MATHEMATICS OF OPERATIONS RESEARCH
Vol. 9, No. 3, August 1984
Printed in U.S.A.

VECTOR MEASURES ARE OPEN MAPS*

DOV SAMET

Northwestern University

Nonatomic vector measures are shown to be open maps from the σ-field on which they are defined to their range, where the σ-field is equipped with the pseudometric of the symmetric difference with respect to a given scalar measure.

The main result we will prove is the following:

MAIN THEOREM. *Let $\lambda, \mu_1, \ldots, \mu_n$ be nonatomic, σ-additive, finite measures on a measurable space (I, Σ), and let λ be a nonnegative measure. Then the vector measure $\mu = (\mu_1, \ldots, \mu_n)$ is an open map from Σ to the range of μ, where Σ is equipped with the topology induced by the pseudometric d_λ defined by $d_\lambda(S, T) = \lambda[(S \setminus T) \cup (T \setminus S)]$, and the range of μ is equipped with its relative topology in R^n.*

Tauman has shown [2, Lemma 2] that when μ is a nonatomic nonnegative finite vector measure then for each x in the range of μ there exists an S with $\mu(S) = x$ such that each neighborhood of S (with respect to d_{μ_i}) is mapped by μ to a neighborhood of x. The Main Theorem strengthens this result mainly by showing that *every* S with $\mu(S) = x$ has the same property. This stronger result is used in [3].

Let us introduce the following notations. For S in Σ we denote by $\bar{S}$ the complementary set $I \setminus S$. The symmetric difference of S and T, $(S \setminus T) \cup (T \setminus S)$ is denoted by $S \Delta T$. The Euclidean norm in R^n is denoted by $\| \ \|$, and the scalar product of ξ and x in R^n is denoted by $\langle \xi, x \rangle$. By the *relative boundary* of a closed set K in R^n we mean the set of all points in K which are not in the relative interior of K. The *face* of a convex closed set K in the direction ξ is the set

$$F(\xi) = \left\{ x \mid x \in K, \langle \xi, x \rangle = \max_{y \in K} \langle \xi, y \rangle \right\}.$$

We say that a closed set K in R^n is *strictly convex* if all the points on the relative boundary of K are extreme, or alternatively if for each $\xi \in R^n$, $F(\xi)$ is either K or a singleton. For a scalar measure λ, we denote by $|\lambda|$ the sum of the positive and the negative parts of λ. For a vector measure $\mu = (\mu_1, \ldots, \mu_n)$, $|\mu|$ is the sum $\sum_{i=1}^n |\mu_i|$. For each S we define $R(\mu, S) = \{ \mu(T) \mid T \subseteq S \}$. Clearly $R(\mu, S) + R(\mu, \bar{S}) = R(\mu, I)$. By Lyapunov Theorem [1], $R(\mu, S)$ is a convex and compact set.

A convenient way to describe $R(\mu, I)$ is as follows. Let f_i be the Radon-Nikodym derivative of μ_i with respect to $|\mu|$ and let $f = (f_1, \ldots, f_n)$. Then $\mu(S) = \int_S f \, d|\mu|$ and for $\xi \in R^n$, $\langle \xi, \mu(S) \rangle = \int_S \langle \xi, f \rangle d|\mu|$. Obviously $\mu(S)$ is in the face of $R(\mu, I)$ in the direction ξ, if and only if $\{ t \mid \langle \xi, f(t) \rangle > 0 \} \subseteq S \subseteq \{ t \mid \langle \xi, f(t) \rangle \geqslant 0 \}$ almost everywhere with respect to $|\mu|$. It follows then that $R(\mu, I)$ is strictly convex if and only if the set $\{ t \mid \langle \xi, f(t) \rangle = 0 \}$ is of $|\mu|$-measure zero for all supporting hyperplanes ξ of $R(\mu, I)$ which do not contain $R(\mu, I)$, or alternatively if for each subspace V of R^n of dimension smaller than that of $R(\mu, I)$, the set $\{ t \mid f(t) \in V \}$ is of $|\mu|$-measure zero.

*Received February 12, 1982; revised September 16, 1982.
AMS 1980 subject classification. Primary: 28B05.
OR/MS Index 1978 subject classification. Primary: 431 Mathematics.
Key words. Measures, vector measures, open maps.

0364-765X/84/0903/0471$01.25

We can prove now:

LEMMA 1. *There is a decomposition* $R(\mu, I) = \sum_i R(\mu, S_i)$ *such that* $\bigcup_i S_i$ *is a partition of* I *and* $R(\mu, S_i)$ *is strictly convex for each* i.

PROOF. The decomposition is built in n stages. In the stages $1, \ldots, k-1$ a family of disjoint sets S_i^j, $1 \leqslant j \leqslant k-1$, $1 \leqslant i < i_j$ is defined (i_j is possibly ∞ or 0) such that $R(\mu, S_i^j)$ is strictly convex and of dimension j. Moreover, for each $k-1$ dimensional subspace of R^n, V, the set $\{t \mid t \in I \setminus \bigcup_{i,j} S_i^j, f(t) \in V\}$ is of $|\mu|$-measure zero. In the kth stage we define the sets S_i^k, $1 \leqslant i < i_k$ which are all the subsets of $I \setminus \bigcup_{i,j} S_i^j$ of the form $\{t \mid f(t) \in V\}$ which have positive $|\mu|$-measure, where V is a k-dimensional subspace of R^n. The disjointness of the sets S_i^k can be guaranteed since the intersection of each such two sets is a set of t's for which $f(t)$ belongs to a subspace of dimension less than k. The strict convexity of $R(\mu, S_i^k)$ follows similarly. Q.E.D.

Let us call a vector measure $\mu = (\mu_1, \ldots, \mu_n)$ *monotonic* if each μ_i $(1 \leqslant i \leqslant n)$ is either nonnegative or nonpositive. We will show now that it suffices to prove the Main Theorem for monotonic μ with strictly convex range $R(\mu, I)$. Indeed, there is a partition $I = \bigcup_{i=1}^{2^n} I_i$ such that the restriction of μ to each I_i is monotonic. We can decompose, furthermore, each I_i according to Lemma 1 to get eventually a partition $I = \bigcup_i S_i$ and a decomposition $R(\mu, I) = \sum_i R(\mu, S_i)$ such that for each i, μ is monotonic on S_i and $R(\mu, S_i)$ is strictly convex. For $\epsilon > 0$ and $S \in \Sigma$ denote

$$\Omega_i(S, \epsilon) = \{T \mid T \subseteq S_i, d_\lambda(T, S \cap S_i) < \epsilon\} \quad \text{and} \quad \Omega(S, \epsilon) = \left\{\bigcup_i T_i \mid T_i \in \Omega_i(S, \epsilon)\right\}.$$

It is easy to verify that the family of sets $\Omega(S, \epsilon)$ where S ranges over Σ and ϵ ranges over the positive reals is a basis to the topology induced by d_λ on Σ. Moreover $\mu(\Omega(S, \epsilon)) = \sum_i \mu(\Omega_i(S, \epsilon))$. But $\mu(\Omega_i(S, \epsilon)) \subseteq R(\mu, S_i)$ and $R(\mu, S_i)$ is strictly convex and the restriction of μ to S_i is monotonic. Therefore by proving the Main Theorem for monotonic μ with strictly convex range we prove that $\mu(\Omega_i(S, \epsilon))$ is relatively open in $R(\mu, S_i)$ which says that $\mu(\Omega(S, \epsilon))$ is relatively open in $R(\mu, I)$.

We assume now that μ is monotonic and that $R(\mu, I)$ is strictly convex. We start by proving the following lemma.

LEMMA 2. *If* $x_0 = \mu(S_0)$ *then for each* $1 \leqslant i \leqslant n$ *and* $\epsilon > 0$ *the set* $\{\mu(S) \mid d_{|\mu_i|}(S, S_0) < \epsilon\}$ *contains a set* $\{x \mid x \in R(\mu, I), \|x - x_0\| < \delta\}$ *for some* $\delta > 0$.

We first prove the lemma in the case that x_0 is in the relative interior of $R(\mu, I)$, using Lemma 3.

LEMMA 3. *If* $x_0 = \mu(S_0)$ *is in the relative interior of* $R(\mu, I)$, *then the intersection of the relative interiors of* $R(\mu, S_0)$ *and* $R(\mu, \bar{S}_0)$ *is not empty.*

PROOF OF LEMMA 3. Indeed, if this intersection is empty then there exists a hyperplane which separates the two sets and for at least one of them, say $R(\mu, S_0)$, contains only points from its relative boundary. Since $0 \in R(\mu, S_0) \cap R(\mu, \bar{S}_0)$ we conclude that there exists $\xi \in R^n$ such that $\langle \xi, x \rangle \geqslant 0$ for $x \in R(\mu, S_0)$ and $\langle \xi, x \rangle \leqslant 0$ for $x \in R(\mu, \bar{S}_0)$ and moreover for some x in the relative interior of $R(\mu, S_0)$, $\langle \xi, x \rangle > 0$. Now let $S \in \Sigma$ and denote $S_1 = S \cap S_0$, $S_2 = S \cap \bar{S}_0$. We have:

$$\langle \xi, \mu(S_2) \rangle \leqslant 0 \leqslant \langle \xi, \mu(S_0 \setminus S_1) \rangle = \langle \xi, \mu(S_0) \rangle - \langle \xi, \mu(S_1) \rangle$$

and therefore,

$$\langle \xi, \mu(S) \rangle = \langle \xi, \mu(S_1) + \mu(S_2) \rangle \leqslant \langle \xi, \mu(S_0) \rangle.$$

This inequality holds for each S in Σ and, moreover, for some S the inequality is strict which shows that $\mu(S_0)$ is in the relative boundary of $R(\mu, I)$, contrary to our assumption. Q.E.D.

PROOF OF LEMMA 2. Assume first that x_0 is in the relative interior of $R(\mu, I)$. Let E_0, E_1 and E_2 be the linear spaces spanned by $R(\mu, I)$, $R(\mu, S_0)$ and $R(\mu, \bar{S}_0)$, respectively, and denote by B_0, B_1 and B_2 the intersection of the unit ball in R^n with E_0, E_1 and E_2, respectively. Since $0 \in R(\mu, S_0) \cap R(\mu, \bar{S}_0)$, we find, using Lemma 3, a point w which belongs to the relative interiors of both $R(\mu, \bar{S}_0)$ and $R(\mu, S_0)$ and for which $\|w\| < \epsilon/4$. Choose now $0 < \eta < \epsilon/4$ such that $w + \eta B_1 \subseteq R(\mu, S_0)$ and $w + \eta B_2 \subseteq R(\mu, \bar{S}_0)$. Clearly $E_0 = E_1 + E_2$ and therefore we can choose $0 < \delta < \epsilon/4$ such that $\delta B_0 \subseteq \eta(B_2 + B_1) = \eta(B_2 - B_1)$. Now let $x \in R(\mu, I)$ with $\|x - x_0\| < \delta$ and denote $z = x - x_0$. Since $z \in \delta B_0$ there exist $z_1 \in \eta B_1$ and $z_2 \in \eta B_2$ such that $z = z_2 - z_1$. There exist also $S_1 \subseteq S_0$, $S_2 \subseteq \bar{S}_0$ such that $\mu(S_1) = w + z_1$ and $\mu(S_2) = w + z_2$. Define $S = (S_0 \setminus S_1) \cup S_2$. We have

$$\mu(S) = \mu(S_0) - \mu(S_1) + \mu(S_2) = x_0 - z_1 + z_2 = x_0 + z = x,$$

and using the monotonicity of μ,

$$d_{|\mu_i|}(S, S_0) \leqslant \| \mu(S \Delta S_0)\| = \| \mu(S_1) + \mu(S_2)\| = \|2w + z_1 + z_2\| < 2\frac{\epsilon}{4} + 2\eta < \epsilon.$$

We continue now to prove Lemma 2 for x_0 on the relative boundary of $R(\mu, I)$. Consider a sequence $x_n = \mu(S_n)$ such that $x_n \to x_0$. We will show that $\mu(S_n \Delta S_0) \to 0$ which is more than we need to complete the proof of Lemma 3. Let $T_n' = S_n \cap S_0$ and $T_n'' = S_n \cap \bar{S}_0$. Since the sequences $\mu(T_n')$ and $\mu(T_n'')$ belong to the compact sets $R(\mu, S_0)$ and $R(\mu, \bar{S}_0)$ we can assume without loss of generality that $\mu(T_n') \to \mu(T')$ and $\mu(T_n'') \to \mu(T'')$ where $T' \subseteq S_0$ and $T'' \subseteq \bar{S}_0$. It follows that $\mu(T' \cup T'') = \mu(S_0)$ and since $R(\mu, I)$ is strictly convex $T' = S_0$ and $T'' = \emptyset$ almost everywhere with respect to μ, which shows that $\mu(S_n \Delta S_0) = \mu(S_0) - \mu(T_n') + \mu(T_n'') \to 0$. Q.E.D.

To complete the proof of the Main Theorem we have to show that d_λ can replace $d_{|\mu|}$ in Lemma 2. There is a partition $I = S_1 \cup S_2$ of I such that the restriction of λ to S_1 is continuous with respect to $|\mu|$ and $|\mu|(S_2) = 0$. Define $\Omega_i(S, \epsilon) = \{T \mid T \subseteq S_i, d_\lambda(T, S) < \epsilon\}$, $i = 1, 2$, and $\Omega(S, \epsilon) = \{T_1 \cup T_2 \mid T_i \in \Omega_i(S, \epsilon), i = 1, 2\}$. Clearly $\mu(\Omega_2(S, \epsilon)) = 0$. But $\Omega_1(S, \epsilon)$ is open in the topology induced by $d_{|\mu|}$ on the σ-field $\{T \mid T \in \Sigma, T \subseteq S_1\}$ and therefore by Lemma 2 $\mu(\Omega(S, \epsilon)) = \mu(\Omega_1(S, \epsilon))$ is relatively open in $R(\mu, S_1) = R(\mu, I)$. Q.E.D.

Although in general the projection of a convex compact set is not necessarily an open map, it is open when the set is the range of a vector measure as follows easily from the Main Theorem.

COROLLARY. *Let $\mu = (\mu_1, \ldots, \mu_{n+1})$ be a nonatomic, σ-additive, finite vector measure. Then the projection π of the range of μ on its first n coordinates is an open map onto the range of $(\mu_1, \ldots, \mu_n)$.*

PROOF. Denote $\hat{\mu} = (\mu_1, \ldots, \mu_n)$. Clearly $\pi\mu = \hat{\mu}$ and $\pi = \hat{\mu}\mu^{-1}$. The result follows since by the Main Theorem $\hat{\mu}$ is an open map with respect to $d_{|\mu|}$ and because μ is continuous with respect to $d_{|\mu|}$. Q.E.D.

The Main Theorem can be stated in terms of the integral of a set valued function as follows. Let λ and ν be nonatomic, positive and finite scalar measures. For a set valued function $F: I \to R^n$ let us denote by $\mathcal{F}$ the set of all ν-integrable functions $\phi: I \to R^n$ such that $\phi(t) \in F(t)$ for each t, and let $\int \mathcal{F} = \{\int \phi \, d\nu \mid \phi \in \mathcal{F}\}$. If $f: I \to R^n$ is a ν-integrable function and $F(t) = \{0, f(t)\}$, then $\int \mathcal{F}$ is the range of the vector measure whose Radon-Nikodym derivative with respect to ν is f. By the Main Theorem we conclude that the map $\phi \to \int \phi$ for $\phi \in \mathscr{F}$ is open when $\mathscr{F}$ is equipped with the norm topology of $L_1^n(\lambda)$. This formulation raises the natural question: how general the set valued function F can be, such that the map $\phi \to \int \phi$ is still open.

Acknowledgment. The author acknowledges Zvi Artstein for a helpful discussion.

References

[1] Lindestrauss, J. (1966). A Short Proof of Liapuunoff's Convexity Theorem. *J. Math. Mech.* **15** 971–972.

[2] Tauman, Y. A Characterization of Vector Measure Games in pNA. *Israel J. Math.* (to appear).

[3] ——— and Reichert, J. (1984). The Space of Polynomials in Measures is Internal. *Math. Oper. Res.* (to appear).

MANAGERIAL ECONOMICS DEPARTMENT, J. L. KELLOGG GRADUATE SCHOOL OF MANAGEMENT, NORTHWESTERN UNIVERSITY, EVANSTON, ILLINOIS 60201