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The M-ellipsoid, Symplectic Capacities and Volume

Shiri Artstein-Avidan, Vitali Milman* and Yaron Ostrover

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Abstract: *In this work we bring together tools and ideology from two different fields, Symplectic Geometry and Asymptotic Geometric Analysis, to arrive at some new results. Our main result is a dimension-independent bound for the symplectic capacity of a convex body by its volume radius.*

1 Short Introduction

In this work we bring together tools and ideology from two different fields, Symplectic Geometry and Asymptotic Geometric Analysis, to arrive at some new results. Our main result is a dimension-independent bound for the symplectic capacity of a convex body by its volume radius. This type of inequality was first suggested by C. Viterbo, who conjectured that among all convex bodies in $\mathbb{R}^{2n}$ with a given volume, the Euclidean ball has maximal symplectic capacity (definition in Section 2 below). More precisely, Viterbo's conjecture states that the best possible constant γ_n such that for any choice of a symplectic capacity c and any convex body $K \subset \mathbb{R}^{2n}$ we have

$$\frac{c(K)}{c(B^{2n})} \leq \gamma_n \left(\frac{\mathrm{Vol}(K)}{\mathrm{Vol}(B^{2n})} \right)^{1/n},$$

is $\gamma_n = 1$, where B^{2n} is the Euclidean unit ball in $\mathbb{R}^{2n}$. The estimate which Viterbo proved in his work [19] was $\gamma_n \leq 32n$, and in the case of centrally symmetric bodies he showed that $\gamma_n \leq 2n$. Hermann showed in [5] that for the special class of convex Reinhardt domains the conjecture holds. The first and third named authors showed in [1] that there exists a universal constant A_1 such that $\gamma_n \leq A_1(\log 2n)^2$, and also presented wide classes of bodies where the inequality holds without the logarithmic term. The methods used for that result came from Asymptotic Geometric Analysis.

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In this work we use some more advanced methods of Asymptotic Geometric Analysis to show that the logarithmic term is not needed at all. That is, there exists a universal constant A_0 for which $\gamma_n \leq A_0$ for any dimension $2n$ and all convex bodies in $\mathbb{R}^{2n}$. Moreover, these strong bounds are obtained by using linear tools only. While this fits with the philosophy of Asymptotic Geometric Analysis, this is less expected from the point of view of Symplectic Geometry where for strong results one expects to need highly nonlinear objects.

Notations: In this paper the letters A_0, A_1, A_2, A_3 and C are used to denote universal positive constants which do not depend on the dimension nor on the body involved. In what follows we identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ by associating to $z = x + iy$, where $x, y \in \mathbb{R}^n$, the vector $(x_1, y_1, \ldots, x_n, y_n)$, and consider the standard complex structure given by complex multiplication by i, i.e. $i(x_1, y_1, \ldots, x_n, y_n) = (-y_1, x_1, \ldots, -y_n, x_n)$. We denote by $\langle \cdot, \cdot \rangle$ the standard Euclidean inner product on $\mathbb{R}^{2n}$. We shall denote by $e^{i\theta}$ the standard action of S^1 on $\mathbb{C}^n$ which rotates each coordinate by angle θ, i.e., $e^{i\theta}(z_1, \ldots, z_n) = (e^{i\theta} z_1, \ldots, e^{i\theta} z_n)$. By $x^{\perp}$ we denote the hyperplane orthogonal to x with respect to the Euclidean inner product. For two sets A, B in $\mathbb{R}^{2n}$, we denote their Minkowski sum by $A + B = \{a + b : a \in A, b \in B\}$. By a convex body we shall mean a convex bounded set in $\mathbb{R}^{2n}$ with non-empty interior. Finally, since affine translations in $\mathbb{R}^{2n}$ are symplectic maps, we shall assume throughout the text that any convex body K has the origin in its interior.

Structure of the paper: The paper is organized as follows. In the next section we recall the necessary definitions from symplectic geometry, describe the history of the problem and state our main theorem. In Section 3 we describe the main tool coming from Asymptotic Geometric Analysis, called the M-ellipsoid. In Section 4 we prove our main result, and in the last section we show an additional result about convex bodies, generalizing a result of Rogers and Shephard.

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2 Symplectic geometry background and the main result.

Consider the $2n$-dimensional Euclidean space $\mathbb{R}^{2n}$ with the standard linear coordinates $(x_1, y_1, \ldots, x_n, y_n)$. One equips this space with the standard symplectic structure $\omega_{st} = \sum_{j=1}^{n} dx_j \wedge dy_j$, and with the standard inner product $g_{st} = \langle \cdot, \cdot \rangle$. Note that under the identification between $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ these two structures are the real and the imaginary part of the standard Hermitian inner product in $\mathbb{C}^n$, and $\omega(v, iv) = \langle v, v \rangle$.

In [19], Viterbo related between the symplectic way of measuring the size of sets using what is called "symplectic capacities", and the classical Riemannian approach, using the canonical volume. Among other things, he conjectured that in the class of convex bodies in $\mathbb{R}^{2n}$ with fixed volume, the Euclidean ball has maximal symplectic capacity. This isoperimetric inequality was proved in the same paper [19] up to a constant γ_n which is linear in the dimension (see Theorem 2.4 below).

In this work, we continue the approach taken in [1], where methods from Asymptotic Geometric Analysis were used to reduce the order of the above mentioned constant γ_n. In [1] it was improved from order n to order $(\log n)^2$, where n is the dimension of the space involved. In this note we improve it further, and prove an upper bound for γ_n which is independent of the dimension. Finding dimension independent estimates is a frequent goal in Asymptotic Geometric Analysis, where surprising phenomena such as concentration of measure (see e.g. [15]) imply the existence of order and structures in high dimension despite the huge complexity it involves. It is encouraging to see that such phenomena also exist in Symplectic Geometry, and although this is just a first example, we hope more will follow. Furthermore, we wish to stress that the tools we use are purely linear and the reader should not expect any difficult symplectic analysis.

In order to state our results we continue with the formal definitions.

Definition 2.1. *A symplectic capacity on* $(\mathbb{R}^{2n}, \omega_{st})$ *associates to each subset* $U \subset \mathbb{R}^{2n}$ *a non-negative number* $c(U)$ *such that the following three properties hold:*

(P1) $c(U) \leq c(V)$ *for* $U \subseteq V$ *(monotonicity)*

(P2) $c\big(\psi(U)\big) = |\alpha|\, c(U)$ *for* $\psi \in \mathrm{Diff}(\mathbb{R}^{2n})$ *such that* $\psi^* \omega_{st} = \alpha\, \omega_{st}$ *(conformality)*

(P3) $c\big(B^{2n}(r)\big) = c\big(B^2(r) \times \mathbb{C}^{n-1}\big) = \pi r^2$ *(nontriviality and normalization),*

where $B^{2k}(r)$ is the open $2k$-dimensional ball of radius r. Note that the third property disqualifies any volume-related invariant, while the first two properties imply that every two sets $U, V \subset \mathbb{R}^{2n}$ will have the same capacity provided that there exists a symplectomorphism sending U onto V. Recall that a *symplectomorphism* of $\mathbb{R}^{2n}$ is a diffeomorphism which preserves the symplectic structure i.e., $\psi \in \mathrm{Diff}(\mathbb{R}^{2n})$ such that $\psi^* \omega_{st} = \omega_{st}$. We will denote by $\mathrm{Symp}(\mathbb{R}^{2n}) = \mathrm{Symp}(\mathbb{R}^{2n}, \omega_{st})$ the group of all symplectomorphisms of $(\mathbb{R}^{2n}, \omega_{st})$.

A priori, it is not clear that symplectic capacities exist. The celebrated non-squeezing theorem of Gromov [4] shows that for $R > r$ the ball $B^{2n}(R)$ does not admit a symplectic embedding into the symplectic cylinder $Z^{2n}(r) := B^2(r) \times \mathbb{C}^{n-1}$. This theorem led to the following definitions:

Definition 2.2. *The symplectic radius of a non-empty set $U \subset \mathbb{R}^{2n}$ is*

$$c_B(U) := \sup \left\{ \pi r^2 \,\middle|\, \textit{There exists } \psi \in \mathrm{Symp}(\mathbb{R}^{2n}) \textit{ with } \psi\left(B^{2n}(r)\right) \subset U \right\}.$$

The cylindrical capacity of U is

$$c^Z(U) := \inf \left\{ \pi r^2 \,\middle|\, \textit{There exists } \psi \in \mathrm{Symp}(\mathbb{R}^{2n}) \textit{ with } \psi(U) \subset Z^{2n}(r) \right\}.$$

Note that both the symplectic radius and the cylindrical capacity satisfy the axioms of Definition 2.1 by the non-squeezing theorem. Moreover, it follows from Definition 2.1 that for every symplectic capacity c and every open set $U \subset \mathbb{R}^{2n}$ we have $c_B(U) \leq c(U) \leq c^Z(U)$.

The above axiomatic definition of symplectic capacities is originally due to Ekeland and Hofer [3]. Nowadays, a variety of symplectic capacities can be constructed in different ways. For several of the detailed discussions on symplectic capacities we refer the reader to [2], [6], [7], [9], [11] and [20].

In this work we are interested in an inequality relating the symplectic capacity of a convex body in $\mathbb{R}^{2n}$ and its volume. Viterbo's conjecture states that among all convex bodies in $\mathbb{R}^{2n}$ with a given volume, the symplectic capacity is maximal for the Euclidean ball. Note that by monotonicity this is obviously true for the symplectic radius c_B. More precisely, denote by $\mathrm{Vol}(K)$ the volume of K and abbreviate B^{2n} for the open Euclidean unit ball in $\mathbb{R}^{2n}$. Following Viterbo [19] we state

Conjecture 2.3. *For any symplectic capacity c and for any convex body $K \subset \mathbb{R}^{2n}$*

$$\frac{c(K)}{c(B^{2n})} \leq \left(\frac{\mathrm{Vol}(K)}{\mathrm{Vol}(B^{2n})} \right)^{1/n}$$

and equality is achieved only for symplectic images of the Euclidean ball.

The first result in this direction is due to Viterbo [19]. Using John's ellipsoid he proved:

Theorem 2.4 (Viterbo). *For a convex body $K \subset \mathbb{R}^{2n}$ and a symplectic capacity c we have*

$$\frac{c(K)}{c(B^{2n})} \leq \gamma_n \left(\frac{\mathrm{Vol}(K)}{\mathrm{Vol}(B^{2n})} \right)^{1/n}$$

where $\gamma_n = 2n$ if K is centrally symmetric and $\gamma_n = 32n$ for general convex bodies.

In [5], Hermann constructed starshaped domains in $\mathbb{R}^{2n}$, for $n > 1$, with arbitrarily small volume and fixed cylindrical capacity. Therefore, in the category of starshaped domains the above theorem with *any* constant γ_n independent of the body K must fail. In addition, he proved the above conjecture for a special class of convex bodies which admit many symmetries called convex Reinhardt domains (for definitions see [5]).

In [1], the first and third named authors showed

Theorem 2.5. *There exists a universal constant A_1 such that for a convex body $K \subset \mathbb{R}^{2n}$ and a symplectic capacity c we have*

$$\frac{c(K)}{c(B^{2n})} \le A_1 (\log 2n)^2 \left(\frac{\mathrm{Vol}(K)}{\mathrm{Vol}(B^{2n})} \right)^{1/n}$$

(so, $\gamma_n \le A_1(\log 2n)^2$).

They also showed that for many classes of convex bodies, the logarithmic term is not needed. Among these classes are all the ℓ_p^n-balls for $1 \le p \le \infty$, all zonoids (bodies that can be approximated by Minkowski sums of segments) and other wide classes of convex bodies, see [1].

In this work we eliminate the logarithmic factor from the above theorem. Before we state our main results we wish to re-emphasize that, as in [1], we work exclusively in the category of linear symplectic geometry. That is, we restrict ourselves to the concrete class of linear symplectic transformations. It turns out that even in this limited category, the tools are powerful enough to obtain a dimension independent estimate for γ_n in Theorem 2.4. More precisely, let $\mathrm{Sp}(\mathbb{R}^{2n}) = \mathrm{Sp}(\mathbb{R}^{2n}, \omega_{st})$ denote the group of linear symplectic transformation of $\mathbb{R}^{2n}$. We consider a more restricted notion of linearized cylindrical capacity, which is similar to c^Z but where the transformation ψ is taken only in $\mathrm{Sp}(\mathbb{R}^{2n})$ namely

$$c^Z_{lin}(U) := \inf \left\{ \pi r^2 \mid \text{ There exists } \psi \in \mathrm{Sp}(\mathbb{R}^{2n}) \text{ with } \psi(U) \subset Z^{2n}(r) \right\}.$$

Of course, it is always true that for every symplectic capacity c we have $c \le c^Z \le c^Z_{lin}$.

Our main result is that for some universal constant A_0 one has that $\gamma_n \le A_0$ for all n. This follows from the following theorem, which we prove in Section 4.

Theorem 2.6. *There exists a universal constant A_0 such that for every even dimension $2n$ and any convex body $K \subset \mathbb{R}^{2n}$*

$$\frac{c^Z_{lin}(K)}{c(B^{2n})} \le A_0 \left(\frac{\mathrm{Vol}(K)}{\mathrm{Vol}(B^{2n})} \right)^{1/n}.$$

3 Asymptotic geometric analysis background: M-position

In this section we work in $\mathbb{R}^n$ with the Euclidean structure, without a symplectic or complex structure. We review some well known theorems from Asymptotic Geometric

Analysis which we will use in later sections. A position of a convex body is equivalent to a choice of a Euclidean structure, or, in other words, a choice of some ellipsoid as the Euclidean unit ball. A fundamental object in Asymptotic Geometric Analysis, which was discovered by the second named author in relation with the reverse Brunn-Minkowski inequality, is a special ellipsoid now called the Milman ellipsoid, abbreviated M-ellipsoid. This ellipsoid has several essentially equivalent definitions, the simplest of which may be the following:

Definition 3.1. *An ellipsoid $\mathcal{E}_K$ is called an M-ellipsoid (with constant C) of K if* $\mathrm{Vol}(\mathcal{E}_K) = \mathrm{Vol}(K)$ *and it satisfies*

$$\mathrm{Vol}(K + \mathcal{E}_K)^{1/n} \leq C\mathrm{Vol}(K)^{1/n}, \quad \text{and} \quad \mathrm{Vol}(K \cap \mathcal{E}_K)^{1/n} \geq C^{-1}\mathrm{Vol}(K)^{1/n}.$$

The fact that there exists a universal C such that every convex body K has an M-ellipsoid (with constant C) was proved in [12] for a symmetric body K. The fact that the body K need not be symmetric, for the existence of an M-ellipsoid with the properties which we use in the proof of our main result, was proved in [13] (see Theorem 1.5 there). A complete extension of all M-ellipsoid properties in the non-symmetric case was performed in [14], where it was shown that the right choice of the origin (translation) in the case of a general convex body is the barycenter (center of mass) of the body.

This ellipsoid was invented in order to study the "reverse Brunn-Minkowski inequality" which is proved in [12], and we begin by recalling this inequality, which we will strongly use in the proof of our main theorem. We then describe some further properties of this ellipsoid. Recall that the classical Brunn-Minkowski inequality states that if A and B are non-empty compact subsets of $\mathbb{R}^n$, then

$$\mathrm{Vol}(A + B)^{1/n} \geq \mathrm{Vol}(A)^{1/n} + \mathrm{Vol}(B)^{1/n}.$$

Although at first sight it seems that one cannot expect any inequality in the reverse direction (imagine, for example, two very long and thin ellipsoids pointing in orthogonal directions in $\mathbb{R}^2$), if one allows for an extra choice of "position", a reverse inequality is possible.

It was discovered in [12] that one can reverse the Brunn-Minkowski inequality, up to a universal constant factor, as follows: for every convex body K there exists a linear transformation T_K, which is volume preserving, such that for any two bodies K_1 and K_2, the bodies $T_{K_1}K_1$ and $T_{K_2}K_2$ satisfy an inverse Brunn-Minkowski inequality. It turned out that the right choice of T_K is such that the ellipsoid $r(T_K)^{-1}B^n$ (for the right choice of r) is an M-ellipsoid of K, which we denote as before by $\mathcal{E}_K$. We then say that the body $T_K K$ is in M-position (or that it is an M-position of K). Thus, a body is in M-position if a multiple of the Euclidean ball B^n is an M-ellipsoid for K. We remark that an M-ellipsoid of a body is far from being unique, and a body can have many different such ellipsoids. For a detailed account about M-ellipsoids we refer the reader to [13] and [17], where there are also proofs of the theorems below. The property of M-position which we use in this paper for the proof of Theorem 2.6 is the following

Theorem 3.2. *There exists a universal constant C such that if $\widetilde{K}_1, \widetilde{K}_2 \subset \mathbb{R}^n$ are two convex bodies in M-position then*

$$\mathrm{Vol}(\widetilde{K}_1 + \widetilde{K}_2)^{1/n} \le C \left(\mathrm{Vol}(\widetilde{K}_1)^{1/n} + \mathrm{Vol}(\widetilde{K}_2)^{1/n} \right). \tag{1}$$

In particular this theorem implies that for a convex body K there exists a transformation T_K, which depends solely on K, such that for any two convex bodies K_1 and K_2, denoting $\widetilde{K}_1 = T_{K_1}(K_1), \widetilde{K}_2 = T_{K_2}(K_2)$, we have that (1) is satisfied. The transformation T_K is the transformation which takes the ellipsoid $\mathcal{E}_K$ to a multiple of B^n. Therefore, it is clear that any composition of T_K with an orthogonal transformation from the left will also satisfy this property.

This ellipsoid $\mathcal{E}_K$ has many more well known intriguing properties. We recall one of them, which we will use in Section 5:

Theorem 3.3. *There exists a universal constant C such that for any convex body K, the ellipsoid $\mathcal{E}_K$ satisfies the following: for every convex body P one has that*

$$C^{-1}\mathrm{Vol}(P + \mathcal{E}_K)^{1/n} \le \mathrm{Vol}(P + K)^{1/n} \le C\mathrm{Vol}(P + \mathcal{E}_K)^{1/n}. \tag{2}$$

4 Proof of the Main Result

We return to $\mathbb{R}^{2n}$ equipped with the standard symplectic structure and the standard Euclidean inner product. We first present the main ingredient needed for the proof of the main theorem. With aid of the M-position, we show that every convex body K has a linear symplectic image $K' = SK$ such that the couple K' and iK' satisfy the inverse Brunn-Minkowski inequality. For this we need to recall a well known fact about the relation between a symplectic form and a positive definite quadratic form. The following theorem by Williamson [21] concerns simultaneous normalization of a symplectic form and an inner product.

Williamson's theorem: *For any positive definite symmetric matrix A there exists an element $S \in \mathrm{Sp}(2n)$ and a diagonal matrix with positive entries D with the property $iD = Di$ (complex linear), such that $A = S^T DS$.*

An immediate corollary (for a proof see [1]) is

Corollary 4.1. *Let T be a volume preserving $2n$-dimensional real matrix. Then there exists a linear symplectic matrix $S \in \mathrm{Sp}(\mathbb{R}^{2n})$, an orthogonal transformation $W \in O(2n)$ and a diagonal complex linear matrix D with positive entries such that*

$$T = WDS.$$

This decomposition, together with Theorem 3.2, implies the following (in the sequel we will only use the special case $\theta = \pi/2$, i.e., multiplication by i)

Theorem 4.2. *Every convex body K in $\mathbb{R}^{2n}$ has a symplectic image $K' = SK$, where $S \in \mathrm{Sp}(2n)$, such that for any $0 \le \theta \le 2\pi$*

$$\mathrm{Vol}(K)^{1/2n} \le \mathrm{Vol}(K' + e^{i\theta}K')^{1/2n} \le A_2 \mathrm{Vol}(K)^{1/2n},$$

where A_2 is a universal constant.

Proof. The first inequality holds trivially for any $K' = SK$ since $K' \subset K' + e^{i\theta}K'$. Next, let K be a convex body in $\mathbb{R}^{2n}$. Set $K_1 = TK$, where T is a volume-preserving linear transformation which takes the body K to an M-position. It follows from Corollary 4.1 that $T = WDS$ where W is orthogonal, S is symplectic, and D is a complex linear transformation. We set $K' = SK$. The remark after Theorem 3.2 implies that we can assume $K_1 = DSK$ where D and S are as above, since an orthogonal image of a body in M-position is also in M-position. Note that the rotated body $e^{i\theta}K_1$ is in M-position as well, since multiplication by a complex number of module 1 is a unitary transformation. Next, it follows from Theorem 3.2 that

$$\mathrm{Vol}(K_1 + e^{i\theta}K_1)^{1/2n} \le C\left(\mathrm{Vol}(K_1)^{1/2n} + \mathrm{Vol}(e^{i\theta}K_1)^{1/2n}\right) = 2C\mathrm{Vol}(K)^{1/2n},$$

where $C > 0$ is a universal constant. Since D is complex linear it commutes with multiplication by $e^{i\theta}$, and using also the fact that it is volume preserving we conclude that

$$\mathrm{Vol}(K' + e^{i\theta}K')^{1/2n} = \mathrm{Vol}(K_1 + e^{i\theta}K_1)^{1/2n} \le 2C\mathrm{Vol}(K)^{1/2n}.$$

The proof is now complete. □

In order to complete the proof of the main theorem, we shall need two more ingredients. The first is the following easy observation

Lemma 4.3. *Let K be a symmetric convex body satisfying $K = iK$, and let $rB^{2n} \subset K$ be the largest multiple of the Euclidean ball contained in K. Then*

$$c^Z_{lin}(K) \le 2\pi r^2.$$

Proof. Since the body K is assumed to be symmetric there are at least two contact points x and $-x$ which belong to ∂K, the boundary of K, and to rS^{2n-1}, the boundary of rB^{2n}. Note that the supporting hyperplanes to K at these points must be $\pm x + x^\perp$ since they are also supporting hyperplanes of rB^{2n} at the tangency points. Thus, the body K lies between the hyperplanes $-x + x^\perp$ and $x + x^\perp$. However, since K is invariant under multiplication by i, the points $\pm ix$ are contact points for ∂K and rS^{2n-1} as well. Thus, the body K lies also between $-ix + ix^\perp$ and $ix + ix^\perp$. Note that the length of the vectors x and ix is r. We conclude that the projection of K onto the plane spanned by x and ix is contained in a square of edge length $2r$, which in turn is contained in a disc of radius $\sqrt{2}r$. Therefore K is contained in a cylinder of radius $\sqrt{2}r$ with base spanned by x and ix. Since this cylinder is a unitary image of the standard symplectic cylinder $Z^{2n}(\sqrt{2}r)$ the lemma follows. $\square$

Remark: The factor 2π above can be replaced by 4 if we replace c^Z_{lin} by c^Z. For this we need only to take a small step out of the linear category and use a non-linear symplectomorphism which is essentially two-dimensional.

The last tool we need is a famous result of Rogers and Shephard [18]. This result, which we generalize in some sense in Section 5 below, states that for a convex body $K \subset \mathbb{R}^n$ the volume of the so called "difference body" $K - K$ is not much larger than the volume of the original body. They show that one has

$$\mathrm{Vol}(K - K) \leq 4^n \mathrm{Vol}(K). \tag{3}$$

We are now in a position to prove our main result:

Proof of Theorem 2.6. Let K be a convex body in $\mathbb{R}^{2n}$ and set $K_1 = K - K$. Note that K_1 is symmetric and by (3) we have $\mathrm{Vol}(K_1) \leq 4^{2n}\mathrm{Vol}(K)$. It follows from Theorem 4.2 that there exists a symplectic map $S \in \mathrm{Sp}(\mathbb{R}^{2n})$ for which $\mathrm{Vol}(SK_1 + iSK_1) \leq A_2^{2n}\mathrm{Vol}(K_1)$. Denote $K_2 = SK_1$, $K_3 = K_2 + iK_2$. Thus $\mathrm{Vol}(K_2) = \mathrm{Vol}(K_1)$ and $\mathrm{Vol}(K_3) \leq A_2^{2n}\mathrm{Vol}(K_2)$. Let $r > 0$ be the largest radius such that $rB^{2n} \subset K_3$. We thus have

$$r^{2n}\mathrm{Vol}(B^{2n}) \leq \mathrm{Vol}(K_3) \leq A_2^{2n}\mathrm{Vol}(K_2) = A_2^{2n}\mathrm{Vol}(K_1) \leq (4A_2)^{2n}\mathrm{Vol}(K).$$

On the other hand, since $K_3 = iK_3$, it follows from the monotonicity property of symplectic capacities and from Lemma 4.3 that

$$c^Z_{lin}(K) \leq c^Z_{lin}(K_1) = c^Z_{lin}(K_2) \leq c^Z_{lin}(K_3) \leq 2\pi r^2.$$

Joining these two together we conclude

$$\frac{c^Z_{lin}(K)}{c(B^{2n})} \leq 2(4A_2)^2 \left(\frac{\mathrm{Vol}(K)}{\mathrm{Vol}(B^{2n})}\right)^{1/n},$$

and the proof of the theorem is complete. $\square$

5 Generalized Rogers Shephard

In this section we again work in $\mathbb{R}^n$ equipped only with the Euclidean structure. The above type of reasoning led us to the following simple generalization of the theorem of Rogers and Shephard (3) above. In this generalization, instead of considering the Minkowski sum and the Minkowski difference of a body and itself, we consider the sum and the difference of two different bodies, and show with the use of M-ellipsoid that both have the same volume radius up to a universal constant. We remark that the constant in (3) is equal to 2 (if we put it in the setting of the theorem below) whereas the constant in the theorem below, although universal, may be much worse.

Theorem 5.1. *There exists a universal constant A_3 such that for any two convex bodies $A, B \subset \mathbb{R}^n$ one has*

$$\mathrm{Vol}(A+B)^{1/n} \le A_3 \mathrm{Vol}(A-B)^{1/n}.$$

Proof. In the case where one of the bodies is centrally symmetric the statement is trivial. In the case where both of them are not symmetric, we will use the property of the M-ellipsoid described in Theorem 3.3 above. Let $\mathcal{E}_B$ be the M-ellipsoid of B, which is of course centrally symmetric. We see that

$$\mathrm{Vol}(A+B)^{1/n} \le C\mathrm{Vol}(A+\mathcal{E}_B)^{1/n} = C\mathrm{Vol}(A-\mathcal{E}_B)^{1/n} \le C^2\mathrm{Vol}(A-B)^{1/n}.$$

□

References

[1] Artstein-Avidan, S., Ostrover, Y. *On Symplectic Capacities and Volume Radius.* Preprint, math.SG/0603411.

[2] Cieliebak, K., Hofer, H., Latschev, J., Schlenk F. *Quantitative symplectic geometry.* Preprint, math.SG/0506191.

[3] Ekeland, I., Hofer, H. *Symplectic topology and Hamiltonian dynamics.* Math. Z. **200** (1989), no. 3, 355–378.

[4] Gromov, M. *Pseudoholomorphic curves in symplectic manifolds.* Invent. Math. **82** (1985), no. 2, 307-347.

[5] Hermann, D. *Non-equivalence of symplectic capacities for open sets with restricted contact type boundary.* Prépublication d'Orsay numéro 32 (29/4/1998).

[6] Hofer, H. *Symplectic capacities.* Geometry of low-dimensional manifolds, 2 (Durham, 1989), 15-34, London Math. Soc. Lecture Note Ser., **151**, Cambridge Univ. Press, Cambridge, 1990.

[7] Hofer, H., Zehnder, E. *Symplectic Invariants and Hamiltonian Dynamics.* Birkhäuser, Basel (1994).

[8] John, F. *Extremum problems with inequalities as subsidiary conditions.* Courant Anniversary Volume, Interscience, New York, 187-204 (1948).

[9] Lalonde, F. *Energy and capacities in symplectic topology.* Geometric topology (Athens, GA, 1993), 328-374, AMS/IP Stud. Adv. Math., **2.1**, Amer. Math. Soc., Providence, RI, 1997.

[10] Lalonde, F., McDuff, D. *The geometry of symplectic energy.* Ann. of Math. 141 (1995) 349-371.

[11] McDuff, D. *Symplectic topology and capacities.* Prospects in mathematics (Princeton, NJ, 1996), 69-81, Amer. Math. Soc., Providence, RI, 1999.

[12] Milman, V.D. *An inverse form of the Brunn-Minkowski inequality with applications to the local theory of normed spaces.* C. R. Acad. Sci. Paris Sér. I Math. 302 (1986), no. 1, 25–28.

[13] Milman, V.D. *Isomorphic symmetrizations and geometric inequalities.* Geometric aspects of functional analysis (1986/87), 107–131, Lecture Notes in Math., 1317, Springer, Berlin, 1988.

[14] Milman, V.D., Pajor, A. *Entropy and asymptotic geometry of non-symmetric convex bodies.* Adv. Math. 152 (2000), no.2, 314-335.

[15] Milman, V.D., Schechtman, G. *Asymptotic theory of finite dimensional normed spaces.* Lectures Notes in Math. 1200, Springer, Berlin (1986).

[16] McDuff, D., Salamon, D. *Introduction to Symplectic Topology*, 2nd edition, Oxford University Press, Oxford, England (1998).

[17] Pisier, G. *The volume of convex bodies and Banach space geometry.* Cambridge University Press, Cambridge, (1989).

[18] Rogers, C.A., Shephard, C. *The difference body of a convex body.* Arch. Math. 8 (1957), 220-233.

[19] Viterbo, C. *Metric and isoperimetric problems in symplectic geometry.* J. Amer. Math. Soc. 13 (2000), no. 2, 411–431.

[20] Viterbo, C. *Capacités symplectiques et applications (d'après Ekeland-Hofer, Gromov).* Séminaire Bourbaki, Vol. 1988/89. Astérisque no. **177-178** (1989), Exp. no. 714, 345-362.

[21] Williamson, J. *On the algebraic problem concerning the normal forms of linear dynamical systems.* Amer. J. of Math. 58 (1963), 141-163.