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The concept of duality for measure projections of convex bodies ☆

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Abstract

We show that an involution $\mathcal{T}$ on some class of functions on $\mathbb{R}^n$, which reverses order (meaning that if $f \leqslant g$ then $\mathcal{T}f \geqslant \mathcal{T}g$) has, often, a very specific form, actually essentially unique. It is done in this paper for the class of s-concave functions, for which this unique formula is derived. These functions are, for integer s, exactly marginals of convex bodies of dimension $n+s$. This understanding is also extended and discussed for other classes of functions, and represents from our point of view the abstract description of the concept of duality.

Keywords: Duality; Convexity; s-Concavity

1. Introduction

In the main theorem of this paper we characterize the concept of duality for the so-called s-concave functions (to be defined below). For $s = 1$ this class consists of non-negative functions on $\mathbb{R}^n$ with convex support which includes 0, and which are concave on their support. This is a second part of the project of characterizing the concept of duality for classes of functions. In the first part, which is presented in [3], we discussed the Legendre transform, and showed that essentially it is the only involution transform on convex functions on $\mathbb{R}^n$ which is order-reversing. We quote this result as Theorem 3 below, for an elaborate discussion and other related

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results see [2]. Actually, it is our understanding that this theory has two different parts. One part, to which correspond the classes which we discuss in this note, is when there is some special point 0, and all functions must include it in their support. The second case, to which corresponds, for example, the class of all convex functions discussed in Theorem 3, the point 0 plays no special role. However, both these classes (and their variants) fit into the scheme of the abstract duality concept, see Definition 2 below.

The class of s-concave functions, which we denote by $\mathrm{Conc}_s(\mathbb{R}^n)$, is, fixing $s > 0$, defined to be the class of upper-semi-continuous non-negative functions on $\mathbb{R}^n$ which are s-concave, namely have convex support which includes 0, and $f^{1/s}$ is concave on the support. (The support is defined, as usual, to be the *closure* of the set $\{y\colon f(y) > 0\}$.) This class was discussed in [5] and [7], and recently in [4]. For a positive integer s, the class of s-concave functions on $\mathbb{R}^n$ arises naturally in geometry when one considers measure projections of convex bodies in $\mathbb{R}^{n+s}$ into $\mathbb{R}^n$ (although, in all the considerations above and below, s need not be an integer). We elaborate on the geometric interpretation in Section 2. As $s \to \infty$, this class converges (in the sense of uniform convergence on compact sets, see [4] for more precise statements) to the class of log-concave functions for which duality was characterized completely in [3] as a consequence of Theorem 3 below. As $s \to 0$, the class of s-concave functions converges, in the same sense, to indicator functions of convex sets.

There are some other variants of the class $\mathrm{Conc}_s(\mathbb{R}^n)$ which are needed in our proofs and are also, we think, of independent interest, and we list them here: The class $\mathrm{Conc}_s^+(\mathbb{R}^n)$ consists of bounded s-concave functions with $f(0) > 0$, that is

$$\mathrm{Conc}_s^+(\mathbb{R}^n) = \{f \in \mathrm{Conc}_s(\mathbb{R}^n)\colon\ f(0) > 0 \text{ and } \exists C \text{ s.t.} f \leqslant C\}.$$

Yet a third, and similar, class, is that of functions in $\mathrm{Conc}_s(\mathbb{R}^n)$ which are compactly supported, and which include 0 in the *interior* of their support. We denote it by $\mathrm{Conc}_s^{(0)}(\mathbb{R}^n)$. Finally, we will work with functions in $\mathrm{Conc}_s^+(\mathbb{R}^n)$ which achieve their maximum at 0, and this class we denote by $\mathrm{Conc}_s^{(m)}(\mathbb{R}^n)$.

Fix some scalar product on $\mathbb{R}^n$ and denote it by $\langle\cdot,\cdot\rangle$. The duality transform which was defined for the class $\mathrm{Conc}_s^+(\mathbb{R}^n)$ in [4] is, up to an irrelevant normalization constant, as follows:

$$(\mathcal{L}_s f)(x) = \inf_{\{y\colon f(y)>0\}} \frac{(1-\langle x,y\rangle)_+^s}{f(y)}. \tag{1}$$

Here we prove that this is essentially the only transform satisfying the abstract duality concept, given in Definition 2 below. That is, we show

Theorem 1. *Assume we are given a transform* $\mathcal{T}:\mathrm{Conc}_s^+(\mathbb{R}^n) \to \mathrm{Conc}_s^+(\mathbb{R}^n)$ (*defined on the whole domain*) *satisfying*

(1) $\mathcal{T}\mathcal{T}f = f$.
(2) $f \leqslant g$ *implies* $\mathcal{T}f \geqslant \mathcal{T}g$.

Then there exists a constant $C_0 \in \mathbb{R}$, *and a symmetric* $B \in GL_n$ *such that*

$$(\mathcal{T}f)(x) = C_0 \inf_{\{y\colon f(By)>0\}} \frac{(1-\langle x,y\rangle)_+^s}{f(By)}.$$

Similarly, the theorem holds when the class $\mathrm{Conc}_s^+(\mathbb{R}^n)$ *is replaced by* $\mathrm{Conc}_s(\mathbb{R}^n)$.

(Notice that $(\mathcal{T} f)(x) = C_0 \mathcal{L}_s(B^{-1}x)$ for the s-duality transform $\mathcal{L}_s$ defined in (1).)

In what follows, we begin by proving Theorem 1 for dimension $n \geqslant 2$, which is in fact simpler than $n = 1$. However, contrary to the case of convex bodies, which was considered by Böröczky and Schneider [6] and which we discuss below, here the answer remains the same also for dimension $n = 1$, and we prove this case separately in Section 5.

Theorem 1 is a clear manifestation of the following 'concept of duality,' which says roughly that the concrete formulae which are used as 'duality formulae' for classes of functions are a direct consequence of two very natural conditions, which we call the 'concept of duality,' and a certain class of functions on which we require the operation to be defined.

Definition 2 *(Concept of duality).* We will say that a transform $\mathcal{T} : \mathcal{S} \to \mathcal{S}$ generates a duality transform on a set of functions $\mathcal{S}$ on $\mathbb{R}^n$ if the following two properties are satisfied:

(1) For any $f \in \mathcal{S}$ we have $\mathcal{T}\mathcal{T} f = f$.
(2) For any two functions in $\mathcal{S}$ satisfying $f \leqslant g$ we have that $\mathcal{T} f \geqslant \mathcal{T} g$.

Another manifestation of the concept, which we quote below for comparison, is as follows: Denote the class of lower-semi-continuous convex functions $\phi : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ by $\mathrm{Cvx}(\mathbb{R}^n)$. Recall the definition of the classical Legendre transform $\mathcal{L} : \mathrm{Cvx}(\mathbb{R}^n) \to \mathrm{Cvx}(\mathbb{R}^n)$ given by

$$(\mathcal{L}\phi)(x) = \sup_y \bigl(\langle x, y\rangle - \phi(y)\bigr). \tag{2}$$

In [3] we proved the following

Theorem 3. *Assume a transform* $\mathcal{T} : \mathrm{Cvx}(\mathbb{R}^n) \to \mathrm{Cvx}(\mathbb{R}^n)$ (*defined on the whole domain* $\mathrm{Cvx}(\mathbb{R}^n)$) *satisfies for all* $\phi, \psi \in \mathrm{Cvx}(\mathbb{R}^n)$

(1) $\mathcal{T}\mathcal{T}\phi = \phi$.
(2) $\phi \leqslant \psi$ *implies* $\mathcal{T}\phi \geqslant \mathcal{T}\psi$.

Then, $\mathcal{T}$ is essentially the classical Legendre transform, namely there exists a constant $C_0 \in \mathbb{R}$, a vector $v_0 \in \mathbb{R}^n$, and an invertible symmetric linear transformation $B \in GL_n$ such that

$$(\mathcal{T}\phi)(x) = (\mathcal{L}\phi)(Bx + v_0) + \langle x, v_0\rangle + C_0.$$

The case of convex sets (which also have a special point 0 which must lie inside the body in order for duality to work properly) is of independent interest, and was settled, for the case of compact sets with 0 in the interior, in [6]. It was shown that in dimension 2 and higher, a transform satisfying that it exchanges intersections with convex hulls and vice versa, must be, up to a linear transformation, either the usual polarity transform of convex geometry or a constant map. One should note that in the papers [8,9] Gruber studied transformations which preserve the lattice of convex bodies, with respect to intersection of bodies and to convex hull of their union, and also for the lattice of norms, which, as was noticed in [6], implies the corresponding duality result for centrally symmetric convex bodies with 0 in their interior. Here we are mainly interested in duality transforms, and so their result (together with the observation in Lemma 4 below) implies that for the class of indicator functions of convex bodies, a transformation satisfying the "concept of duality" must be, up to a symmetric linear transformation, usual polarity. This is discussed in Section 6, where we complement their result by settling the "unbounded" case.

However, Böröczky and Schneider relax the condition of involution (whereas in studying duality it is perhaps the first condition). We may sometimes relax it as well, however the assumption of bijectivity is, in the case we consider, not only natural but sometimes necessary, as can be seen most easily in the example of the Legendre transform, where it is not difficult to define an injective, but not surjective transform which satisfies (2) above, by considering, say, the transform $\mathcal{T}$ defined by $\mathcal{T} f = \mathcal{L}(f + x^2)$.

2. A short discussion of the transform

Before we begin with the proofs, let us make a short computation to understand where the formula for the transformation comes from, and the distinction between the different classes $\mathrm{Conc}_s^+(\mathbb{R}^n)$, $\mathrm{Conc}_s(\mathbb{R}^n)$ and $\mathrm{Conc}_s^{(0)}(\mathbb{R}^n)$.

Let us start with $\mathrm{Conc}_1^{(0)}(\mathbb{R}^n)$. (We will see later that the case $s = 1$ is in fact general, and we omit the index 1 for the rest of the section.) Indeed, every bounded non-negative function on $\mathbb{R}^n$ with compact convex support, which is concave on its support, and such that 0 is in the interior of its support, can be seen as the measure projection of a (compact) convex body in $\mathbb{R}^{n+1}$ with 0 in its interior. If we require it to be symmetric with respect to the subspace $\mathbb{R}^n$, then this body is uniquely defined up to a constant factor in the direction $(\mathbb{R}^n)^\perp$. We may let

$$K_f = \big\{(x, y) \in \mathbb{R}^n \times \mathbb{R}\colon\ |y| \leqslant f(x)\big\}.$$

The dual body to K_f, via the *standard* polarity transform for convex bodies, is

$$(K_f)^\circ = \big\{(x', y') \in \mathbb{R}^n \times \mathbb{R} \text{ such that } \langle x, x'\rangle + yy' \leqslant 1\ \forall (x, y) \in K_f\big\}$$

which in turn is the set

$$\left\{(x', y') \in \mathbb{R}^n \times \mathbb{R} \text{ such that } |y'| \leqslant \inf_{x\in\mathbb{R}^n} \frac{(1 - \langle x, x'\rangle)_+}{f(x)}\right\}.$$

This is a convex body, which is symmetric with respect to the subspace $\mathbb{R}^n$, and it is easily seen to be $K_{f'}$ for $f'(x) = \inf_{z\in\mathbb{R}^n} \frac{(1-\langle z,x\rangle)_+}{f(z)}$, precisely the 1-duality transform we have defined above in (1). This also explains why the formula gives rise to an involution, since the usual polarity of convex bodies is an involution.

Of course, there is more than one way to define polarity for convex bodies (though it follows from [6] that they are all the same up to a choice of some symmetric linear transformation). This accounts for the extra linear transformation in Theorem 6. We note that our proof is different from the proofs of [6] and in particular we do not *use* their Theorem 9 quoted below but rather, we may use our reasoning to give an alternative proof of a similar theorem.

Similarly, for $\mathrm{Conc}^+(\mathbb{R}^n)$, it is not hard to see that this class is also, as $\mathrm{Conc}^{(0)}(\mathbb{R}^n)$, invariant under the transform $\mathcal{L}_1$, since boundedness promises that $(\mathcal{L}_1 f)(0) > 0$ and vice versa. Clearly, any unbounded function f is mapped via $\mathcal{L}_1$ to a function with $(\mathcal{L}_1 f)(0) = 0$, and any bounded function f is mapped via $\mathcal{L}_1$ to some $\mathcal{L}_1 f$ with $(\mathcal{L}_1 f)(0) > 0$. Second, it is necessary to explain why $\mathcal{L}_1$ is an involution also on this class. Indeed, the same arguments of closed convex sets, this time non-compact and with 0 restricted to be inside the set but possibly on its boundary, works, since duality is well defined also for this class of convex sets, and is involutive (by, say, the Hahn–Banach theorem).

3. Interchanging max and min

A first and useful step is to show that for involution, the reversing of order implies exchangement of minimum and maximum. We state it in full generality for later reference.

Let $\mathcal{S}$ be some abstract class of functions, satisfying the following condition: For any two functions $f, g \in \mathcal{S}$, the function $M(x) = \inf\{h(x)\colon\ h \in \mathcal{S},\ h \geqslant f \text{ and } h \geqslant g\}$ also belongs to $\mathcal{S}$, and the function $m(x) = \sup\{h(x)\colon\ h \in \mathcal{S},\ h \leqslant f \text{ and } h \leqslant g\}$ also belongs to $\mathcal{S}$. In this case we denote $\hat{\min}(f, g) := m$ and $\hat{\max}(f, g) := M$.

For example, in the class $\mathrm{Cvx}(\mathbb{R}^n)$ this is true, $\hat{\max}$ is the usual maximum, and $\hat{\min}$ is a supremum of all l.s.c. convex functions which lie below $\min(f, g)$ and thus is also convex and lower-semi-continuous. In the cases discussed in this note, of various types of s-concave functions, we easily see that $\hat{\min}$ is the usual minimum and $\hat{\max}$ exists too, since it is the infimum of a family of concave (or s-concave) functions. The special conditions (of boundedness, compactness, and at 0) should be checked separately. Then we have

Lemma 4. *Assume we are given a bijective transform $\mathcal{T} : \mathcal{S} \to \mathcal{S}$ satisfying*

(1) *$f \leqslant g$ implies $\mathcal{T} f \geqslant \mathcal{T} g$.*
(2) *$\mathcal{T} f \leqslant \mathcal{T} g$ implies $f \geqslant g$.*

Then $\mathcal{T}(\hat{\min}(f, g)) = \hat{\max}(\mathcal{T} f, \mathcal{T} g)$, and vice versa, $\mathcal{T}(\hat{\max}(f, g)) = \hat{\min}(\mathcal{T} f, \mathcal{T} g)$.

(Notice that if $\mathcal{T}$ is an involution, $\mathcal{T} = \mathcal{T}^{-1}$ and in this case condition (2) is a consequence of condition (1).)

Proof. Indeed, by 'onto' we have that $\hat{\max}(\mathcal{T} f, \mathcal{T} g) = \mathcal{T} h$ for some $h \in \mathcal{S}$, so, $\mathcal{T} h \geqslant \mathcal{T} f$ and $\mathcal{T} h \geqslant \mathcal{T} g$, and so by (2) we must have $h \leqslant f$ and $h \leqslant g$, and since $h \in \mathcal{S}$ we have $h \leqslant h' = \hat{\min}(f, g)$. On the other hand, $h' \leqslant f$ and $h' \leqslant g$, so that $\mathcal{T} h' \geqslant \mathcal{T} f$ and $\mathcal{T} h' \geqslant \mathcal{T} g$, hence $\mathcal{T} h' \geqslant \hat{\max}(\mathcal{T} f, \mathcal{T} g) = \mathcal{T} h$, and we have that $h' \leqslant h$ as well, so they are equal. □

Similarly, if the class $\mathcal{S}$ satisfies that for a certain selection of functions $f_\alpha \in \mathcal{S}$, the function $\mathrm{M}(x) = \inf\{h(x)\colon\ h \in \mathcal{S},\ h \geqslant f_\alpha\ \forall\alpha\}$ also belongs to $\mathcal{S}$, or, similarly, that the function $\mathrm{m}(x) = \sup\{h(x)\colon\ h \in \mathcal{S},\ h \leqslant f_\alpha\ \forall\alpha\}$ also belongs to $\mathcal{S}$, we may state the corresponding lemma for the regularized infimum and supremum. In this case we denote $\hat{\inf}_\alpha(f_\alpha) := \mathrm{m}$ and $\hat{\sup}_\alpha(f_\alpha) := \mathrm{M}$. It has exactly the same proof as the lemma above.

Lemma 5. *Assume we are given a bijective transform $\mathcal{T} : \mathcal{S} \to \mathcal{S}$, such that for any $\phi, \psi \in \mathcal{S}$*

(1) *$f \leqslant g$ implies $\mathcal{T} f \geqslant \mathcal{T} g$.*
(2) *$f \leqslant g$ implies $\mathcal{T}^{-1} f \geqslant \mathcal{T}^{-1} g$.*

Then for any family f_α for which the expressions below are well defined, we have $\mathcal{T}(\hat{\inf}(f_\alpha)) = \hat{\sup}(\mathcal{T} f_\alpha)$, and vice versa.

4. Theorem 1 in dimension $\geqslant 2$

We limit our discussion to the case $s = 1$, but this is in fact general, since for any $s > 0$ and any concave g we have that $(\mathcal{L}_s(g^s))^{1/s} = \mathcal{L}_1 g$, and so any transform $\mathcal{T}$ on s-concave functions can be translated to a transform on 1-concave functions by $\mathcal{T}'g = (\mathcal{T}g^s)^{1/s}$, which, as we will show below, will imply that $\mathcal{T}'$ is up to a linear transform and a constant, $\mathcal{L}_1$, which in turn means $\mathcal{T}$ is, up to a linear transform and a constant, $\mathcal{L}_s$. We denote the class $\mathrm{Conc}_1(\mathbb{R}^n)$, namely positive u.s.c. functions on $\mathbb{R}^n$ with convex support that includes 0 on which they are concave simply by $\mathrm{Conc}(\mathbb{R}^n)$. In this section we are interested in showing the following theorem.

Theorem 6. *Assume $n \geqslant 2$ and we are given a transform $\mathcal{T}: \mathrm{Conc}(\mathbb{R}^n) \to \mathrm{Conc}(\mathbb{R}^n)$ (defined on the whole domain) satisfying for every $f, g \in \mathrm{Conc}(R^n)$ that*

(1) $\mathcal{T}\mathcal{T}f = f$.
(2) $f \leqslant g$ *implies* $\mathcal{T}f \geqslant \mathcal{T}g$.

Then up to a symmetric linear transformation $B \in GL_n$, and a positive constant C_0, $\mathcal{T}$ is the 1-duality transform $\mathcal{L}_1$ defined in [4], *that is,*

$$(\mathcal{T}f)(x) = C_0 \inf_{\{y:\ f(By)>0\}} \frac{(1 - \langle x, y\rangle)_+}{f(By)}. \tag{3}$$

To prove Theorem 6 we begin by considering a subclass of 1-concave functions, namely those which attain their maximum at 0, which we denoted above by $\mathrm{Conc}^{(m)}(\mathbb{R}^n)$. Below we show the following theorem.

Theorem 7. *Assume $n \geqslant 2$ and we are given a transform $\mathcal{T}: \mathrm{Conc}^{(m)}(\mathbb{R}^n) \to \mathrm{Conc}^{(m)}(\mathbb{R}^n)$ (defined on the whole domain) satisfying for every $f, g \in \mathrm{Conc}^{(m)}(R^n)$ that*

(1) $\mathcal{T}\mathcal{T}f = f$.
(2) $f \leqslant g$ *implies* $\mathcal{T}f \geqslant \mathcal{T}g$.

Then up to a symmetric linear transformation $B \in GL_n$, and a positive constant C_0, $\mathcal{T}$ is the 1-duality transform $\mathcal{L}_1$, that is,

$$(\mathcal{T}f)(x) = C_0 \inf_{\{y:\ f(By)>0\}} \frac{(1 - \langle x, y\rangle)_+}{f(By)}.$$

Proof. We proceed in several steps, which can be summarized as follows: the multiples of the delta-type functions at 0 must be mapped to constant functions and vice versa, functions with support contained in an interval must be mapped to functions supported on half-spaces, triangle functions (to be defined) must be mapped to constants on half-spaces, and the linearity of some parts of these maps is then established. Then, we use that triangle functions together with the delta-function and constants, determine everything.

Step 1. Denote by δ_0 the function

$$\delta_0(0) = 1 \quad \text{and} \quad \delta_0(x) = 0 \quad \text{for } x \neq 0.$$

We explain why the constant functions $A > 0$ are mapped to the functions $a\delta_0$ with $a > 0$, and vice versa.

Let $A > 0$ and denote $f = \mathcal{T}(A)$. Then, letting $a' \leqslant a = f(0)$ we have that $a'\delta_0 \leqslant f$ and so $\mathcal{T}(a'\delta_0) \geqslant \mathcal{T}f = A$, but the only concave functions greater than a constant function are other constant function, so we have that $\mathcal{T}(a'\delta_0) = A'$ for some $A' \geqslant A$, and this is true for any $a' \leqslant f(0)$.

Assume that there is some a (necessarily greater than $f(0)$) for which $\mathcal{T}(a\delta_0)$ is *not* a constant function, say $\mathcal{T}(a\delta_0) = h$. In particular, the support of h is not the whole $\mathbb{R}^n$, and so, since the support is convex, it does not include some half-space. Fix some other $a' \leqslant f(0)$ and $A' = \mathcal{T}(a'\delta_0)$. Then, for *any* function $h \leqslant g \leqslant A'$ we have that $a'\delta_0 \leqslant \mathcal{T}g \leqslant a\delta_0$, which means that $\mathcal{T}g$ is also of the form $a''\delta_0$ for some a''. This cannot hold for all such g, since it would mean that *all* functions $h \leqslant g \leqslant A'$ are comparable (any two such functions g_1, g_2, satisfy $\mathcal{T}g_1 \leqslant \mathcal{T}g_2$ or $\mathcal{T}g_1 \geqslant \mathcal{T}g_2$, hence either $g_1 \geqslant g_2$ or $g_1 \leqslant g_2$). However, between the constant function and any other function it is easy to construct two non-comparable functions, using for example the fact that some half-space is not in the support of h.

Thus, we have shown that for any a we have that $\mathcal{T}(a\delta_0)$ is a constant function. Similarly, this means that for some constant function A' we have $\mathcal{T}(A') = a'\delta_0$. Moreover, also for any $A'' > A'$ we must have $\mathcal{T}(A'') < a'\delta_0$ so it is also a multiple of the delta function. Assume there is some constant function A (necessarily smaller than A') for which $\mathcal{T}(A) = h$ where h is not a multiple of the delta function. Then between h and $a'\delta_0$ we may construct two non-comparable functions, but since their images lie between A and A', they must be comparable, which gives us a contradiction. Thus, $\mathcal{T}(A)$ is a multiple of δ_0, for any constant A.

We denote by $g : \mathbb{R}^+ \to \mathbb{R}^+$ the function such that $\mathcal{T}(a\delta_0) = g(a)$. Then, by the involution property, $\mathcal{T}(A) = g^{-1}(A)\delta_0$, and by anti-monotonicity, g is decreasing. We will show later (in Step 9) that g must be of a very special form, namely $g(a) = g(0)/a$.

Step 2. The image of a function whose support is contained in a ray with endpoint 0 has support which is a half-space, and vice versa.

We start with functions supported on intervals of the type $[0, z]$. Consider two positively-linearly independent non-zero vectors x, y (by this we mean that $x \neq ly$ for all $l > 0$), and two functions $f_x, f_y \in \mathrm{Conc}^{(m)}(\mathbb{R}^n)$, with supports $[0, x]$ and $[0, y]$ respectively, and bounded respectively by $c_x = f_x(0)$ and $c_y = f_y(0)$. Denote also, for convenience, $c = \min(c_x, c_y)$, $C_x = g(c_x)$, $C_y = g(c_y)$, and $C = \max(C_x, C_y) = g(c)$.

Clearly $\mathcal{T}f_x \leqslant C_x$ and $\mathcal{T}f_y \leqslant C_y$. However, $\min(f_x, f_y) = c\delta_0$, and therefore $\hat{\max}(\mathcal{T}f_x, \mathcal{T}f_y) = \mathcal{T}(c\delta_0)$ is the constant function C. Neither the support of $\mathcal{T}f_x$ nor the support of $\mathcal{T}f_y$ can be all of $\mathbb{R}^n$ because the only concave functions with full support are the constant ones, and we know by Step 1 that those are images of multiples of δ_0. Since the supports of $\mathcal{T}f_x$ and $\mathcal{T}f_y$ are convex, each support must be contained in some half-space (simply separate it from a point outside of it). This also implies that neither of the supports can be compact, since then the union of their supports would be contained in some half-space, and the maximum would also have support in some half-space (thus so will its regularization).

Next we claim that the support of, say, f_x, must be a half-space. To fix notation, for a vector $w \in \mathbb{R}^n$ let H_w denote the half-space

$$H_w = \{z \in \mathbb{R}^n : \langle z, w \rangle \leqslant 1\}.$$

(We will say that two half-spaces are parallel if they correspond to w' and w with $w = lw'$ for some $l > 0$, and anti-parallel if $w' = -lw$.) Notice that $H_{2w} \subset H_w$ and not the other way around.

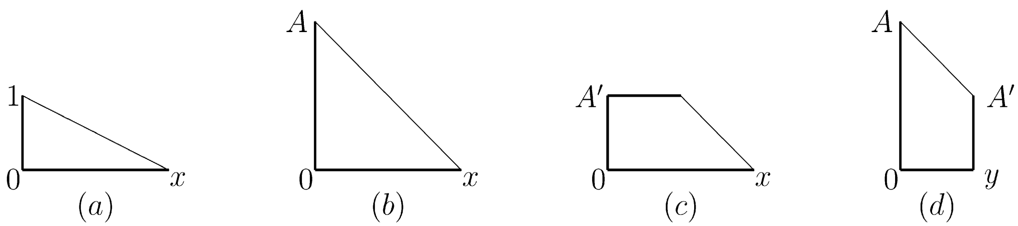
Indeed, if not, then the support of f_x is some other convex set, which is included in two non-parallel half-spaces, say H_{w_1} and H_{w_2}. Consider the functions which are constant C_x on these half-spaces, call them $F_1 = C_x \chi_{H_{w_1}}$ and $F_2 = C_x \chi_{H_{w_2}}$. Both are greater than $\mathcal{T} f_x$, and so their images $\mathcal{T} F_1$ and $\mathcal{T} F_2$ must be less than f_x, which means in particular that the supports of both of them are contained in the interval $[0, x]$ (and less than f_x there). Secondly, since both are less than the constant function C_x, we have that $\mathcal{T} F_1 \geqslant c_x \delta_0$ and $\mathcal{T} F_2 \geqslant c_x \delta_0$. Now look at $\hat{\max}(F_1, F_2)$. Because the half-spaces are non-parallel, we have that $\hat{\max}(F_1, F_2) = C_x$! Therefore, $\min(\mathcal{T} F_1, \mathcal{T} F_2) = c_x \delta_0$. But for two concave functions, with supports contained in $[0, x]$, both greater than $c_x \delta_0$, to have minimum equal $c_x \delta_0$ means that one of them must be a multiple of δ_0, which is a contradiction.

Thus, for a function $f_x \in \mathrm{Conc}^{(m)}(\mathbb{R}^n)$ supported on $[0, x]$, the support of the function $\mathcal{T} f_x$ is a half-space. Moreover, we have shown that if $x \neq ly$ for any $l > 0$, then the supports of f_x and f_y correspond to non-parallel half-spaces. In Step 6 below we will continue this argument to show that the supports of $\mathcal{T} f_x$ and $\mathcal{T}(f_{(-x)})$ are anti-parallel.

Notice that all the above considerations work if instead of support in the interval, we assume support in a ray $\mathbb{R}^+ x$. Of course, since we are assuming the maximum is at 0, the only functions supported in the ray and not in some interval are the constant functions on the ray.

Next we show the opposite fact, namely that a function whose support contains (and so, equals to) a half-space must be mapped to a function supported on a ray. Indeed, let f be supported on some set including a half-space, and consider $\mathcal{T} f$. Let $x, y \in \mathrm{supp}(\mathcal{T} f)$, then there is a function in $\mathrm{Conc}^{(m)}(R^n)$ supported on $[0, x]$ which is smaller than $\mathcal{T} f$, so its image is greater than f, and similarly on $[0, y]$. Thus the support of f is contained in the intersection of two half-spaces, and contains a half-space. Therefore all the half-spaces must be parallel, which as we saw above implies $x = ly$ for some $l > 0$, that is, the support of $\mathcal{T} f$ is included in a ray.

In what follows we will consider the following four types of functions:



Step 3. We start with the function of type (a), and we denote by $\lhd_x$ this 'triangle' function defined on $[0, x]$ by $\lhd_x(tx) = 1 - t$ for $0 \leqslant t \leqslant 1$, and zero elsewhere. This is the smallest function with support $[0, x]$ and which is greater than δ_0. We use it to define a map $F : \mathbb{R}^n \to \mathbb{R}^n$, as follows: for a point $0 \neq x \in \mathbb{R}^n$ let $F(x) = w$ where $\mathrm{supp}(\mathcal{T} \lhd_x) = H_w$. Define $F(0) = 0$.

We claim now that the image of *any* function with support equal to $[0, x]$ is supported on the half-space $H_{F(x)}$. To this end, we continue with the function of type (b). Note that for $A > 1$, this function, which is $A \lhd_x$, a higher triangle, is simply $\hat{\max}(\lhd_x, A\delta_0)$, and thus by Lemma 4 its image is $\min(\mathcal{T} \lhd_x, g(A))$, and so it has the same support as $\mathcal{T}(\lhd_x)$.

Similarly, a function of type (c), which we might call an 'ambulated' triangle, is of the form $\min(A \lhd_x, A')$ for some $A > A'$, and so is mapped to $\hat{\max}(\mathcal{T}(A \lhd_x), g^{-1}(A')\delta_0)$, which also has the same support as $\mathcal{T}(\lhd_x)$. We would like the functions of type (c) to include *all* 'ambulated' triangles, however because of the restriction $A \geqslant 1$ this is not so: the slope at x must be steeper than -1. To capture the more general ambulated triangle we may look at the following: Fix $A < 1$ and look at

$$A\lhd_x = \inf_{A'}\bigl(\hat{\max}\bigl(A\delta_0, \min(\lhd_x, A')\bigr)\bigr).$$

This is a triangle of type (b) with $A < 1$. Clearly

$$\mathcal{T}(A\lhd_x) = \hat{\sup_{A'}}\bigl(\min\bigl(\mathcal{T}(A\delta_0), \mathcal{T}\bigl(\min(\lhd_x, A')\bigr)\bigr)\bigr) = \hat{\sup_{A'}}\bigl(\min\bigl(g(A), \mathcal{T}\bigl(\min(\lhd_x, A')\bigr)\bigr)\bigr).$$

Since for all A' the inner function has the same support $H_{F(x)}$, so does the function $\mathcal{T}(A\lhd_x)$ for $A < 1$.

However, any concave function whose support *equals* $[0, x]$, and with maximum attained at 0, can be written as the supremum of 'ambulated' triangles:

$$f = \sup_t\left(\min\left(\frac{f(tx)}{1-t}\lhd_x, f(tx)\right)\right),$$

and therefore $\mathcal{T}f = \inf_t(\mathcal{T}(\min(\frac{f(tx)}{1-t}\lhd_x, f(tx))))$, and in particular is too has the same support $H_{F(x)}$.

Also note that for $x = ly$ for some $l \geqslant 1$ we have that $F(x)$ and $F(y)$ point in the same direction (the supports are parallel) since $\lhd_x > \lhd_y$ and so $\mathcal{T}\lhd_x \leqslant \mathcal{T}\lhd_y$, therefore there is containment of the supports.

Step 4. Fix some $x \in \mathbb{R}^n$, $A > 0$, and consider all functions with support equal to $[0, x]$ and which are greater than $A\delta_0$. Then $A\lhd_x$ is the smallest in this family. Look at the images of this family, all of which are supported on some half-space $H_{F(x)}$, then they are all below $g(A) = \mathcal{T}(A\delta_0)$, and the maximal one is $\mathcal{T}(A\lhd_x)$. Therefore (using the fact that all the functions whose support includes this half-space are images of functions supported on $[0, x]$, a fact we have shown above in Step 2), we must have that $\mathcal{T}(A\lhd_x) = g(A)\chi_{H_{F(x)}}$.

Step 5. Let us write some formulae: For any $A > 0$

$$\mathcal{T}(A\lhd_x) = g(A)\chi_{H_{F(x)}},$$

and therefore, for $A' < A$,

$$\mathcal{T}\bigl(\min(A\lhd_x, A')\bigr) = \hat{\max}\bigl(g(A)\chi_{H_{F(x)}}, g^{-1}(A')\delta_0\bigr).$$

We claim that $g = g^{-1}$. Indeed, since $A \geqslant A\delta_0$ we have that $g^{-1}(A)\delta_0 \leqslant g(A)$ which means $g^{-1}(A) \leqslant g(A)$. However, if we have, for some A, that $g^{-1}(A) = g(A) - \varepsilon < g(A)$ then we have $(g^{-1}(A) + \varepsilon)\delta_0 \leqslant g(A)$ and so $g(g^{-1}(A) + \varepsilon) \geqslant A\delta_0$, which is impossible since g is strictly decreasing.

We may now rewrite that for $A' < A$

$$\mathcal{T}\bigl(\min(A\lhd_x, A')\bigr) = \hat{\max}\bigl(g(A)\chi_{H_{F(x)}}, g(A')\delta_0\bigr).$$

This function equals to $g(A')$ on the half-space $\{z\colon \langle z, F(x)\rangle \leqslant 0\}$, and for $0 \leqslant t \leqslant 1$, on $\{z\colon \langle z, F(x)\rangle = t\}$ it equals $tg(A) + (1-t)g(A')$. For z with $\langle z, F(x)\rangle > 1$ it is zero.

We thus have the general form of the images of ambulated triangles, up to determining F and g. Since any concave function with maximum at zero can be written as the supremum of ambulated triangles, once we determine the form of F and that of g we will have a concrete formula for $\mathcal{T}$. We start with finding the form of the function F.

Step 6. We will show that F above maps intervals to intervals, namely that if $z = \lambda x + (1-\lambda)y$ then $F(z)$ belongs to the interval between $F(x)$ and $F(y)$.

First, we do this for linearly independent x and y: Let $z = \lambda x + (1 - \lambda y)$. Clearly $\lhd_z \leqslant \hat{\max}(\lhd_x, \lhd_y)$, and it is the largest function supported on $[0, z]$ satisfying this condition. Therefore $\mathcal{T}\lhd_z \geqslant \min(\mathcal{T}\lhd_x, \mathcal{T}\lhd_y) = g(1)\chi_{(H_{F(x)} \cap H_{F(y)})}$. On the other hand, we know of course that $\mathcal{T}\lhd_z = g(1)\chi_{H_{F(z)}}$. Moreover, it is the smallest function, supported on $H_{F(z)}$, satisfying the condition above. This means that $H_{F(z)}$ must be a half-space, contained in the intersection $(H_{F(x)} \cap H_{F(y)})$ and 'touching' the boundary. This precisely means that $F(z)$ is a convex combination of $F(y)$ and $F(z)$.

Next, we must show that the same is true for intervals containing 0, and it is enough to look at the interval $[-x, x]$, since we have shown in Step 2 that for $x = ly$ with $l > 0$ the half-spaces are parallel. In fact, for the same reason, it is enough to show that $F(x) = -F(-x)$, which implies $F(0) = 0 \in [F(x), F(-x)]$, and all the interval is mapped accordingly from Step 2.

To this end, consider the interval $[-x, x]$ and take the function $f = \max(\lhd_x, \lhd_{-x})$ which is supported on $[-x, x]$. Notice that for no y which is linearly independent of x do we have a function with support $[0, y]$ which is below the function f. Now clearly $\mathcal{T}f = g(1)(H_{F(x)} \cap H_{F(-x)})$. Assume $F(-x)$ and $-F(x)$ are not parallel. Then there is some other (many other) half-space H_w which contains this intersection which is not parallel to either $H_{F(x)}$ or $H_{F(-x)}$. This means that the pre-image of $g(1)\chi_{H_w}$ is less than or equal to f, in particular, it is supported on $[-x, x]$. We may now take some other function, which is concave, smaller than $\mathcal{T}(g(1)\chi_{H_w})$, and supported on $[0, x]$ (or on $[-x, 0]$) only. Then its image will have a support which contains H_w, but is of the form $H_{F(tx)}$ for some $0 < t \leqslant 1$, and by Step 2 we know that $H_{F(tx)}$ is parallel to $H_{F(x)}$. This cannot be true for w not in the direction of $F(x)$, and so we see that the intersection must be contained only in half-spaces parallel to $H_{F(x)}$, which happens only if $F(-x)$ and $-F(x)$ are parallel (the option that $F(-x)$ and $F(x)$ point at the same direction was ruled out in Step 2).

Step 7. The fundamental fact of affine geometry implies that an injective function from $\mathbb{R}^{n+1}$ to itself which preserves lines must be an affine linear transformation (see [1] and also a direct proof of a different fact which is sufficient for our goals, in [3]). Notice that indeed all lines are kept in place, including ones passing through 0. Thus, F is affine linear, and moreover, by definition $F(0) = 0$ so it must be a mapping $B \in GL_n$. Thus,

$$\mathcal{T}(A\lhd_x) = g(A)\chi_{H_{Bx}},$$

and for ambulated triangles

$$\mathcal{T}\big(\min(A\lhd_x, A')\big) = \hat{\max}\big(\mathcal{T}(A\lhd_x), g(A')\delta_0\big) = \hat{\max}\big(g(A)\chi_{H_{Bx}}, g(A')\delta_0\big).$$

Step 8. To give the general form of the transform it is more convenient to consider box functions: Denote by $\Box_x$ the function which is 1 on $[0, x]$ and 0 elsewhere. Consider the rectangle function

$A'\Box_x$, the maximal function which is less that the constant A', and is supported on $[0,x]$. Then $A'\Box_x = \max_A(\min(A\lhd_x, A'))$. Clearly

$$\mathcal{T}(A'\Box_x) = \mathcal{T}\Big(\max_A\big(\min(A\lhd_x, A')\big)\Big) = \min_A\big(\hat{\max}\big(g(A)\chi_{H_{Bx}}, g(A')\delta_0\big)\big).$$

This is the function which is supported on H_{Bx}, constant $g(A')$ on the half-space $\langle z, Bx\rangle \leqslant 0$, and equaling $g(A')(1-t)$ if $\langle z, Bx\rangle = t$ for $0 \leqslant t \leqslant 1$ (and zero elsewhere). That is, we may write it as follows:

$$(\mathcal{T}A'\Box_x)(z) = g(A')\min\big(1, \big(1-\langle z, Bx\rangle\big)_+\big).$$

(Another way to show this is to notice that $\mathcal{T}(A'\Box_x)$ is the smallest function supported on H_{Bx} which is above $g(A')\delta_0$.)

Each concave function with maximum at zero can be written as the supremum of such rectangle functions, namely

$$f = \sup_{\{y:\ f(y)>0\}} f(y)\Box_y.$$

Therefore

$$(\mathcal{T}f)(z) = \inf_{\{y:\ f(y)>0\}}\big(\mathcal{T}\big(f(y)\Box_y\big)\big)(z) = \inf_{\{y:\ f(y)>0\}}\big(g\big(f(y)\big)\min\big(1, \big(1-\langle z, By\rangle\big)_+\big)\big).$$

One last thing to notice before moving to the last step is that since we are in the class $\mathrm{Conc}^{(m)}(\mathbb{R}^n)$, for each y we have $f(y) \leqslant f(0)$, and thus $g(f(y)) \geqslant g(f(0))$, and therefore in the infimum above, one never attains the term $g(f(y))$ for $y \neq 0$, and so the element 1 participating in the minimum is redundant, and we may rewrite

$$(\mathcal{T}f)(z) = \inf_{\{y:\ f(y)>0\}}\big(g\big(f(y)\big)\big(1-\langle z, By\rangle\big)_+\big).$$

Step 9. We will now show that g must be of the form $g(t) = C_0/t$, and that B must be symmetric, using the fact that $\mathcal{T}$ is an involution. This will complete the proof of the theorem.

For $f = \lhd_x$ we get

$$g(1)\chi_{Bx}(z) = \big(\mathcal{T}(\lhd_x)\big)(z) = \inf_{\{0\leqslant t<1\}}\big(g(1-t)\big(1-t\langle Bx, z\rangle\big)_+\big).$$

Consider z with $0 < \langle Bx, z\rangle = s < 1$. We are supposed to get $g(1)$, and on the left-hand side we get $\inf_{0\leqslant t<1}(g(1-t)(1-ts))$. Thus we must have $g(1-t) \geqslant g(1)/(1-ts)$, for any $0 \leqslant s < 1$, that is, $g(1-t) \geqslant g(1)/(1-t)$.

Next, consider the function $\varphi = \min(\lhd_x, \Box_{\lambda x})$, for some $0 < \lambda < 1$, and apply the transform. On the one hand, we know we must get

$$\hat{\max}\big(\mathcal{T}(\lhd_x), \mathcal{T}(\Box_{\lambda x})\big) = g(1)\hat{\max}\big(\chi_{H_{Bx}}, \min\big(1, \big(1-\lambda\langle\cdot, Bx\rangle\big)_+\big)\big),$$

which is the following function:

$$(\mathcal{T}\varphi)(z) = \begin{cases} g(1), & \langle z, Bx\rangle \leqslant 1, \\ g(1)(1-\frac{\lambda}{1-\lambda}t), & \langle z, Bx\rangle = 1+t,\ 0<t<\frac{1-\lambda}{\lambda}, \\ 0, & \langle z, Bx\rangle \geqslant \frac{1}{\lambda}. \end{cases}$$

On the other hand, we may use the formula to get that its transform must be

$$(\mathcal{T}\varphi)(z) = \inf_{0\leqslant s\leqslant\lambda}\big(g(1-s)\big(1-s\langle z, Bx\rangle\big)_+\big).$$

Considering z with $\langle z, Bx\rangle = 1+t$ with $0<t<\frac{1-\lambda}{\lambda}$ we get (using the inequality we had above)

$$\begin{aligned}(\mathcal{T}\varphi)(z) &= \inf_{0\leqslant s\leqslant\lambda}\big(g(1-s)\big(1-s(1+t)\big)\big) \\ &\geqslant \inf_{0\leqslant s\leqslant\lambda}\left(g(1)\frac{(1-s(1+t))}{1-s}\right) = g(1)\left(1-\frac{\lambda}{1-\lambda}t\right).\end{aligned}$$

Note that, since we have equality of the end results, we must have equality also in the inequality. In particular, since we do have continuity of g (it is onto and monotone), then at $s=\lambda$ we cannot have a strict inequality. We see thus that for any $0<\lambda<1$ we have $g(1-\lambda) = g(1)/(1-\lambda)$. However, $g = g^{-1}$ and so this determines the complete form of g, namely $g(t) = C_0/t$ for some fixed $C_0 > 0$.

Finally, we show that B must be symmetric. Consider

$$\begin{aligned}\big(g(1)\lhd_{B^{-1}x}\big)(z) = \big(\mathcal{T}(\chi_{H_x})\big)(z) &= \inf_{\{y:\ \langle y,x\rangle<1\}}\big(g(1)\big(1-\langle By, z\rangle\big)_+\big) \\ &= g(1)\inf_{\{y':\ \langle y',(B^*)^{-1}x\rangle<1\}}\big(\big(1-\langle y', z\rangle\big)_+\big).\end{aligned}$$

The function is clearly 0 for all z in directions different than $(B^*)^{-1}x$, which means it must be the same as $B^{-1}x$, and this for every x, hence $B = B^*$. □

Proof of Theorem 6. This proof works equally well for the class $\mathrm{Conc}^+(\mathbb{R}^n)$ and the class $\mathrm{Conc}(\mathbb{R}^n)$, since all we will be using is that $\mathcal{L}_1$ is involutive and order reversing on both these classes.

We would like to use Theorem 7, and for this we must show that the class of concave functions with maximum at 0 is an invariant subclass under the transform $\mathcal{T}$. We continue with the notation of the previous proof. Step 1 of the previous proof does not use the fact that the maximum is attained at 0, and so we have a (strictly) decreasing function $g:\mathbb{R}^+\to\mathbb{R}^+$ such that $\mathcal{T}(a\delta_0) = g(a)$, and $\mathcal{T}(A) = g^{-1}(A)\delta_0$.

For a constant C consider all functions f in the subclass of functions attaining their maximum at 0, with $f(0) = C$. These are

$$\{f:\ f\leqslant C \text{ and } f\geqslant C\delta_0\}.$$

They are transformed to the set

$$\{h\colon\ h \geqslant g^{-1}(C)\delta_0 \text{ and } h \leqslant g(C)\}.$$

Thus, the step where we showed $g = g^{-1}$, is in fact the step in which we established that this is an invariant subclass. Let us repeat the argument: Since $A \geqslant A\delta_0$ we have that $g^{-1}(A)\delta_0 \leqslant g(A)$ which means $g^{-1}(A) \leqslant g(A)$. However, if we have, for some A, that $g^{-1}(A) = g(A) - \varepsilon < g(A)$ then we have $(g^{-1}(A) + \varepsilon)\delta_0 \leqslant g(A)$ and so, applying $\mathcal{T}$ we get that $g(g^{-1}(A) + \varepsilon) \geqslant A\delta_0$, which is impossible since g is strictly decreasing and so the left-hand side must be $< A$.

We see thus that

$$\mathcal{T}\{f\colon\ f \leqslant C \text{ and } f \geqslant C\delta_0\} = \{h\colon\ h \geqslant g(C)\delta_0 \text{ and } h \leqslant g(C)\},$$

and in particular, the class of functions attaining their maximum at 0 is an invariant subclass.

From this we may, by applying Theorem 7 to $\mathcal{T}|_{\mathrm{Conc}^{(m)}(\mathbb{R}^n)}$, conclude the form of the transform on the subclass.

Next we must show that the transform on the whole class is given by the same formula. To this end, we compose the transform $\mathcal{T}$ with the standard $\mathcal{L}_1$ transform (modified by B). We let $\mathcal{F}:\mathrm{Conc}(\mathbb{R}^n) \to \mathrm{Conc}(\mathbb{R}^n)$ be given by

$$\mathcal{F}f = \mathcal{L}_1 \circ \mathcal{T}f$$

where

$$\mathcal{L}_1 f = C_0 \inf_{\{y\colon\ f(y)>0\}} \frac{(1-\langle x, y\rangle)_+}{f(By)}.$$

We see that $\mathcal{F}$ is identity of the subclass of functions attaining their maximum at 0. We also see, because each of the transforms is order-reversing, that $\mathcal{F}$ is order preserving. (It is also invertible, and its inverse, $\mathcal{T} \circ \mathcal{L}_1$, is order preserving as well.) We must show that it is everywhere the identity.

We start with "cones," which are functions of the form (for some fixed $x \in \mathbb{R}^n$)

$$h(tx) = at + b \quad \text{for } -(b/a) \leqslant t \leqslant 1$$

and 0 elsewhere. We ask $a > 0$ and of course require $h(0) = b > 0$. We claim that on this class of functions the transform $\mathcal{F}$ is the identity. Indeed, the function h defined above has the following maximality property: If some concave function $h' > h$, that is $h' \geqslant h$ and for some y, $h'(y) > h(y)$, then also $\min(h'(0), h') > \min(h(0), h)$. Indeed, if $y \in \mathrm{supp}(h)$ then simply $h'(0) > h(0)$, and if $y \notin \mathrm{supp}(h)$ then the support of $\min(h'(0), h')$ is strictly larger than that of $\min(h(0), h)$.

Consider the function ψ which satisfies $\mathcal{F}\psi = h$. (Note that $\mathcal{T}$ is bijective, and so is $\mathcal{L}_1$, hence so is their composition $\mathcal{F}$.) Since $h \geqslant \min(h(0), h) \in \mathrm{Conc}^{(m)}(\mathbb{R}^n)$ we have that $\psi = \mathcal{F}^{-1}h \geqslant \min(h(0), h)$. However, it cannot be that $\psi > h$ since then $\min(\psi(0), \psi) > \min(h(0), h)$ and we get that

$$h = \mathcal{F}\psi \geqslant \mathcal{F}\min(\psi(0), \psi) > \min(h(0), h)$$

whereas the largest function with maximum at 0 which is less than h is $\min(h(0), h)$, a contradiction. Thus, $\psi \leqslant h$, and so $h = \mathcal{F}\psi \leqslant \mathcal{F}h$. Therefore, again by the maximality property of h, we have that unless $h = \mathcal{F}h$ it must be that $\min(h(0), h) < \min((\mathcal{F}h)(0), \mathcal{F}h) \in \mathrm{Conc}^{(m)}(\mathbb{R}^n)$ (and so if mapped to itself, and is less than or equal to $\mathcal{F}h$). Taking now $\mathcal{F}^{-1}$ we see that

$$\min\big((\mathcal{F}h)(0), \mathcal{F}h\big) \leqslant h$$

which cannot hold as $\min(h(0), h)$ is the largest function in the subclass which is below h. We conclude that $h = \mathcal{F}h$.

Thus, we arrived at the following fact: for all "cones" of the form of h as above, $\mathcal{F}$ acts as identity. However, any function with $f(0) > 0$ is the supremum of functions of the form

$$H(tx) = at + b \quad \text{for } 0 \leqslant t \leqslant 1$$

and 0 elsewhere (letting $b = f(0)$ and $a = f(x) - b$), and these functions H are the minimum of a function in the subclass attaining its maximum at 0, say $\max(b, b + a)\Box_x$, and a "cone" function h as before. Since we know that $\mathcal{F}$ is identity on both, and we know that the infimum of two functions is mapped by $\mathcal{F}$ to the infimum of the images, we have that $\mathcal{F}$ is identity also on functions of the form H, and so on all functions in the full class considered. □

5. The one-dimensional case

So far we have proven Theorem 1 for dimension 2 and higher, however the same theorem remains true when the dimension is 1. This is contrary to the corresponding fact for convex bodies (one can define a duality transform for intervals including 0 in $\mathbb{R}$ using any decreasing function f with $f(f(x)) = x$). The reason is, as in the case of convex bodies and the Legendre transform, that because we are dealing with *functions* the dimension is "essentially" higher. The proof of this case, however, is slightly different than the proof in higher dimensions, since one cannot use the fact that transformations which preserve lines have a special form. We give it below for the case of functions with maximum at 0 only, but the other cases follow in the same way as in higher dimensions.

Theorem 8. *Assume we are given a transform $\mathcal{T} : \mathrm{Conc}^{(m)}(\mathbb{R}) \to \mathrm{Conc}^{(m)}(\mathbb{R})$ (defined on the whole domain) satisfying*

(1) $\mathcal{T}\mathcal{T}f = f$.
(2) $f \leqslant g$ *implies* $\mathcal{T}f \geqslant \mathcal{T}g$.

Then up to constants $C_0 > 0$ and $0 \neq C_1 \in \mathbb{R}$ $\mathcal{T}$ is

$$(\mathcal{T}f)(x) = C_0 \inf_{\{y:\ f(y)>0\}} \frac{(1 - xy)_+}{f(C_1 y)}.$$

Proof. We refer to the step-numbers as in the proof of Theorem 7. *Step* 1 remains unchanged, that is, the constant functions $A > 0$ are mapped to the functions $a\delta_0$ with $a > 0$, and vice versa, since this claim relied only on the fact that comparable pairs are mapped to comparable pairs. We denote by $g : \mathbb{R}^+ \to \mathbb{R}^+$ the function such that $\mathcal{T}(a\delta_0) = g(a)$. Then, by anti-monotonicity,

g is decreasing. Also, as in the proof above, we have that $g = g^{-1}$ so that $\mathcal{T}(A) = g(A)\delta_0$. *Step* 2 also does not change: images of functions which are supported in the ray $[0, +\infty)$ are mapped to functions which include, in their support, either $[0, +\infty)$ or $(-\infty, 0]$, and the same for functions supported inside the ray $(-\infty, 0]$. Also the opposite holds: the support of the image of a function whose support includes one of the two rays, is included in one of the two rays.

In *Step* 3 again we may proceed as before, to show that all functions with support equal to $[0, x]$ are mapped to functions all having the same support. (This is done by first considering special triangle functions, and then constructing, from them, all other functions.) We thus define the function $F(x)$ in the following way—take some function with support $[0, x]$, for example, the triangle function $\triangleleft_x(y)$ which equals $1 - t$ for $y = tx$ with $0 \leqslant t \leqslant 1$ and 0 elsewhere, take its image, $\mathcal{T}\triangleleft_x$, and its support, which is a ray, has one infinite endpoint, and the other endpoint we denote $F(x)$.

The function $F|_{\mathbb{R}^*}$ (here we write $\mathbb{R}^*$ for $\mathbb{R} \setminus \{0\}$) is bijective. From anti-monotonicity we see that $|F(x)|$ is decreasing in x, and that $F(x)/x$ has some fixed sign. It is of no loss of generality to assume this sign is $+$, since changing the sign simply amounts to one similar other transform, which we will take into account in the final stage.

Next we claim that these two functions already determine the whole form of the transform. Indeed $A\square_x$ is the maximal function below A which is supported on $[0, x]$ and so $\mathcal{T}(A\square_x)$ is the minimal function above $g(A)\delta_0$ which is supported on $(-\infty, F(x)]$, which is the function

$$(\mathcal{T}A\square_x)(z) = \begin{cases} g(A), & z \leqslant 0; \\ g(A)(1 - z/F(x)), & 0 < z \leqslant F(x); \\ 0, & z > F(x). \end{cases}$$

Every function which is concave and has maximum at 0 is of the form $f(z) = \sup_x f(x)\square_x(z)$, and so for such functions, namely functions in $\mathrm{Conc}^{(m)}(\mathbb{R})$, we have the formula

$$(\mathcal{T}f)(z) = \inf_{\{x:\ f(x)>0\}} \mathcal{T}\big(f(x)\square_x\big)(z)$$

which in turn equals

$$(\mathcal{T}f)(z) = \inf_{\{x:\ f(x)>0\}} g\big(f(x)\big) \min\big(1, \big(1 - z/F(x)\big)_+\big).$$

One last thing to notice is that for functions in $\mathrm{Conc}^{(m)}(\mathbb{R})$, for each y we have $f(y) \leqslant f(0)$, and thus $g(f(y)) \geqslant g(f(0))$, and therefore the element 1 participating in the minimum is redundant, and we may rewrite (with the convention $1/F(0) = 0$)

$$(\mathcal{T}f)(z) = \inf_{\{x:\ f(x)>0\}} \big(g\big(f(x)\big)\big(1 - z/F(x)\big)_+\big).$$

Next we will use the formula above, together with the involution condition $\mathcal{T}\mathcal{T}f = f$ for certain functions, to get some conditions on the functions F and g. Let us begin with the function $f = A\mathbf{1}_{[0,T]}$ for some fixed $A, T > 0$. Then

$$(\mathcal{T}f)(z) = g(A) \inf_{\{0 \leqslant x \leqslant T\}} \big(1 - z/F(x)\big)_+ = \begin{cases} g(A), & z < 0; \\ g(A)(1 - z/F(T)), & 0 \leqslant z \leqslant F(T); \\ 0, & z > F(T). \end{cases}$$

We now apply the transform again

$$(\mathcal{T}\mathcal{T}f)(z) = \inf_{\{x<F(T)\}}\Big(g\big(g(A)\min\big(1, 1 - x/F(T)\big)\big)\big(1 - z/F(x)\big)_+\Big).$$

For $z > 0$ we have

$$\min\Big(\inf_{0\leqslant x<F(T)} g\big(g(A)\big(1 - x/F(T)\big)\big)\big(1 - z/F(x)\big)_+, 1\Big)$$

where we have used that for $x < 0$ the expression is infimal at $x \to 0$ and equal to 1. For $z > T$ this should equal to 0, that is, for some $0 \leqslant x < F(T)$

$$g\big(g(A)\big(1 - x/F(T)\big)\big)\big(1 - z/F(x)\big) \leqslant 0$$

which can only be if $z \geqslant F(F(T))$ for each such z, that is, $T \geqslant F(F(T))$. Of course, it cannot be a strict inequality since then this would be zero also for some $z < T$ and so we see $F(F(T)) = T$, this is the first condition.

Second, look at $0 < z \leqslant T$ then the expression must equal A, so for $A \leqslant 1$, the condition amounts to $\forall 0 \leqslant x < F(T)$ and $\forall 0 \leqslant z < T$,

$$g\big(g(A)\big(1 - x/F(T)\big)\big)\big(1 - z/F(x)\big) \geqslant A$$

which is equivalent to

$$g\big(g(A)\big(1 - x/F(T)\big)\big) \geqslant A/\big(1 - T/F(x)\big)$$

and so to $\forall 0 < x < y < +\infty$

$$g(A)(1 - x/y) \leqslant g\left(\frac{A}{1 - F(y)/F(x)}\right).$$

We now use this fact for a special $A = (1 - F(y)/F(x))$, and get that $g(1 - F(y)/F(x)) \leqslant g(1)/(1 - x/y)$, which, by letting $w = F(x)$ and $v = F(y)$ translates to:

$$g(1 - v/w) \leqslant g(1)/\big(1 - F(w)/F(v)\big), \quad \forall v < w. \tag{4}$$

Next, we will show that there is in fact equality in this inequality. This is similar to the case of higher dimension, only here we do not yet know that F is linear. We consider the function $\varphi = \min(\lhd_x, \square_{\lambda x})$, for some $0 < \lambda < 1$, and apply the transform. On the one hand, we know we must get

$$(\mathcal{T}\varphi)(z) = g(1)\,\hat{\max}\big(\mathbf{1}_{(-\infty,F(x)]}, \min\big(1, \big(1 - z/F(\lambda x)\big)_+\big)\big),$$

which is the following function:

$$(\mathcal{T}\varphi)(z) = g(1)\min\left(1, \frac{F(\lambda x) - z}{F(\lambda x) - F(x)}\right)_+.$$

On the other hand, we may use the formula to get that its transform must be (for $F(x) < z < F(x/2)$, say)

$$(\mathcal{T}\varphi)(z) = \inf_{0\leqslant s\leqslant\lambda}\big(g(1-s)\big(1-z/F(sx)\big)_+\big).$$

Using the inequality (4) above with $v = sx$, $w = x$ we get that

$$\begin{aligned}(\mathcal{T}\varphi)(z) &= \inf_{0\leqslant s\leqslant\lambda}\big(g(1-s)\big(1-z/F(sx)\big)\big)\\ &\leqslant \inf_{0\leqslant s\leqslant\lambda}\left(g(1)\frac{F(sx)-z}{F(sx)-F(x)}\right) = g(1)\frac{F(\lambda x)-z}{F(\lambda x)-F(x)}\end{aligned}$$

(where we have use that fact that $z > F(x)$). As before, since we have equality of the end results, we must have equality also in the inequality. In particular, since we do have continuity of g and F (both are onto and monotone), then at $s = \lambda$ we cannot have a strict inequality. We see thus that for any $0 < \lambda < 1$ we have

$$g(1-\lambda) = g(1)/\big(1 - F(w)/F(v)\big), \quad \forall v = \lambda w. \tag{5}$$

In particular, we see that F must satisfy for every w and every $0 < \lambda < 1$ that

$$\frac{F(w)}{F(\lambda w)} = 1 - g(1)/g(1-\lambda).$$

Next let $G(\lambda) = g(1-\lambda)/(g(1-\lambda) - g(1))$ so that $G(\lambda)F(w) = F(\lambda w)$. Plugging in $w = 1$ we see that $G(\lambda) = F(\lambda)/F(1)$ and we conclude that

$$F(\lambda)F(w)/F(1) = F(\lambda w).$$

This in turn implies that $E(t) = \log(F(e^t))$ satisfies

$$E(s+t) = \log\big(F\big(e^t e^s\big)\big) = \log\big(F\big(e^t\big)F\big(e^s\big)/F(1)\big) = E(s) + E(t) - E(0)$$

which implies (since F is continuous) that E is linear, so F is of the form $F(x) = C_1 x^\alpha$, and since $F = F^{-1}$ and is decreasing we see that $\alpha = -1$. We conclude that $g(x) = C_0/x$ for some $C_0 > 0$ and $F(x) = C_1/x$ for some $C_1 \neq 0$ (under our assumptions above, namely that a functions supported on $[0, x]$ is mapped to one supported on $(-\infty, F(x)]$, we get that $C_1 > 0$, but we have restricted to this case for simplicity, and as stated, the other case where it is mapped to one supported on $[F(x), \infty)$, is completely analogous, and we then get $C_1 < 0$). □

6. Convex bodies

As stated in the introduction, the theorem characterizing duality for compact convex sets with 0 in the interior was established recently by Böröczky and Schneider in [6]. Denote, as do they, the class of compact convex sets in $\mathbb{R}^n$ with 0 in the interior by $\mathcal{K}^n_{(0)}$. Denote the class of not-necessarily-bounded, closed convex sets, with 0 inside the body (but possibly at its boundary) by $\mathcal{K}^n$. They showed

Theorem 9 *(Böröczky–Schneider). Let* $\psi : \mathcal{K}^n_{(0)} \to \mathcal{K}^n_{(0)}$ *be a mapping satisfying*

$$\psi\psi(K) = K,$$

$$\psi(K \cap L) = \mathrm{conv}\big(\psi(K) \cup \psi(L)\big)$$

for all $K, L \in \mathcal{K}^n_{(0)}$. *Then there exists a selfadjoint linear transformation* $g \in GL_n$ *such that* $\psi(K) = gK^\circ$ *for all* $K \in \mathcal{K}^n_{(0)}$.

(In fact, they show a more general fact, namely if the involution condition is replaced by the condition

$$\psi\big(\mathrm{conv}(K \cup L)\big) = \psi(K) \cap \psi(L)$$

then the only possibilities for ψ are as above, with a general g, not necessarily symmetric, or a constant map.)

Here we state a characterization of duality for the class $\mathcal{K}^n$, and outline the proof which is relatively straightforward and also quite similar to the proof of our main theorem.

Theorem 10. *Let* $n \geqslant 2$ *and* $\mathcal{T} : \mathcal{K}^n \to \mathcal{K}^n$ *satisfy*

(1) $\mathcal{T}\mathcal{T}K = K$.
(2) $K_1 \subset K_2$ *implies* $\mathcal{T}K_1 \supset \mathcal{T}K_2$.

Then there exists a symmetric $B \in GL_n$ *such that*

$$\mathcal{T}K = BK^\circ = \big\{x\colon \langle x,\ By\rangle \leqslant 1\ \forall y \in K\big\}.$$

Sketch of proof. The proof goes along the same lines as the proof of Theorem 6, only that a convex set corresponds to a very simple kind of concave function: a function with values 0 or 1. We thus start by considering the function which is constant 1, and the function δ_0, which are the smallest and largest functions, and so are mapped each to the other. We continue with indicators of intervals of the form $[0, x]$, which must be mapped to half-spaces $H_{w(x)}$ by the same arguments as in the proofs above. Next, one shows that the map $x \to w(x)$ (where we let $w(0) = 0$) preserves intervals, and keeps 0 in place, and so must be linear. Finally, every convex set including 0 can be realized as the union of such intervals, and so we have the general form of the transform, as in intersection of half-spaces $\{x\colon \langle x,\ By\rangle \leqslant 1\}$ which is exactly what the theorem claims. We only need to show that B is symmetric, which one gets directly by applying the transform twice and using the involution condition. □

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