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Topology-Free Typology of Beliefs

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In their seminal paper, Mertens and Zamir (*Int. Game Theory* **14** (1985), 1–29) proved the existence of a universal Harsanyi type space which consists of all possible types. Their method of proof depends crucially on topological assumptions. Whether such assumptions are essential to the existence of a universal space remained an open problem. Here we prove that a universal type space does exist even when spaces are defined in pure measure theoretic terms. Heifetz and Samet (mimeo, Tel Aviv University, 1996) showed that coherent hierarchies of beliefs, in the measure theoretic case, do not necessarily describe types. Therefore, the universal space here differs from all previously studied ones, in that it does not necessarily consist of all coherent hierarchies of beliefs. *Journal of Economic Literature* Classification Numbers: D80, D82.

1. INTRODUCTION

We study here the foundations of that part of the theory of games with incomplete information that deals with players' beliefs. We study it in the broadest and most natural setup, that of probability (or measure) theory without any topological notions, which have always been used for this purpose until now. We show that even under this general setup there is a largest measure theoretic type space—*a universal space*—which contains all possible measure theoretic types.

Our study of beliefs in the most general setup leads us to a deeper understanding of the notions of types and universal spaces. The previously studied universal spaces consist of all hierarchies of beliefs that satisfy simple *coherence* conditions. This is why the universal space has become

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almost synonymous with the set of coherent hierarchies of beliefs. But, as demonstrated by Heifetz and Samet [10], this happy coincidence is restricted to the topological case. In the measure theoretic framework some coherent hierarchies of beliefs fail to describe types. This raises the question whether it is still possible to have a universal space in the general, measure theoretic, case. Here we answer it in the affirmative. This answer shows that the role of coherence in the theory of belief types turns out to be trickier than previously assumed; the universal space is not defined in terms of coherence alone.

Before describing our results in more detail we survey briefly the background.

A Single Agent's Beliefs. The starting point for the modeling of beliefs of interacting agents is, naturally, the model of a single agent's belief. The most prevalent such model, and the one that has proved most applicable, is the Bayesian model. Beliefs, in this model, are about *events*, that is, measurable subsets of a given measurable space, the elements of which are *states of nature*. Belief is expressed as a σ-additive probability measure on this space. Belief *types*, in this case, are simply all such probability measures.

A Multi-Agent Setup. When several agents interact it is not enough, of course, to model their beliefs by assigning each of them a probability measure on the set of states of nature. A belief type in such an interaction should account for beliefs of agents regarding each other's beliefs. Here two approaches are possible.

Explicit Description of Beliefs. The first approach is that of describing agents' beliefs *explicitly*. Such a description specifies beliefs about states of nature, beliefs about combinations of states of nature and beliefs about them, and so on. Accordingly, a belief type consists of a whole hierarchy of beliefs. Clearly, the different beliefs that are lumped together to define a type must fit according to some consistency rules. The explicit descriptions that obey these rules are said to be *coherent*.

Advantages and Drawbacks. Explicitly describing belief types seems to be the straightforward approach to modeling agents' beliefs in interaction. A type is defined and constructed solely in terms of the set of states of nature. This kind of construction tells us what are *all* the possible types. The main disadvantage of the approach is that this description of belief types does not fit, prima facie, into the classical Bayesian paradigm of a single agent's belief type—one that is described by one probability measure over some space. It lacks, therefore, all the advantages that the Bayesian model offers. For example, it does not enable integration with respect to belief. In short, the entangled web of beliefs that explicitly describe types in interactive situations does not lend itself to making a workable model. For a long time, it was the explicit description of types that hindered the analysis of games with incomplete information. A remedy to this drawback of explicit descriptions can be found only if it is possible to identify each type with a single σ-additive probability measure on some space.

Implicit Description of Beliefs. It was Harsanyi [7] who showed how to cut the Gordian knot of interacting beliefs, by introducing the notion of type space. Such spaces provide *implicit* description of a belief types. With each point of a type space, called a *state* (or a *state of the world*), are associated a state of nature (which can be thought of as a specification of the parameters of a game), as well as a probability measure, for each agent, on the type space itself. The agent's probability measure is his *type* in the state. It describes his beliefs about the states of the world. But since states of the world are associated with the types of all agents, the agent's type in a state describes his beliefs about all agents' types. Thus, a type is described implicitly; it is, essentially, the belief that the agent has about types and the state of nature.

Advantages and Drawbacks. Type spaces encapsulate the intricacy of interacting beliefs in one tractable and manageable mathematical object. Unlike the explicit description of types, the implicit one, as given in a type space, is the classical Bayesian description of belief; it is given by one probability distribution. It is little wonder then, that for the last three decades type spaces have played a major role in economic theory and game theory.

But unlike the explicit description of types, the implicit one is not expressed directly in terms of the set of states of nature. It is defined in terms of one particular state space. The notion of "all possible types" in this case is not well defined. Moreover, using type spaces may be restrictive in the following sense. By modeling a specific game with incomplete information using a fixed type space we may "miss" some types that are not presented in this type space and can be found only in a larger one that contains the former. If this is true for any type space, then the concept of type space is necessarily restrictive.

Universal Spaces. Types defined in type spaces would not suffer from these drawbacks only if there is a *universal* type space, one that "contains" all type spaces. If such a type space exists then the notion of "all types" is formally defined, while being a type is still defined in terms of type spaces. The existence of a universal type space guarantees that in principle a game with incomplete information can be modeled using a type space without any loss of generality.

To formalize the idea that the universal space "contains" all type spaces, *type morphisms* are first defined. A type morphism from one type space to another is a map that preserves the structure of these spaces; that is, it preserves the way states of nature and types are associated with states of the world. A universal space is one into which each type space can be mapped by a type morphism in a unique way. It is straightforward to show that when there exists a universal space it is necessarily unique up to type isomorphism.

To summarize, the explicit description of belief types provides us, prima facie, with a complicated object, different from the classical Bayesian modeling of belief as a probability measure; the implicit description of types, using type spaces, raises the question of the existence of a universal type space. Finally one wonders how these two approaches are related.

The topological case. Mertens and Zamir [12] addressed these issues and studied the relationship between the explicit and the implicit descriptions of types in the special case where the set of states of nature is a compact topological space. They showed that in this case the two approaches amount to the same definition of belief types. More specifically, they proved that the set of all coherent explicit descriptions of types is a universal type space.

The topological aspect of their work went beyond the compactness of the set of states of nature. Indeed, all probability measure spaces involved are topologized to yield compact spaces, and all the functions involved are assumed to be continuous.

The strategy of the proof in Mertens and Zamir [12] is the following. Each coherent hierarchy of beliefs—comprising an explicit description—defines, in a natural way, a finitely additive probability measure over the set of all coherent hierarchies. In order to make the set of all such hierarchies into a type space it is required to show that the associated probability measure can be extended to a σ-additive one. This is precisely where the topological properties of the spaces (in particular compactness) are used.

The results of Mertens–Zamir have been extended to cover more general cases. Brandenburger and Dekel [4] showed it is enough to assume that set of states of nature is a complete, separable metric space. Heifetz [8] showed it is enough to assume that the set of states nature is a Hausdorff space, provided beliefs are regular probability measures. Mertens, Sorin, and Zamir, [11] showed that various other topological assumptions will do as well. But in all these works the strategy of the proof remains the same: the use of topology to show that the finitely additive probability measure associated with a coherent hierarchy can be extended to a σ-additive one.

The Measure Theoretic Case. The question remains whether the equivalence of explicit and implicit descriptions still holds when no topological assumptions are made, that is, when all spaces are assumed to be general measurable space and all functions are assumed to be measurable. No way has been found to carry out the above mentioned crucial step of the proof without topological assumptions.

We now know that this cannot be done. In a companion paper, Heifetz and Samet [10], we construct a measurable space of states of nature and a coherent hierarchy of beliefs on it, which cannot be extended to a σ-additive probability measure over the set of all coherent hierarchies. Thus the natural candidate for a universal space, the one that played this role in all previously studied cases—the space of all coherent hierarchies—is not a type space at all in the general measure theoretic case. The Mertens–Zamir program breaks down in this case. The explicit description approach now seems much less attractive. The attractiveness of the alternative approach, by type spaces, hinges on the question of whether there exists a universal space in the measure theoretic case. The answer given by Mertens and Zamir is not applicable in this case; it can not be the set of all coherent explicit descriptions.

In this work, we show that a universal space exists, nevertheless, even in the general measure-theoretic case. The basic idea is surprisingly simple. The elements of the universal space, like those in Mertens–Zamir [12] are explicit descriptions. However unlike in Mertens–Zamir [12] these descriptions are not created in accordance with some preconceived notion of coherence, but are rather found ready made. In each given type space and a state in it, it is possible to describe explicitly the beliefs of the players in the state. Their beliefs concerning the state of nature, beliefs about those beliefs and so on. We show that the set of all explicit descriptions that arise in this way in type spaces is a universal space. It is easy to associate with each element of this space a finitely additive measure on it. The extension of this measure to a σ-additive one is done here by using the probability measure defined on one of the type space that gave rise to this element of the universal one.

Two Ways to Describe Beliefs Explicitly. The explicit description of beliefs that an agent has in a state of a type space can be given in two ways. It is possible to give it as a sequence (or a hierarchy) of probability measures: one over the states of nature, another over combinations of states of nature and beliefs about them, and so on. Another way to describe beliefs in a state would be by specifying the events that contain the state. We consider only those events, which we call *expressions*, that are expressed in terms of nature and p-belief operators similar to those used in Monderer and Samet [13]. Thus each state in the universal space can be given in terms of either a hierarchy of measures, or a set of expressions.

In the next section we give measure theoretic and notational preliminaries. The definitions of type spaces, type morphisms and universal spaces are given in Section 3. We construct the universal space in Section 4, using type descriptions in terms of expressions, and again in Section 5, with descriptions in terms of hierarchies. We present the two different constructions, which necessarily lead to the same space, because we find that their complete equivalence merits our attention. We discuss our results and compare them to other works in Section 6.

2. PRELIMINARIES

Let X be a measurable space with a σ-field Σ. We refer to the measurable sets in Σ as *events* in X. The set of all σ-additive probability measures on X is denoted by $\Delta(X)$. We consider $\Delta(X)$ as a measurable space with the σ-field $\sum_{\Delta}$ that is generated by all sets of the form $\beta^p(E)=\{\mu \mid \mu(E)\geqslant p\}$, for an event E in X and $0\leqslant p\leqslant 1$.

For $x\in X$ we denote by δ_x the unique measure in $\Delta(X)$ which is concentrated at x. That is, δ_x is the measure μ for which $\mu(E)=1$ for all events E in X such that $x\in E$.

For measurable spaces X and Y and a measurable function $\varphi: X\to Y$, we denote by $\hat{\varphi}$ the function $\hat{\varphi}: \Delta(X)\to\Delta(Y)$ defined by $\hat{\varphi}(\mu)=\mu\circ\varphi^{-1}$ (that is, for each event F in Y, $\hat{\varphi}(\mu)(F)=\mu(\varphi^{-1}(F))$. It is easy to check that $\hat{\varphi}$ is a measurable function.

We fix a set I to be the set of *players*. The set $I_0=I\cup\{0\}$ includes all players and "0" which stands for "nature". For a family of sets $(X_i)_{i\in I_0}$ we denote by X the product $\prod_{i\in I_0} X_i$, and by X_{-i}, for $i\in I_0$, the product $\prod_{j\in I_0\backslash\{i\}} X_j$. If $(Y_i)_{i\in I_0}$ is another family of sets, and $(f_i)_{i\in I_0}$ a family of functions, $f_i: X_i\to Y_i$, then the *induced* function $f: X\to Y$ is defined by $f((x_i)_{i\in I_0})=(f_i(x_i))_{i\in I_0}$. We consider any product, finite or infinite, of measurable spaces as a measurable space with the product σ-field.

3. MEASURE-THEORETIC TYPE SPACES

Type spaces. Fix a measurable space S the elements of which are called *states of nature*.

DEFINITION 3.1. A *type space* on S is a pair $\langle (T_i)_{i\in I_0}, (m_i)_{i\in I}\rangle$, or $\langle T, m\rangle$ for short, where

(1) $T_0=S$, and T_i, for $i\in I$, is a measurable space.

(2) For each $i\in I$, m_i is a measurable function $m_i: T_i\to\Delta(T)$.

(3) For each $i \in I$ and $t_i \in T_i$, the marginal of $m_i(t_i)$ on T_i is δ_{t_i}.

The points of T are called *states* or *states of the world.* A point of T_i is called an *i-type.*

The function m_i endows the types with their content—it specifies for each type t_i its beliefs concerning states of the world. The claim that i assesses a probability at least p to the event E in T is described by the set:

$$B_i^p(E) = \{ t \in T \mid m_i(t_i)(E) \geqslant p \}.$$

Clearly, $B_i^p(E) = m_i^{-1}(\beta^p(E)) \times T_{-i}$ and therefore is measurable.

Remark. Condition (3) in the definition of type spaces expresses players' self consciousness. Each player is certain of his type. That is, the mass of the marginal probability of a given player's type on his type space concentrates on that type. This condition plays a very minor role in this paper. If we were to omit it only few lines in the paper would change. There is another way to model self consciousness, by requiring that the measure $m_i(t_i)$ associated with an i-type is in $\Delta(T_{-i})$ rather than $\Delta(T)$. We preferred the latter for notational advantages.

Type morphisms. Let $\langle T, m \rangle$ and $\langle T', m' \rangle$ be type spaces on S. We define maps from one space to the other that preserve the structure of the spaces as given by the functions m_i and m'_i.

DEFINITION 3.2. Let $(\varphi_i)_{i \in I_0}$ be an I_0-tuple, of measurable functions φ_i: $T_i \to T'_i$. The induced function φ: $T \to T'$ is called a *type morphism* if,

(1) φ_0 is the identity on S;

(2) for each $i \in I$, $m'_i \circ \varphi_i = \hat{\varphi} \circ m_i$.

The morphism is a *type isomorphism* if φ is an isomorphism (or equivalently, if φ_i is an isomorphism for each $i \in I_0$).

Condition (2) in this definition means that for each $i \in I$ and $t_i \in T_i$, $\varphi_i(t_i) = t'_i$ implies that $\hat{\varphi}(m_i(t_i)) = m'_i(t'_i)$. In other words, using the definition of $\hat{\varphi}$, for each $i \in I$, $t_i \in T_i$ and event $E \subseteq T'$:

$$m'_i(\varphi_i(t_i))(E) = m_i(t_i)(\varphi^{-1}(E)). \tag{3.1}$$

It can be easily verified that condition (2) is equivalent to saying that φ preserves belief operators. That is, for each $0 \leqslant p \leqslant 1$, $i \in I$ and event $E \subseteq T'$

$$B_i^p(\varphi^{-1}(E)) = \varphi^{-1}(B_i^p(E)). \tag{3.2}$$

DEFINITION 3.3. A type space T^* on S is *universal* if for every type space T on S there is a unique type morphism from T to T^*.

Our first main result, which we prove in two different ways in the next couple of sections, is:

THEOREM 3.4. *For any measurable space S there exists a universal type space on S.*

Before we discuss this result, we prove the following observation.

PROPOSITION 3.5. *There is at most one universal type spaces on S up to type isomorphism.*

Proof. Let T and T' be universal type spaces on S. Then there is a type morphism φ from T to T' and also a type morphism φ' from T' to T. Thus, $\varphi' \circ \varphi$ is a type morphism from T to T. But the identity map from T to T is also a type morphism, and therefore by the uniqueness of type morphisms to universal type spaces it follows that $\varphi' \circ \varphi$ is the identity on T, which proves that φ is a type isomorphism. ▮

The basic idea of the construction of the universal type space on S is as follows. Using the functions m_i, in a given type space, we provide for each i-type a *description* of his beliefs. We then show that the set of all descriptions forms a universal space. This construction is carried out in two different ways corresponding to different formalization of the term "description." In the next section we formalize it by a family of events, that we call *expressions*, in the type space. In section 5, descriptions are formalized in terms of a hierarchy of probability distributions.

4. THE UNIVERSAL SPACE IN TERMS OF EXPRESSIONS

Expressions. The building blocks we use in this section to construct the universal space are formulas that express events defined solely in terms of nature and beliefs. We call such formulas *expressions* and define formally the set $\mathscr{E}$ of all expressions as follows:

(1) Each event E in S is an expression.

(2) If e and f are expressions then $\neg e$ (read: the complement of e) and $(e \cap f)$ are expressions.

(3) If e is an expression, then $B_i^p(e)$ for $i \in I$ and $0 \leqslant p \leqslant 1$, is an expression.

An *i-expression*, for $i \in I$, is one obtained by applying set theoretic operations, only, to expressions of the form $B_i^p(e)$. A 0-expression is an event E in S. The set of all i-expressions, for $i \in I_0$, is denoted by $\mathscr{E}_i$.

Descriptions. Obviously, for any given type space T, every expression e can be viewed as an event in T (with the caveat that 0-expressions E are identified with $E \times T_{-0}$). Thus with some abuse of notation, but no ambiguity, we will speak about event e in T where e is an expression. We say that an expression e holds in $t \in T$ if t belongs to the event e in T. Let $D(t)$ the set of all expressions that hold in t. Denote by $d_i(t)$ the set of all i-expressions that hold in t, i.e., $d_i(t) = D(t) \cap \mathscr{E}_i$. Clearly, if for t and t' in T, $t_i = t'_i$, then $d_i(t) = d_i(t')$. Thus $d_i(t)$ depends only on t_i and we can consider d_i as a function $d_i : T_i \to 2^{\mathscr{E}_i}$. We denote by d_0 the identity on S. By convention, $d = (d_i)_{i \in I_0}$, is the induced function from T to $\prod_{i \in I_0} 2^{\mathscr{E}_i}$. We call $d_i(t_i)$ the *i-description* of t_i. Likewise, $d(t)$ is the *description* of t. d_i and d are the *i-description map* and *description map*, respectively.

PROPOSITION 4.1. *Type morphisms preserve description maps.*

Proof. To show that descriptions are preserved we show first that for every type morphism $\varphi: T \to T'$, $D(\varphi(t)) = D(t)$ for every $t \in T$, i.e., that for every expression $e \in \mathscr{E}$, $t \in e$ in T iff $\varphi(t) \in e$ in T'. This is equivalent to saying that φ^{-1} preserves expressions. That is, $\varphi^{-1}(e)$ for event e in T' is the event e in T. Indeed, φ^{-1} preserves 0-expressions since φ_0 is the identity on nature. If e and f are preserved by φ^{-1} then so are $\neg e$ and $(e \cap f)$. Also by (3.1),

$$m'_i(\varphi_i(t_i))(e) = m_i(t_i(\varphi^{-1}(e)) = m_i(t_i)(e),$$

so t belongs to $B_i^p(e)$ in T iff $\varphi(t)$ belongs to $B_i^p(e)$ in T', as required. Since for each $i \in I_0$, $d_i(t) = D(t) \cap \mathscr{E}_i$, it is obvious that i-descriptions maps are also preserved by type morphisms. ∎

The universal Type Space. We define now a type space T^* on S, by setting $T_0^* = S$ and letting T_i^*, for each $i \in I$, to be the set of *all i-descriptions that appear in some type space over S.*

We define for each expression e a subset $[e]$ of T^* as follows. If e is a 0-expression E, $[e]$ is E in T^*, i.e.,

$$[e] = \{t^* \in T^* \mid t_0^* \in E\}.$$

For every $i \in I$ and i-expression $e \in \mathscr{E}_i$,

$$[e] = \{t^* \in T^* \mid e \in t_i^*\}.$$

To extend the definition to expressions in $\mathscr{E}$, it is enough to define for $e, f \in \mathscr{E}$, $[\neg e] = \neg[e]$ and $[e \cap f] = [e] \cap [f]$.

We let the σ-field of T_i^*, for $i \in I$, be the one generated by sets of the form $\{t_i^* \mid e \in t_i^*\}$. Thus, the product σ-field of T^* is the one generated by the field $[\mathscr{E}] = \{[e] \mid e \in \mathscr{E}\}$.

Notice that by the definition of $[e]$, $d^{-1}([e]) = e$ for each i-expression. Since such expressions generate $\mathscr{E}$, we record this result and conclude:

LEMMA 4.2. *For each type space T, the description map d on T satisfies $d^{-1}([e]) = e$ for each expression $e \in \mathscr{E}$. In particular, d is measurable.*

The following proposition helps to define the functions m_i^* on T^*.

PROPOSITION 4.3. *There exists, for each $i \in I$, a function $m_i^* : T_i^* \to \Delta(T^*)$ such that for each type space T with description function d, and $t_i \in T_i$,*

$$m_i^*(d_i(t_i)) = m_i(t_i) \circ d^{-1}. \tag{4.1}$$

Proof. For a given t_i^* fix a space type T and an i-type t_i in T_i, such that $d_i(t_i) = t_i^*$. Define $m_i^*(t_i^*) = m_i(t_i) \circ d^{-1}$. Since d is measurable, $m_i^*(t_i^*)$ is well defined. In particular, by Lemma 4.2,

$$m_i^*(t_i^*)([e]) = m_i(t_i)(d^{-1}([e])) = m_i(t_i)(e). \tag{4.2}$$

We show now that the definition is independent of the particular type space T and type t_i. Note that $m_i(t_i)(e) = \max\{p \mid t_i \in B_i^p(e)\}$. The right-hand side of this expression is, by definition of t_i^*, $\max\{p \mid B_i^p(e) \in t_i^*\}$. Thus by (4.2),

$$m_i^*(t_i^*)([e]) = \max\{p \mid B_i^p(e) \in t_i^*\}. \tag{4.3}$$

By (4.3), $m_i^*(t_i^*)$ is determined on the generating field $[\mathscr{E}]$ independently of T and t_i and therefore it is independent of them. Now, (4.1) holds by the definition of m_i^*. ∎

PROPOSITION 4.4. *The space $\langle T^*, m^* \rangle$ is a type space on S.*

To prove this proposition we use the following lemma, which plays the central measure theoretic role in the construction of the universal space in this section and in the next one.

LEMMA 4.5. *Let (X, Σ) be a measurable space with a σ-field Σ. Let $\mathscr{F}$ be a field on X that generates Σ, and $\mathscr{F}_\Delta$ the σ-field on $\Delta(X)$ generated by sets of the form*

$$\{\beta^p(E) \mid E \in \mathscr{F}, 0 \leqslant p \leqslant 1\}.$$

Then, $\mathscr{F}_\Delta = \Sigma_\Delta$.

Proof. Denote by $\mathscr{F}'$ be the set of all events F in X, such that $\beta^p(F) \in \mathscr{F}_\Delta$ for all $0 \leqslant p \leqslant 1$. We prove that $\mathscr{F}'$ contains Σ, which shows that $\mathscr{F}_\Delta$ contains all the generators of Σ_Δ. Since $\mathscr{F}'$ contains the field $\mathscr{F}$ that generates Σ, it is enough to show that $\mathscr{F}'$ is a monotone class (See, e.g., Dudley [5, Theorem 4.4.2]). That is, we have to show that if $(E_n)_{n=1}^\infty$ is a decreasing (increasing) sequence of events in $\mathscr{F}'$ then $\bigcap_{n=1}^\infty E_n \in \mathscr{F}'$ $(\bigcup_{n=1}^\infty E_n \in \mathscr{F}')$.

If $(E_n)_{n=1}^\infty$ is decreasing, then for any $\mu \in \Delta(X)$, $(\mu(E_n))_{n=1}^\infty$ is a decreasing sequence converging to $\mu(\bigcap_{n=1}^\infty E_n)$. Therefore, by σ-additivity,

$$\beta^p\left(\bigcap_{n=1}^\infty E_n\right) = \bigcup_{n=1}^\infty \beta^p(E_n) \in \mathscr{F}_\Delta.$$

If $(E_n)_{n=1}^\infty$ is increasing then for any $\mu \in \Delta(X)$, $(\mu(E_n))_{n=1}^\infty$ is an increasing sequence converging to $\mu(\bigcup_{n=1}^\infty E_n)$. In this case, σ-additivity implies

$$\beta^p\left(\bigcup_{n=1}^\infty E_n\right) = \bigcap_{m=1}^\infty \bigcup_{n=1}^\infty \beta^{p-1/m}(E_n) \in \mathscr{F}_\Delta. \quad \blacksquare$$

Proof of Proposition 4.4. We have to prove that m_i^* is measurable. By Lemma 4.5 the set of events $\{\beta^p([e]) \mid e \in \mathscr{E}\}$ generates the σ-field of $\Delta(T^*)$. Hence, it is enough to prove that $(m_i^*)^{-1}$ $(\beta^p([e])$ is measurable in T_i^*. That is, we have to show that $\{t_i^* \mid m_i^*(t_i^*)([e]) \geqslant p\}$ is measurable. But by (4.3) this set is precisely the subset $B_i^p(e)$ in T_i^* which is measurable by definition.

We show now that the marginal of $m_i^*(t_i^*)$ on T_i^* is $\delta_{t_i^*}$ Let T be a type space with $d_i(t_i) = t_i^*$ for some $t_i \in T_i$. Suppose that for some $e \in \mathscr{E}_i$, $\{t_i^*\} \times T_{-i}^* \subseteq [e]$. Then $\{t_i\} \times T_{-i} \subseteq e$ in T, and therefore $m_i(t_i)(e) = 1$. By (4.2) $m_i^*(t_i^*)([e]) = 1$ as required. $\blacksquare$

LEMMA 4.6. *For every expression $e \in \mathscr{E}$, $[e] = e$ in T^*.*

Proof. Since the equality holds for 0-expressions, and [ ] preserves set theoretic operations, it is enough to show that it holds for expressions $B_i^p(f)$, whenever it holds for f. But using (4.3) we have,

$$[B_i^p(f)] = \{t^* \in T^* \mid B_i^p(f) \in t_i^*\} = \{t^* \in T^* \mid \max\{q \mid B_i^q(f) \in t_i^*\} \geqslant p\}$$
$$= \{t^* \in T^* \mid m_i^*(t_i^*)([f]) \geqslant p\}$$
$$= \{t^* \in T^* \mid m_i^*(t_i^*)(f) \geqslant p\} = B_i^p(f). \quad \blacksquare$$

LEMMA 4.7. *The description map $d\colon T^* \to T^*$ is the identity.*

Proof. Let t^* be in T^*, $i \in I$ and $e \in \mathscr{E}_i$. Then, by Lemma 4.6,

$$d_i(t_i^*) = \{e \mid t^* \in e\} = \{e \mid t^* \in [e]\} = \{e \mid e \in t_i^*\} = t_i^*. \quad \blacksquare$$

THEOREM 4.8. *The space $\langle T^*, m^* \rangle$ is a universal type space on S.*

Proof. For each type space $\langle T, m \rangle$ the description map $d\colon T \to T^*$ is measurable (Lemma 4.2) and satisfies (4.1). Therefore it is a type morphism. We show that this is the unique type morphism from T to T^*. Suppose that φ is a type morphism from T to T^*. Then for each $i \in I$ and $t_i \in T_i$, $d_i(\varphi_i(t_i)) = d_i(t_i)$, by Proposition 4.1. But by Lemma 4.7, $d_i(\varphi_i(t_i)) = \varphi(t_i)$. Thus $\varphi_i = d_i$. $\blacksquare$

5. THE UNIVERSAL TYPE SPACE IN TERMS OF HIERARCHIES

In this section we offer an alternative construction of the universal space, by defining the description of players' types in terms of hierarchies of beliefs rather than expressions.

Hierarchies. We start by defining spaces of hierarchies H_i^k for each $k \geqslant 0$ and $i \in I_0$. For every $k \geqslant 0$, $H_0^k = S$, and for every $i \in I$, H_i^0 is a singleton. As usual, for each $k \geqslant 0$ we denote $H^k = \prod_{i \in I_0} H_i^k$. We define inductively:

$$H_i^{k+1} = H_i^k \times \Delta(H^k) = H_i^0 \times \prod_{l=0}^{k} \Delta(H^l).$$

The space of *i-hierarchies* for player $i \in I$ is:

$$H_i = H_i^0 \times \prod_{l=0}^{\infty} \Delta(H^l).$$

Thus, H_i is the space of all potential hierarchies of player i's beliefs: about nature, about nature and the players' beliefs about nature, and so on. Denoting for nature, $H_0 = S$, we define $H = \prod_{i \in I_0} H_i$ as the *hierarchies space*. Denote by π_i^k is the projection from H_i to H_i^k. The induced map $\pi^k = (\pi_i^k)_{i \in I_0}$ is the projection from H to H^k.

Descriptions. Given a type space T, we define an *i-description map* $h_i: T \to H_i$, for each $i \in I_0$. The induced map $h = (h_i)_{i \in I_0}$ from T to H, that associates with each $t \in T$ the corresponding hierarchy of beliefs for each i, is the *description map*. The hierarchy $h_i(t_i)$ is the *i-description* of t_i and $h(t)$ is the *description* of t.

To define h_i we define first maps $h_i^k: T_i \to H_i^k$ for $k \geqslant 0$ and $i \in I_0$. For $i = 0$ and every $k \geqslant 0$, let h_i^k be the identity (on S). For $i \in I$, h_i^0 is uniquely defined, since H_i^0 is a singleton. Inductively, define

$$h_i^{k+1}(t_i)(h_i^k(t_i), m_i(t_i) \circ (h^k)^{-1}) = (h_i^0(t_i), m_i(t_i) \circ (h^0)^{-1}, ..., m_i(t_i) \circ (h^k)^{-1}),$$

where, by convention, $h^k = (h_i^k)_{i \in I_0}$. Now, define h_i, for each $i \in I$, as the unique function from T to H_i that satisfies for all $k \geqslant 0$, $h_i^k = \pi_i^k \circ h_i$. That is,

$$h_i(t_i) = (h_i^0(t_i), m_i(t_i) \circ (h^0)^{-1}, ..., m_i(t_i) \circ (h^k)^{-1}, ...).$$

The definition of h is completed by letting h_0 to be the identity on S.

PROPOSITION 5.1. *Type morphisms preserve descriptions and i-descriptions.*

Proof. Let $\varphi: T \to T'$ be a type morphism. We have to show that $h_i'(\varphi_i(t_i)) = h_i(t_i)$ for all $t_i \in T_i$ and $i \in I_0$. For $i = 0$ this is immediate, since $\varphi_0, h_0^k, h_0, h_0'^k$ and h_0' are all the identity on nature. For $i \in I$, $h_i'^0(\varphi_i(t_i)) = h_i^0(t_i)$ because H_i^0 is a singleton. Inductively, if we have already shown that $h_i'^k(\varphi_i(t_i)) = h_i^k(t_i)$ for every $t_i \in T_i$, i.e., that $h_i'^k \circ \varphi_i = h_i^k$ for every $i \in I_0$, then, using (3.1)

$$\begin{aligned} m_i'(\varphi_i(t_i)) \circ (h'^k)^{-1} &= m_i(t_i) \circ \varphi^{-1}(h'^k)^{-}1 \\ &= m_i(t_i) \circ (h'^k \circ \varphi)^{-1} = m_i(t_i) \circ (h^k)^{-1} \end{aligned}$$

and therefore

$$\begin{aligned} h_i'^{k+1}(\varphi_i(t_i)) &= (h_i'^k(\varphi_i(t_i)), m_i'(\varphi_i(t_i)) \circ (h'^k)^{-1}) \\ &= (h_i(t_i), m_i(t_i) \circ (h^k)^{-1}) = h_i^{k+1}(t_i) \end{aligned}$$

as required. ∎

The Universal Type Space. We define now the type space T^* by setting $T_0^* = S$ and T_i^* to be the set of all descriptions in H_i, that is, all hierarchies $t_i^* \in H_i$ for which $t_i^* = h_i(t_i)$ for some $t_i \in T_i$ in some type space T

over S. The σ-field of T_i^* is the one inherited from H_i. We define $m_i^*: T_i^* \to \Delta(T^*)$ by

$$m_i^*(t_i^*) = m_i(t_i) \circ h^{-1}. \tag{5.1}$$

Clearly m_i^* is thus a σ-additive probability measure.

PROPOSITION 5.2. *$\langle (T_i^*)_{i \in I_0}, (m_i^*)_{i \in I} \rangle$ is a type space on S.*

Proof. In order to show that T^* is a type space on S we have to prove first that m_i^* is measurable for each i. For $t_i^* \in T_i^*$ let t_i be the i-type chosen to define $m_i^*(t_i^*)$. Consider the probability measure induced by $m_i^*(t_i^*)$ on H^k, namely, $m_i^*(t_i^*) \circ (\pi^k)^{-1}$. Using the identity $h^k = \pi^k \circ h$ we conclude:

$$\begin{aligned} m_i^*(t_i^*) \circ (\pi^k)^{-1} &= m_i(t_i) \circ h^{-1} \circ (\pi^k)^{-1} \\ &= m_i(t_i) \circ (h^k)^{-1} \\ &= (h_i(t_i))^{k+1} \\ &= (t_i^*)^{k+1}. \end{aligned} \tag{5.3}$$

Thus for each measurable E in H,

$$\begin{aligned} (m_i^*)^{-1} (\beta^p((\pi^k)^{-1} (E))) &= \{ t_i^* \mid m_i^*(t_i^*)((\pi^k)^{-1} (E)) \geqslant p \} \\ &= \{ t_i^* \mid (t_i^*)^{k+1} ((\pi^k)^{-1} (E)) \geqslant p \}. \end{aligned} \tag{5.4}$$

The last set is of course a measurable subset of H_i, as it is defined by an event in H_i^{k+1}. The field of events $(\pi^k)^{-1}(E)$ generates the σ-field on H and hence, by Lemma 4.5, sets of the form $\beta^p((\pi^k)^{-1} (E))$ generate $\Delta(H)$. Thus, by (5.4), m_i^* is measurable.

We have to show that the marginal of $m_i^*(t_i^*)$ on T_i^* is $\delta_{t_i^*}$. Indeed, suppose that $t_i^* \in E$ for some event E in T_i^*. Then

$$m_i^*(t_i^*)(E \times T_{-i}^*) = m_i(t_i)(h^{-1}(E \times T_{-i}^*)) = m_i(t_i)(h_i^{-1}(E) \times T_{-i})).$$

The last term is 1, since $t_i \in h_i^{-1}(E)$. ∎

PROPOSITION 5.3. *For every type space T, the description map $h: T \to T^*$ is a type morphism.*

Proof. The functions h_i, $i \in I_0$ are measurable as functions to H_i. Since the range of h_i is T_i^* it is also measurable as a function to T_i^*. Next, observe that by (5.3) the value of the measure $m_i^*(t_i^*)$ on the field of events $(\pi^k)^{-1} (E)$, that generates the σ-field on H, is independent of the specific type t_i chosen to define $m_i^*(t_i^*)$. That is, on that field $m_i^*(t_i^*)$ coincides with $m_i(t_i)\,h^{-1}$, for any t_i such that $h_i(t_i)=t_i^*$. Since these two measures coincide on a generating field they are equal. Thus (5.1) holds for any t_i such that $h_i(t_i)=t_i^*$, which guarantees that h is a type morphism. ∎

LEMMA 5.4. *The hierarchy description maps h_i: $T^* \to T^*$ are the identity maps.*

Proof. It is enough to show that for each k and $i \in I$, the function h_i^k on T^* is the projection on H_i^k. We prove this by induction on k. It is trivially true for $k=0$. Suppose we proved it for all integers up to k. This implies that $h^k=\pi^k$. By definition $(h_i(t_i^*))^{k+1}=m_i^*(t_i^*)(h^k)^{-1}$. By the induction hypothesis, we can substitute in the last expression π^k for h^k and conclude that $(h_i(t_i^*))^{k+1}=(t_i^*)^{k+1}$. ∎

THEOREM 5.5. *$\langle (T_i^*)_{i\in I_0}, (m_i^*)_{i\in I}\rangle$ is a universal type space.*

Proof. Using Propositions 5.1, 5.3 and Lemma 5.4, the proof is the same as in Theorem 4.8 (with h_i replacing d_i along the proof). ∎

6. DISCUSSION

Belief Spaces. A type space as defined here is a product of the set of states of nature, S, and i-type spaces T_i for each player i. Harsanyi [1] and Mertens, Sorin, and Zamir [11] also define type spaces as products. Mertens and Zamir [12] adopted a different kind of spaces which they called *belief spaces*. A belief space is a triple, $\langle T, \Theta, (m_i)_{i\in I}\rangle$, where T is a measurable state space, and $\Theta: T\to S$, and $m_i: T\to \Delta(T)$ are all measurable functions. Condition (3) in Definition 3.1 is replaced by the requirement that the mass of the probability measure $m_i(t)$ is concentrated on the set $\{t' \mid m_i(t')=m_i(t)\}$. It is obvious that the definition of belief spaces is more comprehensive, since product type spaces can be viewed as a special case of them. Belief morphisms on belief spaces are defined in the obvious way as well as universal belief spaces. Despite the seeming more generality of belief spaces, they do not give rise to a different definition of types. Going through the proof of existence given here, it is easy to see how the same proof, with necessary minor changes, proves the existence of the universal belief space. Moreover, this space has the form of a product space, and hence the universal belief space and the universal type space are one and the same space.

Belief Modeled by Set Functions. Beliefs are modeled here as σ-additive probability measures. But the only properties of σ-additivity that we used in the proof are monotonicity (with respect to set inclusion) and continuity (with respect to increasing and decreasing sequences of events). Specifically we used these properties in Lemma 4.5.

Redefine now the set $\Delta(X)$ to be the set of all monotonic, continuous set functions, μ on the σ-field on X with values in $[0, 1]$, and such that $\mu(\emptyset)=0$ and $\mu(X)=1$ (rather than the set of σ-additive probability measures). The previous sections can be read now, verbatim, as a proof of the existence of a universal type space, when belief is represented by monotonic, continuous set functions. Such functions, as representation of belief, were studied by Shafer [15] and Schmeidler [14].

Knowledge Spaces. Call a set function γ on the σ-field of a space X a *carrier function*, if there exists an event E in X, called the carrier of γ, such that for each event F, $\gamma(F)=1$ when $E \subseteq F$ and $\gamma(F)=0$ otherwise. Suppose we redefine again $\Delta(X)$ to be the set of all carrier functions on X. The definitions of type spaces (and belief spaces) can be applied now to beliefs represented by carrier set functions. If we add also the requirement that each i-type t_i (or t for a belief space) is an element of the carrier of $m_i(t_i)$ (or $m_i(t)$ for a belief space) then type spaces and belief spaces are just the well known partition spaces where the carriers are the elements of the partition. Such spaces are used to model knowledge.

Carrier functions are monotonic and continuous with respect to decreasing sequences of events, but not with respect to increasing sequences. This lack of continuity is the property of knowledge by which one can fail to know any of the statements in a sequence of statements which are less and less informative, and yet know their infinite disjunction. Thus our proof of the existence of a universal space cannot be applied to knowledge spaces. In a set theoretic framework Heifetz and Samet [9] showed, that indeed there is no universal knowledge space.

Coherence. The main feature of all the universal spaces studied up till now in the literature is that of *coherence*. The notion of coherence is mostly known in its hierarchical version. The space of coherent hierarchies of beliefs is the maximal subset C of the set of all potential hierarchies H (described in Section 5) each element of which satisfies the following three requirements. (1) A player's higher order beliefs should agree with his lower order beliefs. (2) Each player is certain about his beliefs. (3) Each player is certain that C holds. This notion of coherence appears in Armbruster and Böge [1], Böge and Eisele [3], and in Mertens and Zamir [12]. In the latter work it was proved that C is indeed a universal space under suitable topological assumptions. The notion of coherence continued, in different equivalent forms, to be the main tool for the construction of the universal type space in all the works that followed.

The notion of coherence can be applied to description by expressions as well. Here we have to give conditions on subsets of $\mathscr{E}$ that make them coherent. These conditions belong to several categories. (1) Set theoretic conditions. For example, the expressions $e \cap f$ and $\neg e$ cannot be part of the same coherent description. (2) Conditions reflecting the structure of S. Thus, if E and F are two disjoint 0-expressions, i.e., disjoint events in S, then they cannot be elements in the same coherent description. (3) Conditions concerning the probabilistic properties of the i-expressions $B_i^p(e)$. For example, if $p > q$ then a coherent description should not include both $B_i^p(e)$ and $\neg B_i^q(e)$. (4) Conditions concerning the σ-additivity of beliefs. For instance, if $B_i^p(e)$ belong to a description for every $p < q$, then $B_i^q(e)$ should belong to the description as well. When such conditions are properly defined, a coherent description is a maximal coherent subset of $\mathscr{E}$. There is a natural one-to-one correspondence of coherent descriptions by hierarchies and coherent descriptions by expressions.

One way to define coherence of descriptions with expressions is by considering expressions as formulas in a formal language. The previously described conditions can be formulated as a deductive system for this language, that is as a set of axioms and inference rules. A coherent set of expressions is defined now as a (deductively) consistent set of formulas. Such deductive systems were described by Fagin and Halpern [6] and Auman [2]. The languages in these works are richer than needed for the expressions we use here. Both include knowledge operators in addition to the belief operators. In Fagin and Halpern [6] they allow also expressions that describe linear combinations of probabilities. Aumann [2] is constructing explicitly the set of all coherent descriptions and shows how to associate with each description and player a probability measure over this space. It is doubtful, however, that his set of coherent descriptions is a universal space for any class of type spaces since, as was shown in Heifetz and Samet [9] there is no universal knowledge space. Furthermore, both Fagin and Halpern and Aumann use finitary formulas, in which the σ-additivity conditions of category (4) above cannot be expressed.

The use of a finitary deductive system to define coherence, limits the kind of spaces of states of nature S that can be handled. This is so because of the compactness property of such deductive systems, by which a set of formulas is consistent whenever any of its finite subsets is. By this property the set $[\mathscr{E}]$ is a base for a topology on the space of coherent descriptions with which it is compact. In particular it also implies that the set $[\mathscr{E}_0]$ is a base for a topology on the space of coherent descriptions S with which S is compact. Defining coherence of descriptions by expressions, without the use of a finitary deductive system, as well as defining coherence of descriptions by hierarchies, avoid any such restrictions on the space of nature states S.

Coherence plays no role in the current paper. It is easy to show that each element of the universal space, presented either as a hierarchy of beliefs or as a set of expressions, is coherent. But the universal space does not consist of *all* coherent descriptions. The failure of a coherent description to serve as a description of a type is demonstrated in Heifetz and Samet [10]. Thus the standard way of proving the existence of a universal space, namely by showing that the space of all coherent description is the desired space, could not possibly work in the general measure-theoretic framework.

REFERENCES

1. W. Armbruster and W. Böge, Bayesian game theory, *in* "Game Theory and Related Topics" (O. Moeschlin and D. Pallaschke, Eds.), North-Holland, Amsterdam, 1979.
2. R. Aumann, "Notes on Interactive Epistemology," Cowles Foundation for Research in Economics working paper, 1989.
3. W. Böge and T. Eisele, On solutions of Bayesian games, *Int. J. Game Theory* **8** (1979), 193–215.
4. A. Brandenburger and E. Dekel, Hierarchies of beliefs and common knowledge, *J. Econ. Theory* **59** (1993), 189–198.
5. R. M. Dudley, "Real Analysis and Probability," Wadsworth, Belmont, CA, 1989.
6. R. Fagin and J. Y. Halpern, Reasoning about knowledge and Probability, *in* "Proceedings of the Second Conference on Theoretical Aspects of Reasoning about Knowledge," Morgan Kaufmann, San Marcos, CA, 1988.
7. J. C. Harsanyi, Games with incomplete information played by Bayesian players, Parts I, II, and III, *Manage. Sci.* **14** (1967), 159–182, 320–334, 486–502.
8. A. Heifetz, The Bayesian formulation of incomplete information—the non-compact case, *Int. J. Game Theory* **21** (1993), 329–338.
9. A. Heifetz and D. Samet, Knowledge spaces with arbitrarily high rank, *Games Econom. Behav.* **22** (1998), 260–273.
10. A. Heifetz and D. Samet, Coherent beliefs are not always types, mimeo, Tel Aviv University, 1996.
11. J. F. Mertens, S. Sorin, and S. Zamir, "Repeated Games: Part A: Background Material," CORE Discussion Paper No. 9420, Université Catholique de Louvain, 1994.
12. J. F. Mertens and S. Zamir, Formulation of Bayesian analysis for games with incomplete information, *Int. J. Game Theory* **14** (1985), 1–29.
13. D. Monderer and D. Samet, Approximating common knowledge with common beliefs, *Games Econom. Behav.* **1** (1989), 170–190.
14. D. Schmeidler, Subjective probability and expected utility without additivity, *Econometrica* **57** (1989), 571–587.
15. G. Shafer, "A Mathematical Theory of Evidence," Princeton Univ. Press, Princeton, NJ, 1976.