ON THE DISPENSABLE ROLE OF TIME IN GAMES OF PERFECT INFORMATION

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Abstract. In Aumann (1995) and Aumann (1998), time is assumed implicitly in the description of games of perfect information, and it is part of the epistemic distinction between ex-ante and ex-post knowledge. We show that ex-post knowledge in these papers can be expressed by ex-ante knowledge and therefore epistemically, time is irrelevant to the analysis. Furthermore, we show that material rationality by weak dominance and by expectation can be expressed in terms of the timeless strategic form of the game.

Time present and time past
Are both perhaps present in time future,
And time future contained in time past.

T. S. Eliot, Burnt Norton—the Four Quartets

1. Introduction

Aumann's (1987) seminal work "Correlated equilibrium as an expression of Bayesian rationality" provided analysis of games in *strategic form* in a given context. By context we mean the knowledge and beliefs of the players. Aumann's (1995) paper "Backward induction and common knowledge of rationality" did the same for games in *extensive form* of perfect information, except that this time only knowledge of the players was involved and not their beliefs.

The two roles of time. Time, in the ordinal sense, is implicitly assumed when a game is described in extensive form: Moves in the game are made, and vertices are reached one *after* the other. In Aumann (1995), time is omnipresent, as rationality is defined per vertices, and thus the dynamic aspect of the extensive form of the game is fully exploited in the analysis.

Time plays a role not only in the objective description of the game, but also in the context. The knowledge of the players changes over time since more information is acquired by the players as the game unfolds. The knowledge of the players before the game starts is called ex-ante knowledge. The knowledge acquired at later times is ex-post knowledge. Aumann (1995) discussed ex-post knowledge and ex-post rationality but did not formalize them. He explained that in order to formalize these notions "we would have had to assign several knowledge operators to each player, one for each of his vertices, and we wanted to keep the formal model as transparent and simple as possible."

However, in Aumann (1998), in his study of *material rationality*, he found it necessary to formally introduce ex-post knowledge and ex-post rationality, stating emphatically that rationality "is *inherently* ex post". The rationality of a player is material when she is required to be rational only in those vertices of her that are reached.[1] The formal definition of ex-post material rationality says that,

> A player is *ex-post materially rational* when for each vertex v and strategy t_i of hers, if v is reached then it is not the case that she knows *ex post* that t_i yields a higher payoff than her strategy.

Our purpose is to show that time plays a less of a role in analyzing games of perfect information than Aumann (1995) and Aumann (1998) imply.

The redundancy of ex-post knowledge. We first show that adding ex-post knowledge operators for each player and each of her vertices does not enrich our language and its expressibility. Any statement that makes use of these operators can be translated into an equivalent statement that makes use of only the ex-ante knowledge operators of the players. In light of this, any definition of rationality can be formulated with ex-ante knowledge operators. There are no notions of rationality that can be formulated in terms of ex-post knowledge only.

In particular, applying this translation to the definition of ex-post material rationality results in the following equivalent description of ex-post material rationality. The changes from the definition are italicized.

> **Proposition:** A player is ex-post materially rational if and only if for each vertex v and strategy t_i of hers, if v is reached then it is not the case that she knows *ex ante* that *if v is reached* then t_i yields a higher payoff than her strategy.

Thus, epistemically, time plays no role in studying games with perfect information in the model of Aumann (1995) and Aumann (1998).

The redundancy of the extensive form. Time still plays a role in the objective description of the game and in the way it is used in the definition of rationality. The very notion of material rationality that requires rationality in each *reached vertex*, seems to indicate that the extensive form is essential for this definition. This is indeed true for the specific notion of material rationality studied in Aumann (1998), but is not true for other notions of material rationality, like the next two notions.

Rationality by weak dominance. Material rationality in the previous definition requires that there is no strategy of the player which she knows to yield a *strictly* higher payoff. Consider the following strengthening of this notion of material rationality, which requires that the player does not even know of another strategy of hers that yields payoffs which are at least as good as her strategy and is not equivalent to it.

> A player is *materially rational by weak dominance* when for each vertex v and strategy t_i of hers, it is not the case that she knows that if v is reached then t_i yields a payoff at least as high as her strategy, unless she knows that if v is reached t_i yields the same payoff as her strategy.

We now write the same definition, except that we omit the word 'material' and any mention of vertices.

> A player is *rational by weak dominance* when for each strategy t_i of hers, it is not the case that she knows that t_i yields a payoff at

[1] The rationality in Aumann (1995) is described as *substantive.* Players are required to be rational even at vertices that are not reached, where the difference between being rational or not is payoff irrelevant.

> least as high her strategy, unless she knows that that t_i yields the same payoff as her strategy.

This definition makes use only of the strategic form of the game. However, we show:

> **Proposition:** Material rationality by weak dominance is the same as rationality by weak dominance.

Thus, time, through the extensive form of the game, plays no role in material rationality by weak dominance.

Rationality by expectation. By adding probabilistic beliefs to the model, as in Aumann (1987), we can define material rationality in terms of the expected payoff of the player at reached vertices.

> A player is *materially rational by expectation* when for each vertex v and strategy t_i of hers, if v is reached, then it is not the case that she knows that conditional on reaching v, playing t_i yields her expected payoff, which is higher than the same expected payoff when she plays her strategy.

Again, we write the same definition without mentioning materiality or vertices.

> A player is *rational by expectation* when for each strategy t_i of hers, it is not the case that she knows that playing t_i yields expected payoff which is higher than her expected payoff.

This last definition is exactly the definition of rationality for games in strategic form in Aumann (1987). And for this notion of rationality too:

> **Proposition:** Material rationality by expectation is the same as rationality by expectation.

Thus, when we change the condition of material rationality from strong dominance to expectation, we go all the way back to the timeless notion of rationality in Aumann (1987).

2. Preliminaries

We use mostly the same notations as Aumann (1995) and Aumann (1998). The set of player i's vertices is denoted by V_i, and the set of i's strategies is S_i. Knowledge is expressed in a standard partition model. The set of states is Ω. Knowledge is described by a set $(\Pi_i)_i$ of partitions of Ω . The knowledge operator K_i, associated with the partition Π_i, is defined by $K_iE = \{\omega \mid \Pi_i(\omega) \subseteq E\}$, where $\Pi_i(\omega)$ is the element of Π_i that contains ω. The event CKE, that E is *common knowledge* is the event that all know E, all know that all know E and so on. The strategy profile at ω is $\boldsymbol{s}(\omega)$. For a strategy $t_i \in S_i$ and and a strategy profile s, we denote by $(s; t_i)$ the strategy profile obtained by replacing s_i by t_i. We assume that each player knows her strategy. This means that $\boldsymbol{s}_i$ is measurable with respect to Π_i. For a vertex v, Ω^v is the event that vertex v is reached.

3. Thinking ahead: ex post turned ex ante

3.1. **Ex-post material rationality.** The event that player i's strategy t_i dominates $\boldsymbol{s}_i$ at v, denoted $[t_i \succ_v \boldsymbol{s}_i]$, consists of all states ω for which $h_i^v(\boldsymbol{s}(\omega); t_i) > h_i^v(\boldsymbol{s}(\omega))$. Ex-post rationality is defined in Aumann (1998) as follows.

Definition 1. *The event that player i is* **ex-post materially rational** *is:*

$$R_i^{\text{epm}} = \bigcap_{v \in V_i} \bigcap_{t_i \in S_i} \neg\Omega^v \cup \neg K_i^v[t_i \succ_v \boldsymbol{s}_i]. \tag{1}$$

Here, K_i^v is the *ex-post* knowledge operator, describing i's knowledge at the time that she learns whether vertex v is reached or not. Formally, the partition corresponding to K_i^v is the coarsest common refinement of Π_i and the partition $\{\Omega^v, \neg\Omega^v\}$.

The reading of (1) is straightforward. Player i is ex-post materially rational when for each vertex v and strategy t_i of i, if v is reached then it is not the case that i knows ex-post that t_i dominates her strategy at v.[2]

3.2. **Getting rid of ex-post knowledge.** Ex-post knowledge is expressible in terms of ex-ante knowledge. The assertion that one knows E ex post, after learning whether v occurred or not, is equivalent to the following assertion: Either v is reached and one knows ex ante that if v is reached then E, or v is not reached and one knows ex ante that if v is not reached then E. In the formal language of the model:

Proposition 1. *For each event E,*

$$K_i^v E = \big(\Omega^v \cap K_i(\neg\Omega^v \cup E)\big) \cup \big(\neg\Omega^v \cap K_i(\Omega^v \cup E)\big). \tag{2}$$

Indeed, by the definition K_i^v, $\omega \in K_i^v E$, if and only if either $\omega \in \Omega^v \cap \Pi_i(\omega) \subseteq E$, or $\omega \in \neg\Omega^v \cap \Pi_i(\omega) \subseteq E$. This holds if and only if $\omega \in \Omega^v$ and $\Pi_i(\omega) \subseteq \neg\Omega^v \cup E$, or $\omega \in \neg\Omega^v$ and $\Pi_i(\omega) \subseteq \Omega^v \cup E$. This is the condition for ω to be in the right hand side of (2).

The operators K_i^v can be used as an abbreviation of the right hand side of (2), but in the case of (1) even this is not required. Using (2) for $E = [t_i \succ_v \boldsymbol{s}_i]$ and substituting in (1) results in the following simple expression for ex-post material rationality.

Corollary 1.

$$R_i^{\text{epm}} = \bigcap_{v \in V_i} \bigcap_{t_i \in S_i} \neg\Omega^v \cup \neg K_i\big(\neg\Omega^v \cup [t_i \succ_v \boldsymbol{s}_i]\big). \tag{3}$$

Thus, ex-post material rationality is simply described without ex-post knowledge operators. Epistemically, time is dispensable in the model of Aumann (1995) and Aumann (1998), and in particular when material rationality is studied.

3.3. **Epistemizing material rationality.** According to (3), i can be rational but fail to know it. This will be the case in a state ω, where v is not reached, i does not know that v is not reached, and she knows that when v is reached, some strategy t_i dominates her strategy at v. As $\omega \in \neg\Omega^v$, i is rational at ω. However, since she does not know $\neg\Omega^v$ there are states in $\Pi_i(\omega)$ where v is reached. As i knows that t_i dominates her strategy at v, i is not rational in these states. Therefore, in some states in $\Pi_i(\omega)$, i is rational, and in some she is not. Thus, she does not know at ω that she is rational. Put differently, a player's rationality depends not only on her behavior *given her knowledge*, but also on some facts that she does not know. This

[2]The event $\neg X \cup Y$ corresponds to the the assertion that either X does not hold, or else Y holds. But it also correspond to the assertion that if X holds then Y holds. In logic, the 'if...then...' construction, in this sense, is called material implication.

diverges from standard definitions of rationality in game theory and economics. We easily fix this problem in the following definition.

Definition 2. *The event that player i is* **materially rational** *is:*

$$R_i^{\mathrm{m}} = \bigcap_{v \in V_i} \bigcap_{t_i \in S_i} (K_i \neg \Omega^v) \cup \neg K_i \big(\neg \Omega^v \cup [t_i \succ_v \boldsymbol{s}_i]\big). \tag{4}$$

The reading of (4) is simple. Player i is materially rational when for each vertex v and strategy t_i of i,

- either player i knows that v is not reached,
- or, it is not the case that player i knows that *if v is reached* then t_i dominates $\boldsymbol{s}_i$ at v.[3]

The reason for the first clause is this. If i knows that v is not reached, then she trivially knows that if v is reached then t_i dominates $\boldsymbol{s}_i$ at v. Thus, without the first clause i would not be rational when she knows that a certain vertex v is not reached, which is, of course, undesirable.

The problem of a player not knowing that she is rational when she is, is solved with this definition: Player i is materially rational if and only if she knows that she is materially rational, by virtue of the positive and negative introspection properties of knowledge. The relation between ex-post material rationality and material rationality is rather simple.

Proposition 2.

$$R_i^{\mathrm{m}} = K_i(R_i^{\mathrm{epm}}).$$

This follows immediately from the fact that K_i is distributed over intersections, and satisfies for each E and F, $\neg K_i E = K_i \neg K_i E$, and $K_i(E \cup K_i F) = (K_i E) \cup (K_i F)$.

As we are interested in the implications of common knowledge of rationality, the difference between the two definitions is completely washed away, since in light of Proposition 2,

Corollary 2.

$$CK(\cap_i R_i^{\mathrm{m}}) = CK(\cap_i R_i^{\mathrm{epm}}).$$

4. Material rationality as strategic-form rationality

The epistemic expression of time, namely, ex-post knowledge, has been shown in the previous section to play a dispensable role in studying material rationality. But time is still present in the definition of material rationality, since it is defined particularly for the extensive form of the game, using the vertices of the game tree. As we see next, the use of the extensive form of the game is peculiar to the specific definition of material rationality, but not to the property of materiality.

[3] We can read (4) alternatively as a conditional. If i does not exclude the possibility that v is reached (that is, if she does not know that v is not reached), then she does not know that if v is reached then t_i dominates $\boldsymbol{s}_i$ at v.

4.1. **Rationality by weak dominance.** Material rationality is defined in (1) and (4) in terms strong dominance, as the event $[t_i \succ_v \boldsymbol{s}_i]$ is defined by strict inequalities. We now define rationality by weak dominance, using the event $[t_i \succcurlyeq_v \boldsymbol{s}_i]$, which is the set of states ω for which $h_i^v(\boldsymbol{s}(\omega); t_i) \geq h_i^v(\boldsymbol{s}(\omega))$. Note that $[t_i \succcurlyeq_v \boldsymbol{s}_i] = [t_i \succ_v \boldsymbol{s}_i] \cup [t_i \sim_v \boldsymbol{s}_i]$, where the last event consists of the states ω for which $h_i^v(\boldsymbol{s}(\omega); t_i) = h_i^v(\boldsymbol{s}(\omega))$.

Definition 3. *The event that player i is* **materially rational by weak dominance** *is:*

$$R_i^{\text{mwd}} = \bigcap_{v \in V_i} \bigcap_{t_i \in S_i} \neg K_i\big(\neg\Omega^v \cup [t_i \succcurlyeq_v \boldsymbol{s}_i]\big) \cup K_i\big(\neg\Omega^v \cup [t_i \sim_v \boldsymbol{s}_i]\big). \tag{5}$$

That is, player i is materially rational by weak dominance when for each v and t_i, if i knows that t_i weakly dominates her strategy when v is reached, then she knows that the two strategies are equivalent when v is reached.

We now define two events in terms of the strategies of the game without referring to vertices of the game tree. The event $[t_i \succcurlyeq \boldsymbol{s}_i]$ consists of all the states ω for which $h_i(\boldsymbol{s}(\omega); t_i) \geq h_i(\boldsymbol{s}(\omega))$. Similarly, $[t_i \sim \boldsymbol{s}_i]$ is the set of states ω for which $h_i(\boldsymbol{s}(\omega); t_i) = h_i(\boldsymbol{s}(\omega))$. The following definition of rationality by weak dominance is for the strategic form of the game.

Definition 4. *The event that player i is* **rational by weak dominance** *is:*

$$R_i^{\text{wd}} = \bigcap_{t_i \in S_i} \neg K_i\big([t_i \succcurlyeq \boldsymbol{s}_i]\big) \cup K_i\big([t_i \sim \boldsymbol{s}_i]\big). \tag{6}$$

It turns out that the the use of the extensive form of the game in the definition of material rationality by weak dominance is superfluous. Material rationality of this type can be described in terms of the strategic form of the game. Time is dispensable for weak dominance rationality.

Proposition 3. *Player i is materially rational by weak dominance if and only if i is rational by weak dominance. That is,*

$$R_i^{\text{mwd}} = R_i^{\text{wd}}.$$

Discussion: A first attempt at defining material rationality by weak dominance would replace the event $[t_i \succ_v \boldsymbol{s}_i]$ in (4) by the event $[t_i \succcurlyeq_v \boldsymbol{s}_i]$, and require that for each t_i and v,

$$(K_i \neg\Omega^v) \cup \neg K_i\big(\neg\Omega^v \cup [t_i \succcurlyeq_v \boldsymbol{s}_i]\big), \tag{7}$$

holds. However this definition has the following problem. By the monotonicity of knowledge, the event in (7) is a subset of $(K_i \neg\Omega^v) \cup \neg K_i\big(\neg\Omega^v \cup [t_i \sim_v \boldsymbol{s}_i]\big)$. But there is no reason for requiring that for player i to be rational she should not know that if v is reached then t_i is not equivalent to her strategy. Thus, we have to amend (7) by allowing a rational player to know that if v is reached t_i is equivalent to her strategy. That is, we require that for each t_i and v,

$$(K_i \neg\Omega^v) \cup \neg K_i\big(\neg\Omega^v \cup [t_i \succcurlyeq_v \boldsymbol{s}_i]\big) \cup K_i^v\big(\neg\Omega^v \cup [t_i \sim_v \boldsymbol{s}_i]\big), \tag{8}$$

Noting further that $(K_i \neg\Omega^v) \subseteq K_i^v\big(\neg\Omega^v \cup [t_i \sim_v \boldsymbol{s}_i]\big)$ we get definition 3.

4.2. **Rationality by expectation.** We examine material rationality when it is expressed in terms of expectation with respect to probabilistic beliefs. Player i's beliefs are given by a type function τ_i which assigns to each state ω a probability function $\tau_i(\omega)$ on Ω called i's *type* at ω. Each type function τ_i is measurable with respect to the partition Π_i (i.e., it is constant on each element of this partition) and satisfies for each ω, $\tau_i(\Pi_i(\omega)) = 1$.[4] For simplicity, we assume *positivity*, by which we mean that for each i and ω $\tau_i(\omega)$ is positive on $\Pi_i(\omega)$, or equivalently, that $\tau_i(\omega)(\omega) > 0$.

Player i's expected payoff given that vertex $v \in V_i$ is reached is a function $\mathbf{E}_i^v$ on Ω. When $\Pi_i(\omega) \cap \Omega^v \neq \emptyset$, $\mathbf{E}_i^v(\omega) = E_{t_i(\omega)}(h_i^v(\boldsymbol{s}) \mid \Omega^v)$, where $E_{t_i(\omega)}(\cdot \mid \Omega^v)$ is the conditional expectation given Ω^v with respect to $t_i(\omega)$. By the positivity axiom this conditional expectation is well defined. For other ω's, $\mathbf{E}_i^v(\omega)$ is arbitrarily defined. Similarly, define $\mathbf{E}_i^v(t_i)$ by $\mathbf{E}_i^v(t_i)(\omega) = E_{t_i(\omega)}(h_i^v(\boldsymbol{s}; t_i) \mid \Omega^v)$, when $\Pi_i(\omega) \cap \Omega^v \neq \emptyset$ and define $\mathbf{E}_i^v(t_i)(\omega)$ arbitrarily otherwise.

Definition 5. *The event that player i is* **materially rational by expectation** *is:*

$$R_i^{\text{mexp}} = \bigcap_{v \in V_i} \bigcap_{t_i \in S_i} K_i(\neg\Omega^v) \cup \neg K_i\big([\mathbf{E}_i^v(t_i) > \mathbf{E}_i^v]\big). \tag{9}$$

As in the previous section we can define rationality by expectation in the strategic form of the game. We define $\mathbf{E}_i$ and $\mathbf{E}_i(t_i)$ as the unconditional expectation of $h_i(\boldsymbol{s})$ and $h_i(\boldsymbol{s}; t_i)$ correspondingly.

Definition 6. *The event that player i is* **rational by expectation** *is:*

$$R_i^{\text{exp}} = \bigcap_{t_i \in S_i} \neg K_i\big([\mathbf{E}_i(t_i) > \mathbf{E}_i]\big). \tag{10}$$

This is the standard definition of rationality for games in strategic form when payoffs are computed by expectation, as in Aumann (1987). Again, as in the previous subsection, material rationality turns out to be the timeless rationality in the strategic form of the game.

Proposition 4. *Player i is materially rational by expectation if and only if i is rational by expectation. That is,*

$$R_i^{\text{mexp}} = R_i^{\text{exp}}.$$

Discussion: We can describe the event that player i is materially rational by expectation similarly to the event that the player is materially rational. Namely,

$$R_i^{\text{mexp}} = \bigcap_{v \in V_i} \bigcap_{t_i \in S_i} K_i(\neg\Omega^v) \cup \neg K_i\big(\neg\Omega^v \cup [\mathbf{E}_i^v(t_i) > \mathbf{E}_i^v]\big). \tag{11}$$

To see this, suppose that $\omega \in K_i\big(\neg\Omega^v \cup [\mathbf{E}_i^v(t_i) > \mathbf{E}_i^v]\big)$. Then, either $\omega \in K_i\neg\Omega^v$, or else, $\Pi_i(\omega) \cap \Omega^v \neq \emptyset$, in which case, by the definition of $\mathbf{E}_i^v$ and $\mathbf{E}_i^v(t_i)$, $\Pi_i(\omega) \subset [\mathbf{E}_i^v(t_i) > \mathbf{E}_i^v]$. Thus, $K_i\big(\neg\Omega^v \cup [\mathbf{E}_i^v(t_i) > \mathbf{E}_i^v]\big) \subseteq (K_i\neg\Omega^v) \cup K_i([\mathbf{E}_i^v(t_i) > \mathbf{E}_i^v])$. The inverse inclusion holds by the monotonicity of K_i. By the equality of the two events (11) and (9) are equivalent.

[4]In the model of knowledge and belief that we use here, the assumption of measurability is tantamount to saying that each player knows her beliefs, and the other assumption is equivalent to saying that each player is certain of whatever she knows.

It is easy to see that the weak and strong versions of rationality by expectation coincide.

5. Hermaphroditic rationality

In light of the redundancy of of ex-ante knowledge, we should clarify how material rationality compares to what Aumann (1998) called *ex-ante material rationality.* According to his definition, the event that player i is ex-ante materially rational is:

$$R_i^{\text{eam}} = \bigcap_{v \in V_i} \bigcap_{t_i \in S_i} \neg\Omega^v \cup \neg K_i([t_i \succ_v \boldsymbol{s}_i]). \tag{12}$$

A comparison of this event to R_i^{epm} in (3) shows that the only difference between them is the missing $\neg\Omega^v$ in the scope of K_i in (12). The following examples show that this rationality is hermaphroditic; sometimes it is material and at other times, substantive.

Consider a partition element of i that contains two states ω_1 and ω_2. The player's strategy in these states is s_i. Suppose that $V_i = \{v\}$, and that v is reached at ω_1 but is not reached at ω_2. Assume, moreover, that no strategy of i yields a higher conditional payoff h_i^v at ω_2, but there is a strategy t_i that yields a higher payoff h_i^v at ω_1.

Player i is substantively rational in ω_1 and ω_2, as there is no strategy that dominates s_i at v in both states. However, she is *not* materially rational, because t_i dominates s_i at the only state in which v is reached, namely, at ω_1.

As $K_i([t_i \succ_v \boldsymbol{s}_i])$ holds true in the said element of the partition, i is rational in this element according to (12). Thus, here, rationality according to (12) coincides with substantive rationality. Player i wins the title of rationality by virtue of conditional payoffs at the state ω_2 *where v is not reached.*

Next, consider an element of i's partition where v is not reached and in which player i knows that a strategy t_i dominates her strategy at v in all the states of the element. Then, player i is not substantively rational in this element, but is materially rational, since $K_i(\neg\Omega^v)$ holds true. As $\neg\Omega^v$ holds true in both states, player i is rational according to (12). Here, rationality by (12) coincides with material rationality.

6. Proofs

Proof of Proposition 3. Suppose $\omega_0 \in R_i^{\text{mwd}}$, and for some t_i, $\omega_0 \in K_i\big([t_i \not\succ \boldsymbol{s}_i]\big)$. We need to show that $\omega_0 \in K_i\big([t_i \sim \boldsymbol{s}_i]\big)$. Suppose to the contrary that $\omega_0 \notin K_i\big([t_i \sim \boldsymbol{s}_i]\big)$. Then,

(a) for some $\omega \in \Pi_i(\omega_0)$, $h_i(\boldsymbol{s}(\omega); t_i) > h_i(\boldsymbol{s}(\omega))$,
(b) for all $\omega' \in \Pi_i(\omega_0)$, $h_i(\boldsymbol{s}(\omega'); t_i) \geq h_i(\boldsymbol{s}(\omega'))$.

There must be some $v \in V_i$ such that $\omega \in \Omega^v$, or else, the path at ω is independent of i's strategy, contrary to (a). Let v be the first such vertex. Then, for any strategy s that reaches v, $(s; t_i)$ also reaches v. Therefore, for each such s, $h_i(s) = h_i^v(s)$ and $h_i(s; t_i) = h_i^v(s; t_i)$. We conclude by (b) that for any $\omega' \in \Pi(\omega_0) \cap \Omega^v$, $\omega' \in [t_i \not\succ_v \boldsymbol{s}_i]\big)$. Hence, $\omega_0 \in K_i\big(\neg\Omega^v \cup [t_i \not\succ_v \boldsymbol{s}_i]\big)$. As $\omega_0 \in R_i^{\text{mwd}}$, it follows that $\omega_0 \in K_i\big(\neg\Omega^v \cup [t_i \sim_v \boldsymbol{s}_i]\big)$. But this is a contradiction, since $\omega \in \Pi_i(\omega_0) \cap \Omega^v$ and therefore by (a), $h_i^v(\boldsymbol{s}(\omega); t_i) > h_i^v(\boldsymbol{s}(\omega))$

Conversely, suppose that $\omega_0 \in R_i^{\mathrm{wd}}$, and for some t_i and $v \in V_i$, $\omega_0 \in K_i\big(\neg\Omega^v \cup [t_i \succcurlyeq_v \boldsymbol{s}_i]\big)$. We need to show that $\omega_0 \in K_i\big(\neg\Omega^v \cup [t_i \sim_v \boldsymbol{s}_i]\big)$. Suppose to the contrary that $\omega_0 \notin K_i\big(\neg\Omega^v \cup [t_i \sim_v \boldsymbol{s}_i]\big)$. Then,

(c) for some $\omega \in \Pi_i(\omega_0) \cap \Omega^v$, $h_i^v(\boldsymbol{s}(\omega); t_i) > h_i^v(\boldsymbol{s}(\omega))$,

(d) for all $\omega' \in \Pi_i(\omega_0) \cap \Omega^v$, $h_i^v(\boldsymbol{s}(\omega'); t_i) \geq h_i^v(\boldsymbol{s}(\omega'))$.

Let $\hat{t}_i$ be the strategy that agrees with t_i on v and all the vertices in V_i that follow v, and with $\boldsymbol{s}_i$ on all other vertices. Then, for all $\omega' \in \Pi_i(\omega_0) \cap \Omega^v$, $h_i^v(\boldsymbol{s}(\omega'); \hat{t}_i) = h_i(\boldsymbol{s}(\omega'); t_i)$, and $h_i^v(\boldsymbol{s}(\omega')) = h_i(\boldsymbol{s}(\omega'))$, and for all $\omega' \in \Pi_i(\omega_0) \cap \neg\Omega^v$, $h_i(\boldsymbol{s}(\omega'); \hat{t}_i) = h_i(\boldsymbol{s}(\omega'))$. Thus, by (d), for all $\omega' \in \Pi_i(\omega_0)$, $h_i(\boldsymbol{s}(\omega'); \hat{t}_i) \geq h_i(\boldsymbol{s}(\omega'))$. Hence, $\omega_0 \in K_i\big([\hat{t}_i \succcurlyeq \boldsymbol{s}_i]\big)$. As $\omega_0 \in R_i^{\mathrm{wd}}$, it follows that $\omega_0 \in K_i\big([\hat{t}_i \sim \boldsymbol{s}_i]\big)$. But this is a contradiction, since $\omega \in \Pi_i(\omega_0) \cap \Omega^v$ and therefore by (c), $h_i(\boldsymbol{s}(\omega); \hat{t}_i) > h_i(\boldsymbol{s}(\omega))$. □

Proof of Proposition 4. Suppose that $\omega \notin R_i^{\mathrm{exp}}$. Then, for some $t_i \in S_i$, $\omega \in K_i\big([\mathbf{E}_i(t_i) > \mathbf{E}_i]\big)$. There must be some vertex $v \in V_i$ such that $\Pi_i(\omega) \cap \Omega^v \neq \emptyset$, because otherwise the strategy of i cannot change her payoff, and thus $\mathbf{E}_i(t_i)(\omega') = \mathbf{E}_i(\omega')$ for all $\omega' \in \Pi_i(\omega)$ which means that $\omega \in K_i\big([\mathbf{E}_i(t_i) = \mathbf{E}_i]\big)$, contrary to our assumption. Let $\bar{V}_i$ be the set of all vertices $v \in V_i$ such that $\Pi_i(\omega) \cap \Omega^v \neq \emptyset$ and there is no $v' \in V_i$ that precede v. The events $(\Omega^v)_{v \in \bar{V}_i}$ are disjoint in pairs. For each $\omega' \in \Pi_i(\omega) \setminus \cup_{v \in \bar{V}_i} \Omega^v$, $h_i(\boldsymbol{s}; t_i) = h_i(\boldsymbol{s})$. Thus, $\mathbf{E}_i(t_i)(\omega) - \mathbf{E}_i(\omega) = \sum_{v \in \bar{V}_i} \tau_i(\Omega^v)(\mathbf{E}_i^v(t_i)(\omega) - \mathbf{E}_i^v(\omega))$. Hence, for some $v \in \bar{V}_i$, $\mathbf{E}_i^v(t_i)(\omega) - \mathbf{E}_i^v(\omega) > 0$. Thus, $\omega \in K_i(\mathbf{E}_i^v(t_i) > \mathbf{E}_i^v)$. In addition $\omega \in \neg K_i \neg\Omega^v$, and therefore, $\omega \notin R_i^{\mathrm{mexp}}$.

Conversely, suppose that $\omega \notin R_i^{\mathrm{mexp}}$. Then, for some $v \in V_i$ and $t_i \in S_i$, $\omega \in (\neg K_i \neg\Omega^v) \cap K_i(\mathbf{E}_i^v(t_i) > \mathbf{E}_i^v)$. Thus, $\Pi_i(\omega) \cap \Omega^v \neq \emptyset$. Let $\hat{t}_i$ be the strategy described in the proof of Proposition 3. It follows from the properties of $\hat{t}_i$ that $\mathbf{E}_i(t_i)(\omega) - \mathbf{E}_i(\omega) = \tau_i(\Omega^v)(\mathbf{E}_i^v(\hat{t}_i)(\omega) - \mathbf{E}_i^v(\omega)) > 0$. Therefore $\omega \notin R_i^{\mathrm{exp}}$. □

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