)$ is a ray emanating from $F(0, y_0)$. Then, there exist two two-dimensional subspaces H_X and H_Y of $\mathbb{R}^n$ such that all rays parallel to the X-axis are mapped to rays parallel to H_X and all rays parallel to the Y-axis are mapped to rays parallel to H_Y.*

Proof Without loss of generality, we may assume that $n = 3$, since the image of F is easily seen to be contained in a three-dimensional subspace; for instance, the one spanned by $F(0, \mathbb{R}_+)$, $F(1, \mathbb{R}^+)$, and $F(\mathbb{R}_+, 0)$. By translating F, we may assume that $F(0, 0) = 0$. Let $x_1, x_2 \in \mathbb{R}_+$ and assume that the rays $F(\mathbb{R}_+, y_1)$ and $F(\mathbb{R}_+, y_2)$ span a two-dimensional subspace S. Then, the image of F is contained in S. Indeed, each ray of the form $F(x_0, \mathbb{R}_+)$ intersects both $F(\mathbb{R}_+, y_1)$ and $F(\mathbb{R}_+, y_2)$ and so each point in $\mathbb{R}^2_+$ is mapped to a point on S. Clearly, in this case, the proof is complete and $H_X = H_Y$. Therefore, we may assume that given three distinct points $y_1, y_2, y_3 \in \mathbb{R}_+$, the rays $F(\mathbb{R}_+, y_1)$, $F(\mathbb{R}_+, y_2)$, and $F(\mathbb{R}_+, y_3)$ are in general position in $\mathbb{R}^3$. Thus, by composing F with a linear invertible transformation, we may assume, without loss of generality, that $F(0, \mathbb{R}_+) = (0, 0, \mathbb{R}_+)$, $F(\mathbb{R}_+, y_1) = (\mathbb{R}_+, 0, z_1)$, $F(\mathbb{R}_+, y_3) = (0, \mathbb{R}_+, z_3)$ and that $F(0, y_2) = (0, 0, z_2)$ with $z_1 < z_2 < z_3$. It remains to show that $F(\mathbb{R}_+, y_2)$ is parallel to the XY- axis. Indeed, since each point on $(\mathbb{R}_+, y_2)$ intersects a ray emanating from a point $(x_0, 0)$ and passing through both (x, y_1) and (x, y_3), it follows that each point on $F(\mathbb{R}_+, y_2)$ intersects a ray connecting a point on $(\mathbb{R}_+, 0, z_1)$ and a point on $(0, \mathbb{R}_+, z_3)$. Hence, the ray $F(\mathbb{R}_+, y_2)$ must not intersect neither of the translates of the XY plane, $(\mathbb{R}, \mathbb{R}, z_1)$ and $(\mathbb{R}, \mathbb{R}, z_3)$. Since $z_1 < z_2 < z_3$, it follows that $F(\mathbb{R}_+, y_2)$ must be parallel to the XY plane. Since the original assumptions were symmetric in X and Y, the proof is complete. □

Remark 2.13 The above proposition is strongly connected with the characterization of doubly ruled surfaces, see e.g., [20], p. 7. The general theorem characterizing doubly ruled surfaces states that if a surface admits two different "rulings," meaning that it has two essentially different parameterizations of the form $x(s,t) = c(s) + tw(s)$, then it is a subset of either a plane, a hyperbolic paraboloid, or a rotational hyperboloid. In our case, the image of F cannot be a subset of a rotational hyperboloid. The difference is that generally doubly ruled surfaces are not assumed to have a parametrization, which is a ruling in each of the two parameters (i.e., a parametrization $x(s,t)$ that gives lines whenever one of the parameters is kept constant) but merely to have two different "rulings," that is, two different realizations as a union of straight lines. However, one should notice that for a compact set, having a simultaneous double ruling does not imply that it is a part of a hyperbolic paraboloid (or a plane) as the example

$$\left\{\cos(s) - \frac{\cos(t)-\cos(s)}{\sin(t)+\sin(s)} \cdot \sin(s),\ \sin(s) + \frac{\cos(t)-\cos(s)}{\sin(t)+\sin(s)} \cdot \cos(s),\ \frac{\cos(t)-\cos(s)}{\sin(t)+\sin(s)}\right\}$$

shows, so that not only the simultaneity but also the unboundedness of the rays plays a role in the above proposition. We remark that Alexandrov also used this classification when he proved his characterization of order isomorphisms preserving the light-cone structure in [4].

3 A characterization of extremal vectors

In this section, we return to the main subject of this paper, which is order isomorphisms, and provide the needed links with maps sending lines to lines (or rays to rays).

We need to show that certain lines are mapped to lines under the given order isomorphism. To this end, we prove a standard characterization lemma that states that extremal vectors (and consequently translates of extremal lines) have a unique property that is preserved under order isomorphisms. Using this property, we shall prove that translates of extremal lines are mapped to lines. Although such arguments are standard, we give a proof in order for this note to be as self-contained as possible. We then pass to the proofs of Propositions 3.4 and 3.5, for which the standard arguments are not enough, and which play a key role in the proof of Theorems 1.3 and 1.7.We remark here that in this section, most of the results hold for any real normed space.

To state our characterization lemma, let us recall a standard notation and some simple facts. Let $(X, \leq_K)$ be an ordered vector space. Let $x, y \in X$. The order interval of x and y is the set $[x, y]_o := \{z \in X : x \leq_K z \leq_K y\}$. The segment $\{x + \lambda(y - x) : 0 \leq \lambda \leq 1\}$ is denoted by $[x, y]$. Note that, by definition, $x \in K$ is an extremal vector if and only if $[0, x]_o = [0, x]$. Moreover, it is easy to check that a line L is a translation of an extremal line of K if and only if $[x, y]_o \cup [y, x]_o = [x, y]$ for every $x, y \in L$.

We remark here that in the following proof, we shall use the basic fact that a nonzero vector e of a closed cone K is an extremal vector if and only if it cannot be written as a sum of two linearly independent vectors in the cone (see e.g., [5, Lemma 1.43]).

Lemma 3.1 *Let X be a real normed space. Let $K \subset X$ be a closed cone and let $e \in K$. Then, e is an extremal vector of K if and only if*

$$z_1, z_2 \in [0, e]_o \Rightarrow z_1 \leq_K z_2 \quad or \quad z_2 \leq_K z_1 \tag{6}$$

Proof Assume first that e is an extremal vector of K. Then, by definition,

$$[0, e]_o = [0, e]$$

and therefore (6) holds. For the converse, assume that e is not an extremal vector (note that we may assume that X is of dimension greater than 1 since otherwise e must be an extremal vector of K). Hence, there exist two linearly independent vectors $x_1, x_2 \in K$ such that $e = x_1 + x_2$. Denote the two-dimensional subspace spanned by x_1 and x_2 by S. Define $K_S = S \cap K$. Clearly, K_S is a two-dimensional non-degenerate closed cone. Notice that such a cone must have exactly two distinct extremal rays. Thus, we may choose two linearly independent extremal vectors u, v of K_S. Let L denotes the one-dimensional subspace spanned by $u - v$. Since neither $u \geq_K v$ nor $v \geq_K u$, we have

$$L \cap K = L \cap (-K) = \{0\}. \tag{7}$$

Since x_1, x_2, and L are lying in one plane, and since $x_1 \notin L$, there exist $w \in L$ and $\alpha > 0$ such that

$$x_2 = w + \alpha x_1.$$

Notice that, indeed, α must be positive, for otherwise we would have that $w = -\alpha x_1 + x_2 \geq_K 0$, a contradiction to (7). Dividing the equation by $1 + \alpha$, we obtain

$$\frac{1}{1+\alpha} x_2 = \frac{1}{1+\alpha} v + \alpha \frac{1}{1+\alpha} x_1.$$

Define $z_1 := \alpha \frac{1}{1+\alpha} x_1$ and $z_2 := \frac{1}{1+\alpha} x_2$. Clearly, $0 \leq_K z_i \leq_K x_i \leq_K e$ for $i = 1, 2$. Moreover, we have that $z_2 - z_1 = \frac{1}{1+\alpha} v \in L$, hence neither $z_1 \leq_K z_2$ nor $z_2 \leq_K z_1$ and so (6) does not hold. □

The following lemma extends the characterization in Lemma 3.1 for translations of extremal lines of closed cones.

Lemma 3.2 *Let X be a real normed space. Let $K \subset X$ be a closed cone. Let $x_0 \in X$ and let $L \subset X$ be a one-dimensional subspace. Then, L is an extremal line of K if and only if there exist two different vectors $x, y \in x_0 + L$ such that $x \leq_K y$ and*

$$z_1, z_2 \in [x, y]_o \Rightarrow z_1 \leq_K z_2 \quad \text{or} \quad z_2 \leq_K z_1 \tag{8}$$

Consequently, in such a case, any two vectors $x, y \in x_0 + L$ are comparable and satisfy Eq. (8).

Proof First, notice that if L is an extremal line of K, then every $x, y \in x_0 + L$ are comparable, that is,

$$x \leq_K y \quad \text{or} \quad y \leq_K x.$$

Next, observe that for every two vectors $z_1, z_2 \in X$, we have that

$$z_1 \leq_K z_2 \iff z_1 - x \leq_K z_2 - x,$$

hence, Eq. (8) holds if and only if

$$z_1', z_2' \in [0, y - x]_o \Rightarrow z_1' \leq_K z_2' \quad \text{or} \quad z_2' \leq_K z_1'.$$

Thus, by Lemma 3.1, Eq. (8) holds if and only if $y - x$ is an extremal vector of K, or in other words, L is an extremal line of K. □

The preceding characterization yields the following result, which states that order isomorphisms (associated with closed cones) must map translates of extremal lines to translates of extremal lines.

Lemma 3.3 *Let X, Y be real normed spaces. Let $K_1 \subset X$, $K_2 \subset Y$ be closed cones. Let $T : (K_1, \leq_{K_1}) \to (K_2, \leq_{K_2})$ be an order isomorphism. Let L be an extremal line of K_1 and let $x_0 \in K_1$. Then, there exists an extremal line L' of K_2 such that*

$$T((x_0 + L) \cap K_1) = (T(x_0) + L') \cap K_2.$$

The same result holds when one assumes $T : (\mathbb{R}^n, \leq_{K_1}) \to (\mathbb{R}^n, \leq_{K_2})$ and general $x_0 \in \mathbb{R}^n$ and for $T : (int(K_1), \leq_{K_1}) \to (int(K_2), \leq_{K_2})$ with $x_0 \in int(K_1)$.

Proof We write the proof in the more general $T : (S_1, \leq_{K_1}) \to (S_2, \leq_{K_2})$ formulation, where S_i is either K_i, $int(K_i)$ or $\mathbb{R}^n$. First, we show that $T((x_0 + L) \cap S_1) \subset (T(x_0) + L') \cap S_2$, for some extremal line L' of K_2. Let $y, z \in S_1 \cap (x_0 + \mathcal{L})$. Without loss of generality, we may assume that $y \leq_{K_1} x_0 \leq_{K_1} z$ and so $T(y) \leq_{K_2} T(x_0) \leq_{K_2} T(z)$. Let L' denote the one-dimensional subspace for which $T(y) + L' = T(z) + L'$. Let $z_1', z_2' \in S_2 \cap [T(y), T(z)]_o$. Note that in our three cases of S_2, we always have by assumption that $S_2 \cap [T(y), T(z)]_o = [T(y), T(z)]_o$. Since T is onto, there exist $z_1, z_2 \in S_1$ such that $T(z_i) = z_i'$ for each $i \in \{1, 2\}$. Moreover, $z_1, z_2 \in S_1 \cap [y, z]_o$. By Lemma 3.2, either $z_1 \leq_{K_1} z_2$ or $z_2 \leq_{K_1} z_1$, and so, either $z_1' \leq_{K_2} z_2'$ or $z_2' \leq_{K_2} z_1'$. Applying Lemma 3.2 once more implies that L' is an extremal line of K_2. Since $T(x_0) \in S_2 \cap [T(y), T(z)]_o$ it follows that $T(x_0) + L' = T(y) + L' = T(z) + L'$, so that L' is independent of the choice of y or of z and therefore $T((x_0 + L) \cap S_1) \subset (T(x_0) + L') \cap S_2$. Applying the same reasoning for T^{-1} implies that $T^{-1}((T(x_0) + L') \cap S_2) \subset (x_0 + L) \cap S_1$. Thus, $T((x_0 + L) \cap S_1) = (T(x_0) + L') \cap S_2$. □

In order to use Theorem 2.2 for the proof of Theorems 1.3 and 1.7, we need to show that order isomorphisms satisfy a parallelism condition, namely not only map extremal lines to extremal lines but also map parallel lines to parallel lines. We do so in the following proposition.

Proposition 3.4 *Let $n \geq 2$. Let $K_1, K_2 \subset \mathbb{R}^n$ be closed, non-degenerate, cones. Let $T : (K_1, \leq_{K_1}) \to (K_2, \leq_{K_2})$ be an order isomorphism. Let L be an extremal line of K_1. Then, there exists an extremal line L' of K_2 such that*

$$T((x_0 + L) \cap K_1) = (T(x_0) + L') \cap K_2,$$

for every $x_0 \in K_1$.

Proof First, we show that on any translate of a two-dimensional subspace spanned by two extremal vectors of K_1, parallel translates of extremal lines are mapped to parallel lines. To this end, fix any point $x_0 \in K_1$ and let L_1 and L_2 be two different extremal lines of K_1, with $e_1 \in K_1 \cap L_1$ and $e_2 \in K_1 \cap L_2$. By Lemma 3.3, for any $s \in \mathbb{R}_+$, T maps $(x_0 + se_2 + L_1) \cap K_1$ to $(T(x_0 + se_2) + L') \cap K_2$ for some extremal line of K_2, L'. Since L_1 and L' are extremal lines and T is an order isomorphism, it follows that T maps any ray of the form $x_0 + se_2 + \mathbb{R}_+e_1$, emanating from $x_0 + se_2$, onto a ray emanating from $T(x_0 + se_2)$, parallel to an extremal ray. Similarly, T maps any ray of the form $x_0 + te_1 + \mathbb{R}_+e_2$, emanating from $x_0 + te_1$, onto a ray emanating from $T(x_0 + te_1)$, parallel to an extremal ray. Define $F(s,t) = T(x_0 + se_1 + te_2)$ and note that $F : \mathbb{R}_+^2 \to \mathbb{R}^n$ satisfies the conditions of Proposition 2.12, and therefore we may use the proposition to conclude that there exist two planes H_1 and H_2 such that T maps rays of the form $x_0 + se_2 + \mathbb{R}_+e_1$, to rays parallel to H_1 and rays of the form $x_0 + te_1 + \mathbb{R}_+e_2$ to rays parallel to H_2. However, there cannot be more than two extremal rays on a given two-dimensional subspace, so that in fact the extremal rays parallel to H_1 are of at most two forms, say L' and L''. However, once we have two parallel lines mapped to parallel lines, this implies that the whole image of $\mathbb{R}_+^2$ is mapped into the plane spanned by these two vectors.

Again using that there cannot be more than two extremal rays on a given two-dimensional subspace, and since the rays $T(x_0 + e_1 + \mathbb{R}_+e_2)$, $T(x_0 + \mathbb{R}_+e_1 + e_2)$ are parallel to two distinct extremal rays, it follows that T maps rays of the form $x_0 + se_2 + \mathbb{R}_+e_1$ to parallel rays and rays of the form $x_0 + te_1 + \mathbb{R}_+e_2$ to parallel rays. To conclude, we have that T maps lines of the form $(x_0 + se_2 + L_1) \cap K_1$ to parallel lines of the form $(T(x_0 + se_2) + L') \cap K_2$ (that is, L' is independent of s).

Next, we show that T maps any parallel translates of extremal lines to parallel lines. Let $x_0 \in K_1$. Then, there are n linearly independent extremal vectors $x_1, \dots, x_n \in K_1$, such that $x_0 = \alpha_1x_1 + \alpha_2x_2 + \cdots + \alpha_nx_n$ for some nonnegative coefficients $\{\alpha_i\}_{i=1}^n$ (follows by Krein-Milman for example). For each i, let L_i denote the extremal line generated by x_i. Let L be an extremal line of K_1. By the first observation of this proof, we have that $T(L \cap K_1)$ is parallel to $T((\alpha_1x_1 + L) \cap K_1)$ which, in turn, is parallel to $T((\alpha_1x_1 + \alpha_2x_2 + L) \cap K_1)$ and so on, which implies that $T(L \cap K_1)$ is parallel to $T((\alpha_1x_1 + \alpha_2x_2 + \cdots + \alpha_nx_n + L) \cap K_1) = T((x_0 + L) \cap K_1)$ □

We also need a version of the above proposition for interiors of cones, in order to employ Theorem 2.1 for the proof of Theorem 1.7.

Proposition 3.5 *Let $n \geq 2$. Let $K_1, K_2 \subset \mathbb{R}^n$ be closed, non-degenerate, cones. Let $T : (int(K_1), \leq_{K_1}) \to (int(K_2), \leq_{K_2})$ be an order isomorphism. Let L be an extremal line of K_1. Then, there exists an extremal line L' of K_2 such that*

$$T((x_0 + L) \cap int(K_1)) = (T(x_0) + L') \cap int(K_2),$$

for every $x_0 \in int(K_1)$.

Proof The first part of the proof, showing that on any translate of a two-dimensional subspace spanned by two extremal vectors of K_1, parallel translates of extremal lines are mapped to parallel lines, can be copied without changes. To show that T

maps any parallel translates of extremal lines to parallel lines, pick $x_0, y_0 \in int(K_1)$. Since $int(K_1)$ is open, there exists some $z_0 \in int(K_1)$, which satisfies $z_0 \leq_{K_1} x_0$ and $z_0 \leq_{K_1} y_0$. Therefore, there are $2n$ linearly independent extremal vectors $x_1, \ldots, x_n, y_1, \ldots y_n \in K_1$, such that $x_0 = z_0+\alpha_1 x_1+\alpha_2 x_2+\cdots+\alpha_n x_n$ for some non-negative coefficients $\{\alpha_i\}_{i=1}^n$ and $y_0 = z_0+\beta_1 x_1+\beta_2 x_2+\cdots+\beta_n x_n$ for some nonnegative coefficients $\{\beta_i\}_{i=1}^n$. For each i, let L_i^x and L_i^y denote the extremal line generated by x_i and y_i respectively. Let L be an extremal line of K_1. By the first observation of this proof, we have that $T((z_0+L)\cap int(K_1))$ is parallel to $T((z_0+\alpha_1 x_1+L)\cap int(K_1))$ which, in turn, is parallel to $T((z_0 + \alpha_1 x_1 + \alpha_2 x_2 + L) \cap int(K_1))$ and so on, which implies that $T((z_0 + L) \cap int(K_1))$ is parallel to $T((x_0 + L) \cap int(K_1))$. The same being true for y_0, we get the desired result that $T((x_0 + L) \cap int(K_1))$and $T((x_0 + L) \cap int(K_1))$ are parallel. □

4 Proof of the main theorems

A proof of Theorem 1.3

Let us first consider the case $A = K$. Let $v_1, v_2, \ldots, v_{n+1} \in K$ be generic extremal vectors of K. By Proposition 3.4, T satisfies the conditions of Theorem 2.9 and so there exist bijective additive functions $f_1, f_2, \ldots, f_n : \mathbb{R} \to \mathbb{R}$, so that for every $x \in K$,

$$T(x) = \sum_{i=1}^{n} f_i(\langle x, v_i\rangle)T(v_i)$$

where $\langle\cdot,\cdot\rangle$ is a scalar product such that $v_1, \ldots, v_n$ are orthogonal. Since T is an order isomorphism and since extremal lines are mapped to extremal lines, each function f_i must be strictly increasing and, being additive, must be linear. Thus, T is a linear transformation.

In order to complete the proof for the cases $A = \mathrm{int}K$ and $A = \mathbb{R}^n$, one may use Rothaus's argument [26, Proposition 2]. In his proposition, Rothaus shows that if φ is an order isomorphism on the interior of the cone K (or on $\mathbb{R}^n$), which, by previous arguments, is continuous, then for any $u \in \mathrm{int}K$, φ induces an order isomorphism on K by $\varphi(u+x)-\varphi(u)$. This in turn implies that the induced mapping must be linear (in his case, this conclusion is drawn for a non-angular cone K) and so may be extended uniquely to a linear map on $\mathbb{R}^n$. Thus, one deduces that the restrictions of the original order isomorphism to different patches of the domain, which are obtained for different choices of u, are affine maps, which agree in pairs on open sets in $\mathrm{int}K$ (or $\mathbb{R}^n$), and so are all restrictions of a single affine map. This argument thus implies the desired conclusion for the cases $A = \mathrm{int}K$ and $A = \mathbb{R}^n$. □

A proof of Theorem 1.7

Let us first consider the case where $A = K$ or $A = \mathbb{R}^n$.

Let $v_1, v_2, \ldots, v_n \in K$ be linearly independent extremal vectors. By Proposition 3.4, T satisfies the conditions of Theorem 2.2 and so there exist bijective functions

$f_1, f_2, \ldots, f_n : \mathbb{R} \to \mathbb{R}$ so that for every $x \in K$,

$$T(x) = \sum_{i=1}^{n} f_i(\langle x, v_i \rangle) T(v_i)$$

where $\langle \cdot, \cdot \rangle$ is a scalar product such that $v_1, \ldots, v_n$ are orthogonal. Since T is an order isomorphism and since extremal lines are mapped to extremal lines, each function f_i must be strictly increasing.

The proof of the case where $A = \mathrm{int} K$ is literally the same, only one should use Proposition 3.5 and Theorem 2.1 instead of Proposition 3.4 and Theorem 2.2. $\square$

5 Order isomorphisms of ellipsoids

In this section, we prove Theorem 1.10. We divide the proof into several steps. The first step is the main observation of this section, which identifies the class $\mathcal{E}_b^n$ of all closed, bounded and centered ellipsoids, equipped with the inclusion order relation, with the cone of symmetric positive semi-definite matrices on $\mathbb{R}^n$, denoted here by PSD, equipped with the standard order for self-adjoint matrices, namely $A \leq B$ whenever $B - A$ is a positive semi-definite matrix. Similarly, we identify the class $\mathcal{E}_c^n$ of all non-degenerate ellipsoids in $\mathcal{E}_b^n$ with the interior of the cone PSD, denoted here by PD.

In Sect. 5.2 we apply our main results for order isomorphisms of PSD and deduce that they are linear maps. We describe known results regarding the form of all linear transformations preserving the cone PSD, which will enable us to conclude that the original mapping on ellipsoids is induced by a linear transformation on $\mathbb{R}^n$. The proof of Theorem1.10 is given in Sect. 5.3, where the case of generalized ellipsoids requires some additional arguments.

5.1 Identification of ordered ellipsoids with the cone PSD

Let $\mathcal{E} \in \mathcal{E}_b^n$ be an ellipsoid and let A be a matrix for which $AD_n = \mathcal{E}$. By polar decomposition $A = RU$ where U is a unitary matrix and R is a symmetric positive definite matrix. We identify the matrix R with $\mathcal{E} = RD_n$ and denote it by $\mathcal{E}_R$. Indeed, this identification is well defined; Assume that A, B are symmetric positive definite matrices that satisfy $AD_n = BD_n$. Then, $B^{-1}AD_n = D_n$ and so, by polar decomposition, $B^{-1}A$ is a unitary matrix U. By the uniqueness of the polar decomposition, $AI = BU$ implies $A = B$.

Next, we show that the inclusion order relation on ellipsoids corresponds to the order relation on positive semi-definite matrices, induced by the cone of symmetric positive semi-definite matrices PSD.

Lemma 5.1 $\mathcal{E}_A \subset \mathcal{E}_B \iff A \leq B$

Proof Let $||\cdot||_{op}$ denote the standard operator norm on operators on $\mathbb{R}^n$. First, we observe that

$$\mathcal{E}_A \subset \mathcal{E}_B \iff A(D_n) \subset B(D_n) \iff B^{-1}A(D_n) \subset D_n \iff ||B^{-1}A||_{op} \le 1$$

Since A and B are symmetric, it follows that

$$||AB^{-1}||_{op} = ||(AB^{-1})^*||_{op} = ||B^{-1}A||_{op} \le 1.$$

By [17, Theorem 7.7.3], $A \le B \iff ||AB^{-1}||_{op} \le 1$. For completeness, we provide the proof: we use the fact that any two symmetric matrices may be simultaneously diagonalized by conjugation, that is, there exist an invertible matrix $C \in GL_n$ so that $B = CIC^*$ and $A = CDC^*$, where $D = \text{diag}\{d_1, \dots, d_n\}$. Indeed, there exists an invertible matrix C' such that $C'^*BC' = I$ (take $(\sqrt{B})^{-1}$), and since C'^*AC' is symmetric, there exists a unitary matrix so that $C'^*AC' = UDU^*$ and so this fact is proved. Now, $A \le B$ if and only if $C[I - D]C^* \ge 0$ which holds if and only if $d_i \le 1$ for all i. But, since $AB^{-1} = CDC^*C^{*-1}C^{-1} = CDC^{-1}$, the eigenvalues of AB^{-1} are precisely $d_1, d_2, \dots, d_n$ and all $d_i \le 1$ if and only if $||AB^{-1}||_{op} \le 1$ □

5.2 Order isomorphisms of the cone PSD and its interior PD

It is a known fact that the extremal vectors of the cone PSD are positive semi-definite matrices of rank 1. This is due to the fact that any positive semi-definite matrix of rank k can be written as a sum of k positive semi-definite matrices of rank 1. Moreover, such matrices are exactly of the form $v \otimes v$, where we use this standard notation for the matrix $A = vv^T$, that is, $Ax = \langle x, v\rangle v$. Next, we explain why the cone PSD is a "good" cone, namely has at least $N + 1$ N-independent extremal vectors, where $N = n(n + 1)/2$ is the dimension of the space of all symmetric matrices in $\mathbb{R}^n$, so that our main theorem may be applied.

To this end, we construct a family of $N + 1$ matrices of the form $v \otimes v$, which are N-independent. Let e_i denote the standard basis of $\mathbb{R}^n$. The first n matrices will be $e_i \otimes e_i$. The next $\binom{n}{2}$ will be $(e_i + e_j) \otimes (e_i + e_j)$ for $i \ne j$. These N matrices are a basis of the space of symmetric $n \times n$ matrices. The $(N + 1)$th matrix we choose is $(\sum_{i=1}^n e_i) \otimes (\sum_{i=1}^n e_i)$. One may easily verify that any N of the above matrices span the space of symmetric $n \times n$ matrices, and are thus linearly independent.

Therefore, the conditions of our main theorems are satisfied and we conclude:

Theorem 5.2 *Let $n \ge 2$. Let A stand for either PSD or PD and let $T : A \to A$ be an order isomorphism. Then, T is a linear transformation.*

To prove Theorem 1.10, we need the following characterization of linear maps on $\mathbb{R}^N$ (identified with space of symmetric $n \times n$ matrices), which preserve the cone PSD. We found this result in [18], and it was kindly pointed out to us by Prof. P. M. Gruber that it first appeared in the paper [27], albeit in Russian. See also [16] for a wider discussion on the isometries of the cone PSD.

Theorem 5.3 (Ryshkov) *Let $F : \mathbb{R}^N \to \mathbb{R}^N$ be an invertible linear transformation preserving the cone PSD, that is, $F(PSD) = PSD$. Then, there exists an invertible matrix $U \in M_{n\times n}$ such that $F(A) = U^T AU$ for every symmetric matrix $A \in \mathbb{R}^N$.*

5.3 Order isomorphisms for ellipsoids

We are now ready to prove Theorem 1.10 for the classes $\mathcal{E}_c^n$ and $\mathcal{E}_b^n$.

Proof of Theorem 1.10 (part I) Let T be an order isomorphism of $\mathcal{E}_b^n$. Let $G \in GL_n$ be an invertible matrix for which $GT(D_n) = D_n$. The map $T' = GT$ is obviously an order isomorphism of $\mathcal{E}_b^n$ as well. Using the identification of an ellipsoid $\mathcal{E}_A$ with a positive semi-definite matrix A given in Sect. 5.1, we define a mapping $F : PSD \to PSD$ by the relation $T'(\mathcal{E}_A) = \mathcal{E}_{F(A)}$. By our identification, it follows that F is an order isomorphisms of PSD, and so by Theorem 5.2, F is a linear map. Theorem 5.3, then implies that $F(A) = U^T AU$ for some invertible matrix U. Note that, by the definition of T', $F(I) = U^T U = I$, and so U is an orthogonal matrix. It follows that

$$T'(\mathcal{E}_A) = \mathcal{E}_{F(A)} = \mathcal{E}_{U^T AU} = U^T AUD_n = U^T AD_n = U^T \mathcal{E}_A$$

which means that U^T induces T', and so $G^{-1}U^T$ induces T, as required. The exact same proof works for the class $\mathcal{E}_c^n$. $\square$

To complete the proof of Theorem 1.10 for the case of generalized ellipsoids $\mathcal{E}_g^n$, we will need the following two geometric propositions.

Proposition 5.4 *Let $T : \mathcal{E}_g \to \mathcal{E}_g$ be an order isomorphism. Then, for all $E \in \mathcal{E}_g$, we have that* $\dim E = \dim T(E)$.

Proof First, we show that one-dimensional ellipsoids, namely symmetric segments, are mapped to symmetric segments. Indeed, let E be a symmetric segment and let $A, B \in \mathcal{E}_g$ be any two ellipsoids that are contained in E. Then, either $A \subset B$ or $B \subset A$. This property, which E satisfies, is obviously satisfied by any one-dimensional ellipsoid and only by one-dimensional ellipsoids. Since T preserves this property, that is, any two ellipsoids A', B' for which $A' \subset T(E)$ and $B' \subset T(E)$ are comparable, it follows that $T(E)$ is one-dimensional. Moreover, it follows that any two linearly dependent segments (i.e., segments in the same direction) are mapped to linearly dependent segments that are also comparable.

Next, we prove that any k linearly independent segments (i.e., segments in linearly dependent directions) are mapped to k linearly independent segments. This would clearly imply our proposition since it would imply that the dimension of both the image and the inverse image of any ellipsoid under T can only grow, and hence must be preserved. The proof is done by induction on k. For $k = 1$, the claim trivially holds. Assume that the claim holds for k and let $s_1, s_2, \ldots, s_{k+1}$ be linearly independent segments. Choose a $k - 1$ ellipsoid E for which $s_i \subset E$ for $i = 1, \ldots, k - 1$. Then, $s_k \cap E = \{0\}$. Assume that $T(s_k)$ is linearly dependent of $T(s_1), T(s_2), \ldots, T(s_{k-1})$ (which by our induction hypothesis are linearly independent). Then, for some $\lambda > 0$, we have that $\lambda T(s_k) \subset T(E)$ and so the inverse image of $\lambda T(s_k)$ is contained in E, which is a contradiction to the fact that $s_k \cap E = \{0\}$ and that $T^{-1}(\lambda T(s_k))$ is linearly dependent of s_k. □

As a consequence of Proposition 5.4, we can prove that under an order isomorphism of $\mathcal{E}_g^n$, the subclass $\mathcal{E}_b^n$ is invariant. In other words, an order isomorphism of $\mathcal{E}_g^n$ maps compact ellipsoids to compact ellipsoids.

Proposition 5.5 *Let $T : \mathcal{E}_g \to \mathcal{E}_g$ be an order isomorphism. Then, for any $E \in \mathcal{E}_b^n$, we have that $T(E) \in \mathcal{E}_b^n$.*

Proof By Proposition 5.4, T preserves dimension. Let E be a non-compact ellipsoid and assume that $T(E)$ is compact. Since E is non-compact, it contains a one-dimensional subspace $L \subset \mathbb{R}^n$. Let $e \subset L \subset E$ be a one-dimensional ellipsoid. Since T is an order isomorphism, it follows that $T(e) \subset T(E)$. Since $T(E)$ is compact, we may choose a one-dimensional ellipsoid e' such that $e' \not\subset T(E)$, and $T(e) \subset e'$. Since T is an order isomorphism, it follows that $e \subset T^{-1}(e')$ and $T^{-1}(e') \not\subset E$. Note that since T preserves dimension and $e \subset T^{-1}(e')$, it follows that $T^{-1}(e') \subset L \subset E$, which is a contradiction. Thus, compact ellipsoids are mapped to compact ellipsoids. □

We are now ready to conclude the proof of Theorem 1.10, as follows.

Proof of Theorem 1.10 (part II) By Proposition 5.5, the class $\mathcal{E}_b^n$ is invariant under our order isomorphism T, and similarly, by Proposition 5.4, the class of all generalized ellipsoids with non-empty interior is invariant under T. This class, which we denote by $(\mathcal{E}_b^n)^\circ$, can also be identified with the class of generalized ellipsoids dual to ellipsoids in $\mathcal{E}_b^n$. By the first part of the proof (for the classes $\mathcal{E}_b^n$ and $\mathcal{E}_c^n$), the restrictions of T to $\mathcal{E}_b^n$ and to $(\mathcal{E}_b^n)^\circ$ are linear maps (for the latter, compose T with polarity from both sides, say). Since the two linear maps coincide on $\mathcal{E}_c^n$, they are identical. This completes the proof. □

Remark 5.6 It is interesting to note that the embedding of $\mathcal{E}_c^n$ into PD, can be done in more than one way. For instance, instead of associating with $A \in PD$, the ellipsoid $\mathcal{E}_A$ given by AD, one may associate $E_1(A) = \sqrt{A}D$ or $E_2(A) = A^{-1}D$. The first of the two is an order preserving isomorphism, and the second is order reversing. In both cases, for an order isomorphism $T : \mathcal{E}_c^n \to \mathcal{E}_c^n$ induces an order isomorphism on PD by $E_i^{-1} \circ T \circ E_i$ for $i = 1, 2$. Note that extending the mapping E_2 to be defined on PSD gives rise to the class $(\mathcal{E}_g^n \setminus \mathcal{E}_b^n) \cup \mathcal{E}_c$, and thus the above proof works for this class as well. Another point to notice is the way each embedding combines with addition. Since the Minkowski sum of two ellipsoids is not an ellipsoid, we get a different kind of addition. Of course, in PD and PSD, addition is basic (and remains in the cone). We recall the 2-Firey sum of two convex bodies K and T with associated support functions h_K and h_T is defined to be the body (denoted $K +_2 T$) with associated support function

$$h_{K+_2T}(x) = (h_k^2(x) + h_T^2(x))^{1/2}.$$

The 2-Firey sum of two ellipsoids is again an ellipsoid. Moreover,

$$\sqrt{A}D +_2 \sqrt{B}D = \sqrt{A+B}D.$$

therefore, the embedding E_1 behaves well with respect to this addition, namely

$$E_1(A+B) = E_1(A) +_2 E_1(B).$$

Proof of Corollary 1.11 By composing the order-reversing isomorphism T with the standard polarity, one obtains an order preserving isomorphism and applies Theorem 1.10. □

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