Matching of like rank and the size of the core in the marriage problem *

Dov Samet

Abstract

When men and women are objectively ranked in a marriage problem, say by beauty, then pairing individuals of equal rank is the only stable matching. We generalize this observation by providing bounds on the size of the rank gap between mates in a stable matchings in terms of the size of the ranking sets. Using a metric on the set of matchings, we provide bounds on the diameter of the core—the set of stable matchings—in terms of the size of the ranking sets and in terms of the size of the rank gap. We conclude that when the set of rankings is small, so are the core and the rank gap in stable matchings.

1 Introduction

1.1 Matching of likes

When considering the dazzling world of stardom and glamor we are not at all surprised to see Angelina Jolie and Brad Pitt as a couple. Both are highly ranked in this world, and their match seems natural. We would be bewildered, on the other hand, to see Jolie matched up with another man of this world whose physical appearance ranks much lower than hers. Such a man, so we expect, would be naturally matched with a woman ranked like him.

Those who are not familiar with the world of entertainment, may find it easier to relate to a similar mating of likes in the academic arena. Highly ranked scholars are affiliated, more often than not, with top-tier universities, while those who are academically less attractive are affiliated with lesser universities.

1.2 Matching of like trait and of like rank

The phenomenon of matched people being similar in terms of traits like beauty, intellect, and education is called (positive) assortative matching. The model used by Becker (1973) to study the economic theory of mariage gives rise, under certain conditions, to assortative matching. The main elements in this model are the quantified traits of individuals and the production function that associates

*Helpful discussions with Vicki Knoblauch and Alvin Roth are acknowledged.

with the traits of a pair of individuals of different gender the output produced when they are matched.

Here we examine the phenomenon of matching likes in the elementary marriage problem introduced by Gale and Shapley (1962). Unlike Becker's model, in a marriage problem, individuals are not endowed with objective traits, and their preferences are not expressed in terms of an objective cardinal production function. Rather, *subjective* preferences are given by specifying for each individual an ordinal ranking of of the opposite gender. The solution for a marriage problem is its core, which consists of the stable matchings. A matching is stable if there is no pair of individuals of different gender who are not matched to each other, but prefer to be matched to each other than to their mates. Since individuals in a marriage problem do not have traits, we cannot describe the phenomenon of matching of likes as matching of like traits. Instead we describe it as matching of like rank, which requires some clarification.

1.3 The case of a universal ranking

Although the ranking of individuals in a marriage problem is not necessarily derived from some objective trait, there is a simple case in which a marriage problem can be thus interpreted, namely, the case that rankings are *universal*. That is, all men rank women in the same way and all women rank men in the same way. Here we can say that the ranking reflects an objective measuring of a trait.

The analysis of this case is simple. Matching pairs of equal rank is the only stable matching for such a problem. Moreover, even if only one side, say the men, are universally ranked there still exists only one stable matching. In this case each man is matched to a woman whom *he* ranks at least as high as his objective rank.

We can conclude that universal ranking is associated with the smallest possible core, and the equality of rank of matched individuals.

1.4 Correlated rankings

Obviously, the assumption of a universal ranking is too strong. We can hardly expect a unanimous, universal ranking in anything that involves human beings. Our purpose here is to quantify the notion of "matching of like rank" so that we can say for *any* marriage problem to what extent individuals are matched with their likes in stable matchings.

We can expect matching of likes to the extent that we can identify some objective component in the rankings, that is, if individual rankings are positively correlated which reflects a partial agreement on some hidden trait. This may indeed often be the case. Beauty, for example, is indeed in the eyes of the beholder. Nevertheless, in a given culture there is a great deal of agreement in the judgement of beauty, and rankings of beauty are positively correlated. Similarly, scholars may differ on the ranking of universities, but in all rankings Harvard is among the top, say, ten universities. Here, we use the bound on the range of ranks of each individual in a set of rankings as a measure of the positive correlation of the rankings in this set. The smaller the bound the more objective are the rankings. Thus, we expect that the smaller this bound, the greater is the similarity between matched pairs in stable matchings, and the smaller is the core.

1.5 Like rank and the size of the core

This notion of like rank is obvious in the case of universal ranking on both genders, because rank in this case is uniquely defined. But when ranking is not universal there is no objective rank to serve as a basis for comparison. Instead we adopt a subjective measure of like rank. The alikeness of rank of two matched individuals is measured by the difference between their ranks *according to the rankings of these two individuals.* We call the absolute value of this difference the *rank gap* for this pair of individuals.[1] To the extent that individual rankings are close and therefore reflect an objective component, the rank gap serves as proxy to the comparison of objective ranks.

The size of the core of a marriage problem can be easily measured. From the point of view of the woman, the core is small if the ranks of the men she is matched to in the two optimal stable matchings are close. The core is small for the women if it is small for each one of them. A similar measure of the size of the core can be defined for men.

1.6 The main results

Equipped with precise definitions of the above mentioned measures we give here bounds on the size of the core and on the rank gap in terms of the size of the sets of rankings. In particular it follows that when these sets are small, that is, when rankings are close to being objective, then the set of stable matchings is small and the rank gap is also small. We show, moreover, that the size of the core has a bound in terms of the rank gap of the two optimal stable matchings.

1.7 Related work

The empirical relation between positively correlated rankings and the size of the core in the college admission problem, was observed by Roth and Peranson (1999):

> "One factor that strongly influences the size of the set of stable matchings (which coincides with the core in this simple model) is the correlation of preferences among programs and among applicants. When preferences are highly correlated (i.e., when similar programs tend to agree which are the most desirable applicants, and applicants

[1] If we think of the ranking as ordering the opposite gender according to "love" then the rank gap of a matched pair measures precisely the term "*more*" when one of them complains to the other "I love you *more* then you love me".

tend to agree which are the most desirable programs), the set of stable matchings is small."

The theoretical result here is one possible formalization of this observation.

Caldarelli and Capocci (2001) and Knoblauch and Boudreau (2008) studied correlation of rankings via statistical simulation. They introduce an objective trait of agents measured numerically, and assigning the value of this trait to individuals by random variables, they generate correlated and intercorrelated rankings. The simulations are restricted to the optimal matchings obtained by the deferred acceptance algorithm. Their main interest is in gender satisfaction, which is the sum of the ranks of the women by their mates in the the optimal matchings.

2 Preliminaries

A ***ranking*** of a nonempty finite set X is a bijection $r : X \to \{1, \ldots, |X|\}$. If $r(x) < r(x')$ we say that r ranks x *higher* than x'.

A ***marriage market*** is a tuple (M, W, R_M, R_W) where M and W are disjoint sets of size $n > 0$ of ***men*** and ***women***, called the two ***sides*** of the market, $R_M = (r_m)_{m \in M}$ is a vector of rankings of W by men, and $R_W = (r_w)_{w \in W}$ is a vector of rankings of M by women. We refer also to R_M and R_W as the *sets* of ranking in each vector correspondingly. No confusion will result.

A ***matching*** is a set of pairs $\mu = \{(m, w)\}$ which is the graph of a bijection of M and W. For each man m we denote by $\mu(m)$ the unique woman w such that $(m, w) \in \mu$. For each woman w, $\mu(w)$ is similarly defined.

A pair (m, w) ***blocks*** the matching μ if $r_m(w) < r_m(\mu(m))$ and $r_w(m) < r_w(\mu(w))$. The matching μ ***stable*** if no pair blocks it. The core, C, of the marriage problem is the set of all its stable matchings. There exists a man-optimal stable matching, μ_M, that satisfies for each m and w, $r_m(\mu_M(m)) = \min_{\mu \in C} r_m(\mu(m))$, and $r_w(\mu_M(w)) = \max_{\mu \in C} r_w(\mu(w))$. Similarly, there exist a woman-optimal stable matching μ_W with the corresponding properties.

3 From universal ranking to general ranking

We say that one of the sides of a marriage problem is ***universally ranked*** if it is ranked in the same way by all the individuals of the other side. The following consequences of universal ranking are well known.

(i) If one of the sides, say M, is universally ranked, then there exists a unique stable matching.

(ii) In this matching each man ranks his spouse at least as she ranks him.

(iii) Consequently, when both sides are universally ranked, then individuals who are matched in the unique stable matching, have the same rank.

In this section we generalize these three results by relating each pair in the following list:

- The size of the sets of ranking;
- the size of the gap between the ranks that spouses rank each other in stable matchings;
- the size of the core.

In particular, it follows that when rankings are close the set of stable matchings is small and the gap between the ranks of the individuals who are matched in a stable matching is also small.

3.1 The ranking sets and the rank gap

Given a set R of rankings of a set X, the ***displacement*** of $x \in X$ is $\delta(x) = \max_{r\in R} r(x) - \min_{r\in R} r(x)$. We use the maximal displacement, $\Delta^{\max}(R) = \max_{x\in X} \delta(x)$ and the average displacement, $\Delta^{\text{av}}(R) = 1/n \sum_{x\in X} \delta(x)$ as measures of the size of R.

The ***rank gap*** of a pair $(m, w) \in M \times W$ is $\gamma(m, w) = |r_m(w) - r_w(m)|$. The disparity of the mutual rankings of spouses in a given matching μ is measured by the maximal rank gap in μ, $\Gamma^{\max}(\mu) = \max_{(m,w)\in\mu} \gamma(m, w)$ and the average rank gap in μ, $\Gamma^{\text{av}}(\mu) = 1/n \sum_{(m,w)\in\mu} \gamma(m, w)$.

The next theorem generalizes (ii).

Theorem 1 *For each stable matching μ and $(m, w) \in \mu$,*

$$r_m(w) - r_w(m) \leq 2\Delta^{\max}(R_W). \tag{1}$$

Proof: Let μ be a stable matching and $(m, w) \in \mu$. Man m ranks $r_m(w) - 1$ women higher than w. By the stability of μ each one of these $r_m(w) - 1$ women is matched to a man she ranks higher than m. We now compute an upper bound on the number of men that can be ranked higher than m. Obviously, $r_m(w) - 1$ cannot exceed such an upper bound.

Let $r_{w_0}(m) = \max_{w'\in W} r_{w'}(m)$. If a man is ranked higher than m by some woman, then his rank is $r_{w_0}(m) - 1$ or higher. For each man m' and woman w', $|r_{w'}(m') - r_{w_0}(m')| \leq \delta(m')$. Thus if $r_{w_0}(m') > r_{w_0}(m) - 1 + \delta(m')$, then for all woman w', $r_{w'}(m') > r_{w_0}(m) - 1$. Thus, all man ranked by w_0 less than $r_{w_0}(m) - 1 + \Delta^{\max}(R_W)$ cannot be ranked by any woman $r_{w_0}(m) - 1$ or higher. Thus, at most $r_{w_0}(m) - 1 + \Delta^{\max}(R_W)$ men can be ranked above m.[2] As $r_m(w) - 1$ men are ranked above m we conclude,

$$r_m(w) - 1 \leq r_{w_0}(m) - 1 + \Delta^{\max}(R_W). \tag{2}$$

[2]For $k \geq 1$ a tighter bound holds. At most $r_{w_0}(m) - 2 + \Delta^{\max}(R_W)$ men can be ranked above m because m is among the $r_{w_0}(m) - 1 + k$ highest ranked men. Thus, for $k \geq 1$ the bound of the ranking mismatch is $2k - 1$. For $k = 0$, m is not included among the the $r_{w_0}(m) - 1 + k$ highest ranked men.

Also,

$$(3) \qquad r_{w_0}(m) - r_w(m) \le \delta(m).$$

Adding (2) and (3) we have $r_m(w) - r_w(m) \le \Delta^{\max}(R_W) + \delta(m) \le 2\Delta^{\max}(R_W)$. ∎

Claim (ii) is a special case of this theorem for $\Delta^{\max}(R_W) = 0$. We next generalize claim (iii).

Theorem 2 *For any stable matching μ,*

$$(4) \qquad \Gamma^{\max}(\mu) \le 2\max\{\Delta^{\max}(R_W), \Delta^{\max}(R_M)\},$$

and

$$(5) \qquad \Gamma^{\text{av}}(\mu) \le 2n^{1/2}[\Delta^{\text{av}}(R_W)^{1/2} + \Delta^{\text{av}}(R_M)^{1/2}] + [\Delta^{\text{av}}(R_W) + \Delta^{\text{av}}(R_M)].$$

Proof: By (1) and the symmetric equation $r_w(m) - r_m(w) \le 2\Delta^{\max}(R_M)$ we conclude that for each stable matching μ and pair $(m, w) \in \mu$, $|r_w(m) - r_m(w)| \le 2\max\{\Delta^{\max}(R_W), \Delta^{\max}(R_M)$, from which (4) follows.

To prove (5), consider a man m and the woman w_0 defined in the proof of Theorem 1. Let $x > 0$ and consider men m' for which $\delta(m') \le x\Delta^{\text{av}}(R_W)$. For such a man m' and woman w', $|r_{w'}(m') - r_{w_0}(m')| \le \delta(m') \le x\Delta^{\text{av}}(R_W)$. Thus if $r_{w_0}(m') > r_{w_0}(m) - 1 + x\Delta^{\text{av}}(R_W)$, then for all woman w', $r_{w'}(m') > r_{w_0}(m) - 1$. Thus, all man m' for which $\delta(m') \le x\Delta^{\text{av}}(R_W)$ and who are ranked by w_0 less than $r_{w_0}(m) - 1 + x\Delta^{\text{av}}(R_W)$ cannot be ranked $r_{w_0}(m) - 1$ or higher by any woman. Hence, at most $r_{w_0}(m) - 1 + x\Delta^{\text{av}}(R_W)$ among these men can be ranked above m. In addition there can be men m' with $\delta(m') > x\Delta^{\text{av}}(R_W)$ who can be ranked higher than m. Let M' be the set off all men for which $\delta(m') > x\Delta^{\text{av}}(R_W)$, then $\Delta^{\text{av}}(R_W) \ge (1/n)\sum_{m' \in M'} \delta(m') > (1/n)|M'|x\Delta^{\text{av}}(R_W)$. Therefore, $|M'| < n/x$. We conclude that for any $x > 0$ there are at most $r_{w_0}(m) - 1 + x\Delta^{\text{av}}(R_W) + n/x$ men that are ranked higher than m. The minimum of $x\Delta^{\text{av}}(R_W) + n/x$ is $2[n\Delta^{\text{av}}(R_W)]^{1/2}$. Thus,

$$(6) \qquad r_m(w) - 1 \le r_{w_0}(m) - 1 + 2[n\Delta^{\text{av}}(R_W)]^{1/2}.$$

Adding (6) and (3) we have $r_m(w) - r_w(m) \le 2[n\Delta^{\text{av}}(R_W)]^{1/2} + \delta(m)$. This inequality with the symmetric inequality obtained by exchanging men and women imply that

$$\begin{aligned} |r_m(w) - r_w(m)| &\le \max\{2[n\Delta^{\text{av}}(R_W)]^{1/2} + \delta(m), 2[n\Delta^{\text{av}}(R_M)]^{1/2} + \delta(w)\} \\ &\le 2[n\Delta^{\text{av}}(R_W)]^{1/2} + \delta(m) + 2[n\Delta^{\text{av}}(R_M)]^{1/2} + \delta(w). \end{aligned}$$

Averaging these inequalities over all pairs $(m, w) \in \mu$ results in (5). ∎

By this theorem, when ranking is universal on both sides, the rank gap of any stable matching vanishes, as claimed in (iii). This also implies the uniqueness of the stable matching, since when rankings are universal there is only one matching for which the rank gap vanishes.

3.2 The size of the core

We now provide bounds on the size of the core in terms of the size of the ranking sets and the rank gap. For this we define two metrics on matchings. The ***woman-metric*** on matchings, d_W, is defined for each pair of matching μ_1 and μ_2 by

$$d_W(\mu_1, \mu_2) = 1/n \sum_w |r_w(\mu_1(w)) - r_w(\mu_2(w))|.$$

The ***man-metric***, d_M is similarly defined. The diameter of the core with respect to the metrics d_W and d_M are denoted by $D_W(C)$ and $D_M(C)$ correspondingly. For stable matchings μ_1 and μ_2, $|r_w(\mu_1(w)) - r_w(\mu_2(w))| \leq r_w(\mu_M(w) - r_w(\mu_W(w)$ for each w. Thus, $D_W(C) = 1/n \sum_w r_w(\mu_M(w)) - r_w(\mu_W(w))$, and similar equation holds for $D_M(C)$

The first theorem generalizes (i).

Theorem 3

$$D_W(C) \leq \Delta^{\text{av}}(R_W).$$

Proof: $D_W(C) = 1/n \sum_w r_w(\mu_M(w)) - r_w(\mu_W(w)) = 1/n \sum_m r_{\mu_M(m)}(m) - r_{\mu_W(m)}(m) \leq 1/n \sum_m |r_{\mu_M(m)}(m) - r_{\mu_W(m)}(m)| \leq 1/n \sum_m \delta(m) = \Delta^{\text{av}}(R_W)$. ∎

When one side of the market is universally ranked, then by Theorem 3, $D_W(C) = 0$ or $D_M(C) = 0$, and in either case C is a singleton. Thus, claim (i) is a special case of the theorem.

In the next theorem, the size of the core is bounded in terms of the average gap of the woman and man optimal matchings.

Theorem 4

$$D_M(C) + D_W(C) \leq \Gamma^{\text{av}}(\mu_M) + \Gamma^{\text{av}}(\mu_W).$$

Proof: Define $S_{MM} = \sum_m r_m(\mu_M(m))$ and $S_{MW} = \sum_m r_m(\mu_W(m))$, and define S_{WW} and S_{WM} similarly. Then $D_M(C) = 1/n[S_{MW} - S_{MM}]$ and $D_W(C) = 1/n[S_{WM} - S_{WW}]$. Next, observe that

$$\begin{aligned} |S_{WM} - S_{MM}| &= |\sum_w r_w(\mu_M(w)) - \sum_m r_m(\mu_M(m))| \\ &= |\sum_w r_w(\mu_M(w)) - \sum_w r_{\mu_M(w)}(w)| \\ &\leq \sum_w |r_w(\mu_M(w)) - r_{\mu_M(w)}(w)| \\ &= n\Gamma^{\text{av}}(\mu_M) \end{aligned}$$

and similarly, $|S_{MW} - S_{WW}| \leq n\Gamma^{\text{av}}(\mu_W)$. Thus,

$$
\begin{aligned}
D_M(C) + D_W(C) &= 1/n[S_{MW} - S_{MM} + S_{WM} - S_{WW}] \\
&\leq 1/n[|S_{WM} - S_{MM}| + |S_{MW} - S_{WW}|] \\
&= \Gamma^{\text{av}}(\mu_M) + \Gamma^{\text{av}}(\mu_W).
\end{aligned}
$$

∎

The following is an immediate corollary of this theorem.

Corollary 1 *If the rank gaps in the man-optimal and the woman-optimal matchings vanish, then there exists a unique stable matching.*

References

Becker, G. S. (1973), A Theory of Marriage: Part I., *Journal of Political Economy*, 81(4), 813–46.

Caldarelli, G. and A. Capocci (2001) Beauty and distance in the stable marriage problem, *Physica A: Statistical Mechanics and its Applications* , 300, 325–331.

Gale, D., and L.S. Shapley (1962), College admissions and the stability of marriage, *Am. Math. Monthly*, 69, 9–15.

Knoblauch, V., and J. W. Boudreau (2008), Marriage matching and intercorrelation of preferences, University of Connecticut, *Department of Economics Working Paper Series*, Working Paper 2008-27.

Roth, A. E., and E. Peranson (1999), The redesign of the matching market for American physicians: Some engineering aspects of economic design, *American Economic Review*, American Economic Association, 89(4), 748–780.