ON THE CORE AND DUAL SET OF LINEAR PROGRAMMING GAMES*†

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We study the relation between the core of a given LP-game and the set of payoff vectors generated by optimal dual solutions to the corresponding linear program. It is well known that the set of dual payoffs is contained in the core, and that cores of games in which players are replicated converge to the set of dual payoffs when the number of replications tends to infinity. We give a necessary and sufficient condition for finite convergence. As corollaries we strengthen a sufficient condition due to Owen and obtain new conditions as well. We also study conditions in which the core and the set of dual payoffs coincide even without replication. We give a necessary and sufficient condition for this phenomenon and present two classes of LP-games with this property which properly subsumes all examples of this type discussed in the literature.

1. Introduction. Optimization problems with several independent decision makers can be modeled often as cooperative games with side payments. In games of this type the worth of each coalition equals the optimal value of an optimization problem associated with this coalition. A well-known example of games of this type is the class of market games of Shapley and Shubik [12]. In these games each player has an initial bundle of commodities and the worth of a coalition equals the maximal value of the aggregate utility which can be generated by trading commodities between members of this coalition.

In this paper we study a family of games which we call LP-games and which are generated by linear programming optimization problems. Individual players, in an LP-game, are each endowed with a bundle of inputs from which outputs can be produced under linear constraints. The worth of a coalition is the maximal value of outputs that can be produced by the coalition. The first example of games of this type that was studied is the assignment games of Shapley and Shubik [11]; the formulation used in this paper is due to Owen [7].

An LP-game can be viewed as a market game in which all players have the same utility function which is defined by the linear programming problem. Thus, the family of LP-games is contained in the family of market games. In fact, the two families actually coincide, and are equal to the family of totally balanced games. Dubey and Shapley [4] and, independently, Kalai and Zemel [6], have presented extensions of LP-games which are still totally balanced but in which the objective function and constraints are nonlinear. Samet, Tauman and Zang [9] studied LP-games in which not all the resources are controlled by the players. These games may have an empty core.

Our main interest in this paper is the relation between the core of an LP-game and the set of dual optimal solutions for the optimization problem faced by the entire set of players (the grand coalition). We can view such a dual optimal solution as a vector of (shadow) prices on the various resources. Thus, each such vector can be used to define

a payoff vector where each player is paid an amount which corresponds to the value of his resources, according to the given price vector. We call payoff vectors generated in this form *dual payoffs*. In terms of market games dual payoffs correspond to competitive payoffs. It turns out that the set of dual payoffs is always contained within the core of a given LP-game but the two sets need not be identical. However, if the set of players is refined (or equivalently, duplicated) many times, the cores of the refined games converge in a certain sense to the set of dual payoffs. This result follows from a well-known theorem of Debreu and Scarf [3] which states the equivalence of the core and the set of competitive payoffs for market games. However, for LP-games the proof is especially easy [7]. Billera and Raanan [2] considered games with a continuum of players and proved that for such games the core and the set of dual payoffs coincide in accordance with the equivalence theorem of Aumann and Shapley [1] for market games with a continuum of traders.

Owen [7] has noticed a property which is peculiar to LP-games which market games in general do not possess. He has shown that if there exists a unique optimal dual solution, then the coincidence of the dual set and the core occurs after a *finite* number of replications. Rosenmüller [8] studied the fineness of the refinement required to achieve this coincidence. We study the finite convergence property in more detail in §3. Theorem 2 in this section provides a necessary and sufficient condition for finite convergence in LP-games. Using corollaries of this theorem we can conclude that such convergence occurs if the set of dual payoffs is a singleton (even if the set of dual optimal solutions is not). Finite convergence is also guaranteed if the underlying LP-problem has rational data or if there are only two players in the original game. In §4 we examine the conditions under which the core and dual set coincide even without replication. This phenomenon actually occurs in most of the special instants of LP-games which were considered in the literature ([5], [11], [13]). We describe two families of LP-games with this property which properly subsume all the known examples of [5], [11] and [13].

2. Definitions and preliminaries. Let $N = \{1, 2, \ldots, n\}$ be a set of players. A *coalition* is a nonempty subset of N. We associate with each coalition $S \subseteq N$ a 0-1 vector $t^S \in R^n$ in the obvious way and reserve the letter e to denote the vector $(1, 1, \ldots, 1) \in R^n$ which corresponds to the grand coalition N.

Assume that each player j is endowed with a vector of resources $b^j = (b_1^j, b_2^j, \ldots, b_m^j) \in R^m$. Let $B = (b_{ij})$ be the $(m \times n)$ matrix whose jth-column is b^j. For each coalition $S \subseteq N$, let $b(S) = Bt^S$. Thus $b(S)$ is the vector of the total amount of resources available to the coalition S. Let A be an $m \times p$ matrix and $c \in R^p$. For each vector $t \in R^n$ consider the linear program:

$$P(t): \quad \begin{aligned} F(t) = \text{maximum} \quad & cy \\[1ex] \text{subject to} \quad & Ay \leqslant Bt \end{aligned}$$

where $F(t) = -\infty$ if $P(t)$ is infeasible. It is well known that F is piecewise linear, concave and homogeneous of degree one on R^n. Let $P = \{P(t) \mid t \in R^n\}$. We call a system of linear programs of this type a *Linear Programming System* (LP-system). We assume in the sequel that for each coalition S the linear program $P(t^S)$ is feasible and bounded so that the optimal objective function values for the various coalitions are finite. Under these conditions the function F is finite over the entire nonnegative orthant R_+^n.

Consider the game V_P generated by the system P via the relation $V_P(S) = F(t^S)$. Thus, $V_P(S)$ is the value of outputs which can be generated using the resources of coalition S only. We refer to games which arise in this fashion as *Linear Programming*

Games (LP-games). In cases where the LP-system associated with a given game is clear from the context, we suppress the subscript P and refer to the game simply as V. This convention is also used for all other constructs which are defined with respect to a given LP-system P.

The core of a game V (not necessarily an LP-game) is the set of all payoff vectors $x = (x_1, \ldots, x_n)$ such that $\sum_{i=1}^{n} x_i = v(N)$ and $\sum_{i \in S} x_i \geqslant v(S)$ for every coalition $S \subseteq N$. We denote the core of the game V_P by C_P.

Let $u = (u_1, \ldots, u_m)$ be an optimal dual solution for the linear program $P(e)$. Consider the vector $x = uB$. The n dimensional vector x can be considered as a payoff vector which endows each player the value of his resources vector according to the price vector u. We call a vector x obtained in this way, a *dual payoff vector*, and denote by D_p the set of all dual payoff vectors, i.e.,

$$D_p = \{ x \in R^n \mid x = uB \text{ for some dual optimal vector } u \text{ for } P(e) \}.$$

It can easily be shown that D_p is the superdifferential of F at e, namely it contains all points x for which $xe = F(e)$ and $xt \geqslant F(t)$ for every $t \in R^n$. In particular, for each $S \subseteq N$ and $x \in D_p$, $xt^S \geqslant F(t^S) = V(S)$ which shows that $D_p \subseteq C_p$ for every LP-game and the latter set is therefore not empty.

There are several classes of linear programming systems which yield games with $D_p = C_p$. However, this is not the case in general. To illuminate the relation between these sets it is instructive to observe their behavior when the players are split (or equivalently replicated).

Let P be a given LP-system. The *r-refinement* of P, denoted P^r, is obtained by splitting each original player (column of B) into r identical players each receiving b^j/r as his initial endowment. We call the set of r identical players which replaces one original player a *suit*. Let V_{P^r} be the LP-game generated by the system P^r. It is well known that members of the same suit are equally treated in the core, i.e., they are always paid the same by core allocations. Therefore, the payment that a coalition S in V_{P^r} gets in a core allocation is determined by its profile, i.e., by the number of players from each suit it contains. More precisely, let G^r be the grid of side $1/r$ in the unit cube of R^n, i.e.,

$$G^r = \left\{ x \in R^n \mid x_i = \frac{q_i}{r}, q_i \text{ an integer}, 0 \leqslant q_i \leqslant r, i = 1, \ldots, n \right\}.$$

With each coalition S in V_{P^r} we associate a point $t^S = (q_1/r, \ldots, q_n/r)$ in G^r where q_i is the number of players from the ith suit in S.

Clearly, because of the equal treatment property each allocation in the core of V_{P^r} is uniquely determined by the amount allocated to each suit and therefore this core can be described by a point in R^n (rather than $R^{n \times r}$). Explicitly, define the *r-refinement core*, C^r by:

$$C^r = \{ x \in R^n \mid xe = F(e), xt \geqslant f(t) \text{ for each } t \in G^r \}.$$

Clearly, $D \subseteq C^r$ and for each r and therefore $D \subseteq \bigcap_{r=1}^{\infty} C^r$. Since $\bigcup_{r=1}^{\infty} G^r$ is dense in the unit cube we conclude by the continuity and homogeneity of F that $\bigcap_{r=1}^{\infty} C^r$ is the superdifferential of F at e, i.e., $D = \bigcap_{r=1}^{\infty} C^r$. This observation can be viewed as a special case of the limit theorem by Debreu and Scarf [3] (see also Owen [7]). However, in the linear case, one can show an even stronger result, namely *finite* convergence of C^r to D.

THEOREM 1 (Owen [7]). *If the linear program $P(e)$ has a unique dual optimal solution, then for sufficiently large r, $D = C^r$.*

Below, we examine in more detail the conditions under which such a finite convergence is achieved.

3. Finite convergence of the core. We study in this section conditions which ensure finite convergence of C^r to D. We open the discussion with a necessary and sufficient condition for equality of D and C^r for a given integer $r > 0$.

Let $u \in D$. Denote by $T_r(u)$ the convex cone generated by the points in G^r for which $ut = F(t)$. Obviously, $ut = F(t)$ for every $t \in T_r(u)$, i.e., these cones are regions in which F is linear.

THEOREM 2. *The r-refinement core C^r coincides with the set of dual payoff vectors D if and only if $\bigcup_{u \in D} T_r(u)$ contains a neighborhood of e.*

PROOF. Since F is finite on R_+^n, D is a bounded polyhedral set and thus can be convexly spanned by the finite set $\{u^1, \ldots, u^l\}$, of its extreme points. It is straightforward to verify that for every $u \in D$ there exists a vertex u^i, $1 \leqslant i \leqslant l$, such that $T_r(u) \subseteq T_r(u^i)$, and thus $\bigcup_{u \in D} T_r(u) = \bigcup_{i=1}^l T_r(u^i)$.

Assume that $\bigcup_{i=1}^l T_r(u^i)$ contains a neighborhood of e. Let $x \in C^r$, i.e., $xe = F(e)$ and $xt \geqslant F(t)$ for every $t \in G^r$. We have to show that $xt \geqslant F(t)$ for every $t \in R^n$. Since F is homogeneous and concave, it is enough to show that this inequality holds for the neighborhood of e, which is contained in $\bigcup_{i=1}^l T_r(u^i)$. Let t_0 be a point in this neighborhood. Then by our assumption, $t_0 \in T_r(u^i)$ for some index i, $1 \leqslant i \geqslant l$. Hence $F(t_0) = u^i t_0$. But since $xt \geqslant F(t) = u^i t$ for each $t \in G^r \cap T_r(u^i)$, and since $G^r \cap T_r(u^i)$ (conically) span $T_r(u^i)$ it follows that $xt_0 \geqslant F(t_0)$.

Conversely, assume that $\bigcup_{i=1}^l T_r(u^i)$ does not contain a neighborhood of e. Since each one of these cones is a polyhedral cone, there exists a direction vector $z \in R^n$ such that for each positive α, $e + \alpha z \notin \bigcup_{i=1}^l T_r(u^i)$. Let $T(u^i) = \{t \in R_+^n \mid u^i t = F(t)\}$. Clearly, $T(u^i)$ is a convex cone which contains e for each $i = 1, \ldots, l$. Moreover, since F is piecewise linear and finite in a neighborhood of e, we get that $\bigcup_{i=1}^l T(u^i)$ contains a neighborhood of e. Furthermore, $T_r(u^i) \subseteq T(u^i)$ for $i = 1, \ldots, l$. Let $\hat{\alpha}$ be a small enough positive number so that $\hat{t} = e + \hat{\alpha} z$ belongs to $\bigcup_{i=1}^l T(u^i)$. In particular $\hat{t} \in T(u^j)$ for some $1 \leqslant j \leqslant l$. Obviously, $\hat{t} \notin T_r(u^j)$. Note that the intersection of $T_r(u^j)$ with the convex cone generated by $\hat{t}$ and e contains only the ray generated by e. We can therefore separate these two cones by a hyperplane which passes necessarily through the points 0 and e. Let $ht = 0$ be the equation defining this hyperplane with $h\hat{t} < 0$, and $ht \geqslant 0$ for every $t \in T_r(u^j)$. We can normalize h in such a way that $ht \geqslant F(t) - u^j t$ for each t in the finite set $G_r \cap T_r(u^j)$ (note that the right side is strictly negative for t in this set).

Consider now the payoff vector $u^j + h$. We claim that this vector belongs to C_r but not to D. To check the first assertion note that $(u^j + h)t \geqslant u^j t = F(t)$ for each $t \in T_r(u^j)$. Furthermore, for each $t \in G_r \cap T_r(u^j)$ it holds that

$$(u^j + h)t \geqslant u^j t + F(t) - u^j t = F(t);$$

and finally at the point e, $(u^j + h)e = u^j e = F(e)$. On the other hand, to see that $u^j + h \notin D$ we note that $(u^j + h)\hat{t} < u^j \hat{t} = F(t)$. Q.E.D.

REMARK. The existence of an integer r_0 such that $D = C_{r_0}$ is clearly a sufficient condition for finite convergence but it is also a necessary one. Indeed if $D = \bigcap_{r=1}^m C_r$ for some m then for $r_0 = m!$ $C_{r_0} \subseteq C_r$ for $r = 1, \ldots, m$ and therefore $C_{r_0} \subseteq D$; on the other hand $D \subseteq C_{r_0}$ and therefore $C_{r_0} = D$.

Theorem 2 provides as corollaries special cases of finite convergence of the refined cores to the set of dual payoff vectors.

COROLLARY 1. *If the set of dual payoff vectors D is a singleton then for sufficiently large r, $C^r = D$.*

PROOF. Let $D = \{u\}$. Then $T(u)$ contains a neighborhood of e. For sufficiently large r, the vectors $t^i = e - t^{\{i\}}/r$, $i = 1, \ldots, n$, are in this neighborhood. It follows that for those r's, $T_r(u)$ contains a neighborhood of e and the corollary follows by Theorem 2. Q.E.D.

This corollary strengthens Owen's result (Theorem 1) since it requires only the uniqueness of the dual payoff vector and not the uniqueness of the dual optimal solution. Rosenmüller [8] gives bounds on the refinement needed to achieve coincidence of the refined core with the dual payoff vector when the optimal dual solution is unique. (This is done in a more general case where players are not equally refined.) In terms of Corollary 1, these bounds are given by the size of the neighborhood of e which should be generated by $T_r(u)$.

COROLLARY 2. *If the matrices A, B and c (which define a given LP-system) contain only rational numbers, then for some integer $r > 0$, $D = C^r$.*

PROOF. Observe first that each of the extreme points $u^1, \ldots, u^l$ of D is rational. It follows that each of the convex cones $T(u^i) = \{t \in R_+^n \mid u^i t = F(t)\}$, $i = 1, \ldots, l$, can be generated by a finite set of rational points in the unit cube of R^n. We can therefore find large enough r such that G^r contains all these points. Clearly for such r, $T_r(u^i) = T(u^i)$ and therefore $\bigcup_{i=1}^l T_r(u^i)$ contains a neighborhood of e. Q.E.D.

COROLLARY 3. *If there are only two players ($n = 2$), then for sufficiently large r, $D = C^r$.*

PROOF. In this case $D \subseteq R^2$ and for $u \in D$, $u_1 + u_2 = F(e)$ and therefore there are at most two extreme points in D. By Corollary 1 it is enough to consider the case of two extreme points u^1 and u^2. Then $T(u^1) \cup T(u^2)$ contain a neighborhood of $(1, 1)$. Without loss of generality assume that $(t, 1) \in T(u^1)$ implies $t \leqslant 1$ and $(1, t) \in T(u^2)$ implies $t \leqslant 1$. For sufficiently large r,

$$\left(\frac{r-1}{r}, 1\right) \in T(u^1) \quad \text{and} \quad \left(1, \frac{r-1}{r}\right) \in T(u^2).$$

Therefore $T_r(u^1) \cup T_r(u^2)$ contains a neighborhood of $(1, 1)$. Q.E.D.

Theorem 2 can be specialized to the case $r = 1$. This yields a necessary and sufficient condition for equality of the core and the set of dual payoff vectors even without refinement. An alternative statement of this condition is given also in Theorem 3 below.

Consider the LP-system $\tilde{P}$ given by:

$$\tilde{P}(t): \quad \tilde{F}(t) = \text{maximum} \quad \sum_{S \subseteq N} y_S V(S)$$

$$\text{subject to} \quad \sum_{S \subseteq N} y_S t^S = t,$$

$$y_S \geqslant 0; \quad S \subseteq N.$$

$\tilde{F}(t)$ is called the cover of V and for an LP-game V (which is therefore also a market game), $\tilde{F}(t)$ is the least homogeneous concave function which has the value $V(S)$ at t^S for each S [10].

THEOREM 3. *The core of V_p coincides with the set of dual payoff vectors D_p, if and only if F coincides with $\tilde{F}$ in a neighborhood of e.*

PROOF. By Theorem 1 of [10] it follows that the core C coincides with $D_{\tilde{P}}$, the set of dual payoff vectors defined by $\tilde{P}$. ($D_{\tilde{P}}$ is actually the set of dual optimal solutions for $\tilde{P}(e)$.) But $D_{\tilde{P}}$ and D_P are the superdifferentials of $\tilde{F}$ and F respectively, at e. Since

the two functions are piecewise linear, the equality $D_{\tilde{P}} = D_P$ holds if and only if $\tilde{F}$ and F coincide in a neighborhood of e.   Q.E.D.

4. Coincidence of the core and the set of dual allocations. There are several known classes of LP-games for which $D = C$, even without refinement. These include the optimal assignment games of Shapley and Shubik [11], simple network games, Kalai and Zemel [5], and location games on tree networks, Tamir [13]. Below we study in more detail the conditions which give rise to the equality of D and C. We examine two classes of LP-systems for which this equality holds and which together properly subsume all classes of LP-systems mentioned earlier. For both types of systems, it is possible to derive the equality $D = C$ directly from Theorem 2 or Theorem 3. However, we use here a direct approach which illuminates the specific features of these systems. In order to highlight the structure of the system involved, we modify the convention used to describe LP-games in the preceding sections. In particular, we break the matrices A and B into several blocks and introduce equality as well as greater than or equal to constraints. These modifications still leave us within the class of LP-games as defined previously.

We first consider an LP-system of the form

$$P(t): \quad \begin{array}{ll} \text{maximize} & cy \\[4pt] \text{subject to} & A^1 y \leqslant B^1 t, \\[4pt] & A^2 y \geqslant B^2 t, \\[4pt] & A^3 y = B^3 t, \\[4pt] & y \geqslant 0. \end{array}$$

Let

$$\tilde{A} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}.$$

We say that the system P is *simple zero-one* if $\tilde{A}$ is composed of zeros and ones and $\tilde{B}$ is the identity matrix. Under the last condition each player has complete individual control over one of the resources and therefore with each player we can associate one of the constraints defined by the rows of $\tilde{A}$.

THEOREM 4.   *If P is a simple zero-one system, then $D_P = C_P$.*

PROOF.   Observe that if player i is associated with a "$\leqslant$" constraint then $V(\{i\}) \geqslant 0$ and if he is associated with a "$\geqslant$" constraint then $V(N \setminus \{i\}) \geqslant V(N)$. It follows that every core allocation $x \in R^n$ must satisfy:

$$x_i \geqslant 0 \text{ if the } i\text{th constraint is "} \leqslant \text{."}$$

$$x_i \leqslant 0 \text{ if the } i\text{th constraint is "} \geqslant \text{."}$$

Consider the kth column of A, and let $S^k = \{i \mid a_{ik} = 1\}$. Then, the (primal) solution $y_i^* = 0$, $i \neq k$, $y_k^* = 1$ is feasible for $P(t^{S_k})$. Thus, x must satisfy $\sum_{i \in S^k} x_i \geqslant V(S^k) \geqslant c_k$. But these conditions, together with the sign convention on the x_i's established previously, are precisely the constraints defining the dual set of $P(N)$. Since $\sum_{i \in N} x_i = V(N)$ which is the value of $P(N)$, it follows that x is an optimal dual solution.

                                                                                Q.E.D.

Assignment games [11], simple network games [6] (in the path flow formulation) and location games [13] are all examples of games generated by simple zero-one LP-

systems. In all three cases, the matrix A possesses some special additional structure which ensures that the linear problem involved has solution in integers. However, as is clear from the proof of Theorem 4, the only requirement for the coincidence of C and D is that the matrix be zero-one.

It was shown in [6] that $D = C$ for simple network games also when the arc flow formulation of these games is taken. In the path flow formulation mentioned earlier, there is a variable associated with each *path* of the network while in the arc flow formulation there is a variable associated with each *arc* of the network. The arc flow formulation typically involves much fewer variables and is more tractable computationally. For the equivalence between these two formulations, see [6]. The following class is a generalization of such systems.

$$P(t): \quad \begin{array}{ll} \text{maximize} & cy \\ \\ \text{subject to} & A^1 y \leqslant B^1 t, \\ \\ & A^2 y \leqslant 0, \end{array}$$

where the matrix B^1 is the identity matrix. Under these conditions we can identify the rows of A^1 with the players of V. Note that in this case

$$D = \{ x \in R^n \,|\, \text{for some } w, (x, w) \text{ is an optimal dual solution for } P(e) \}.$$

An interesting necessary and sufficient condition for a given $x \in R^n$ to be contained in D is given in the next lemma.

LEMMA 1.   *Let $x \in R^n$. Then $x \in D$ iff*
(i) $x_i \geqslant 0$, $i = 1, \ldots, n$.
(ii) $\sum_{i=1}^n x_i = V(N)$.
(iii) *For every primal solution y feasible to $P(e)$ we have $(c - xA^1)y \leqslant 0$.*

PROOF.   The necessity of conditions (i)–(iii) is obvious. For the sufficiency note that the third condition is equivalent to the implication $A^2 y \leqslant 0 \Rightarrow (c - xA^1)y \leqslant 0$. Thus, by Farkas' lemma, there exists $w \geqslant 0$ such that $wA^2 = c - xA^1$. But this condition, together with (i) and (ii), implies that (x, w) is a dual optimal solution for $P(e)$.
                                                                                          Q.E.D.

Note that for $x \in C$ and for a primal optimal solution y for $P(e)$

$$(c - xA^1)y = cy - xA^1 y = V(N) - \sum_{i=1}^n x_i (A^1 y)_i \geqslant V(N) - \sum_{i=1}^n x_i = 0.$$

Using Lemma 1 we conclude:

LEMMA 2.   *Let $x \in C$. Then $x \in D$ iff the optimal value of the following program is equal to zero:*

$$(\hat{P}_x): \quad \begin{array}{ll} \text{maximize} & (c - xA^1)y \\ \\ \text{subject to} & A^1 y \leqslant e, \\ \\ & A^2 y \leqslant 0. \end{array}$$

It follows that a necessary and sufficient condition for $D = C$ is that for every $x \in C$ the condition of Lemma 2 holds. A class of LP-systems where this indeed is the case is the following:

THEOREM 5.   *Let the matrices A^1, A^2 be such that for every objective vector c there*

exists an optimal solution for the program

$$\text{maximize} \quad cy$$

$$\text{subject to} \quad A^1 y \leqslant e,$$

$$A^2 y \leqslant 0,$$

with $A^1 y$ a 0-1 vector. Then $D = C$.

PROOF. Let $x \in C$, and consider the problem $\hat{P}_x$. By Lemma 2 it is sufficient to show that the optimal value of this program is not positive. Let y^* be an optimal solution for this program with the integrality property (i.e., $A^1 y^*$ is a zero-one vector). Let $S = \{i \,|\, (A^1 y^*)_i = 1\}$. Assume, on the contrary, that $(c - xA^1)y^* > 0$. Note, that y^* is feasible to $P(S)$ and thus $V(S) \geqslant cy^*$. Hence,

$$0 < (c - xA^1)y^* = cy^* - xA^1 y^* \leqslant V(S) - \sum_{i \in S} x_i$$

which contradicts our assumption that $x \in C$. Q.E.D.

The stipulations of Theorem 5 hold, for instance, if A^2 is a totally unimodular matrix which contains (implicitly or explicitly) the nonnegativity constraints on the variables and where the matrix A^1 is the identity matrix. The path flow formulation of simple network games falls into this category.

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