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AN AXIOMATIZATION OF THE EGALITARIAN SOLUTIONS

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The egalitarian solutions for general cooperative games which were defined and axiomatized by Kalai and Samet, are compared to the Harsanyi solution. It is shown that axioms used by Hart to characterize the Harsanyi solution can be used to characterize the (symmetric) egalitarian solution. The only changes needed are the omission of the scale covariance axiom and the inclusion, in the domain of the solution, of games which lack a certain smoothness requirement.

Key words: Cooperative games; the egalitarian solution; the Harsanyi solution.

1. Introduction

Axiomatic characterization of solutions for cooperative games has a long history, but until recently only solutions for cooperative games with transferable utility and bargaining problems have enjoyed axiomatic treatment. Lately, the axiomatic approach was applied to three solution concepts for cooperative games without transferable utility. Aumann (1983) provided an axiomatic characterization of the nontransferable utility (NTU) value introduced by Shapley (1969). Hart (1983), using axioms which are closely related to those of Aumann, gave an axiomatic definition of the Harsanyi solution (1959, 1963). Kalai and Samet (1983) defined and characterized axiomatically the family of egalitarian solutions for games without transferable utility.

The egalitarian solutions differ essentially from the NTU value and the Harsanyi solution. While the latter two solutions are covariant with utility rescaling applied separately to each one of the players, egalitarian solutions are not. As such, egalitarian solutions are said to admit interpersonal comparisons of utility.

The axioms used by Kalai and Samet are also different from those used by Aumann and Hart.

The monotonicity axiom used for the egalitarian solutions is a stronger axiom (when supported by a Pareto optimality axiom) than the independence of irrelevant alternatives (IIA) axiom which is shared (in different versions) by both the NTU value and the Harsanyi solution. Also, the additivity required by Kalai and Samet ('additivity of endowments') is of a different nature than the conditional additivity used by Aumann and Hart. The axioms of monotonicity and additivity of endowments used to characterize the family of the egalitarian solutions highlight the difference between these solutions and the NTU value and the Harsanyi solution since the latter solutions do not satisfy these axioms.

In this paper we demonstrate the similarities between the egalitarian solutions and the Harsanyi solution by showing that the symmetric egalitarian solution (from which the other egalitarian solutions can be derived by rescaling of utility) can be characterized by axioms used to define the Harsanyi solution. To this end we have only to drop one axiom, the scale covariance axiom, and to enlarge the domain of games on which the solution is defined by allowing games for which the set of feasible outcomes of the grand coalition, $V(N)$, is not smooth. This axiomatization illuminates the differences and similarities of these two solutions and shows how narrow (albeit deep) is the gap between them. A possible interpretation of this result is that intrinsic interpersonal comparisons of utility, which are part of both the Harsanyi solution and the NTU value, are possible only when transfer rates for the grand coalition can be unambiguously determined (this is the case where $V(N)$ is smooth), otherwise these comparisons should be externally given.

2. Preliminaries

Most of the notations and definitions follow Hart (1983).

A finite set N is the set of *players*. Each nonempty subset S of N is a *coalition*. Let $\mathbb{R}^S$ be the $|S|$-dimensional Euclidean space with coordinates indexed by the players in S. Thus, for $x\in\mathbb{R}^S$ we write $x=(x^i)_{i\in S}$. In particular we denote by 0_S the origin of $\mathbb{R}^S$ and by 1_S the vector $(1,1,\dots,1)$ in $\mathbb{R}^S$. Inequalities between elements of $\mathbb{R}^S$ are interpreted coordinate-wise. $\mathbb{R}^S_+$ is the non-negative orthant of $\mathbb{R}^S$ and $\mathbb{R}^S_{++}$ consists of all positive vectors in $\mathbb{R}^S$. For $A, B\subseteq\mathbb{R}^S$, $A+B$ is the closure of the set $\{a+b\,|\,a\in A, b\in B\}$. For λ, $x\in\mathbb{R}^S$ we denote $\lambda x=(\lambda^i x^i)_{i\in S}$. For a closed set A in $\mathbb{R}^S$, ∂A is the boundary of A.

A *nontransferable utility game* (a *game* for short) is a function V which assigns to each coalition S a subset $V(S)$ of $\mathbb{R}^S$. For two games V and W, $V+W$ is the game defined by $(V+W)(S)=V(S)+W(S)$ for each coalition S. For $\lambda\in\mathbb{R}^N_{++}$ and a game V we define λV by $\lambda V(S)=\{\lambda x\,|\,x\in V(S)\}$ for each S. For a real number c the game cV is the game $\bar{c}V$, where $\bar{c}=(c,c,\dots,c)$.

We denote by Γ_0 the space of all games V for which $V(S)$ is closed, convex and comprehensive for each coalition S. (The comprehensiveness of $V(S)$ means that if $x\in V(S)$ and $y\leq x$, the $y\in V(S)$.)

The space Γ_1 consists of the games in Γ_0 for which: (i) $V(N)$ is smooth, i.e. $V(N)$ has a unique supporting hyperplane at each point on its boundary $\partial V(N)$, and (ii) $V(N)$ is not level, i.e. if $x, y\in V(N)$ and $x\geqq y$, then $x=y$.

A transferable utility (*TU*) *game* is a real function v defined on all the coalitions.

For each TU game v there is a corresponding game V in Γ_0 defined by

$$V(S)=\left\{x\in \mathbb{R}^S \,\middle|\, \sum_{i\in S} x^i \le v(S)\right\}$$

for each coalition S. The set of all games corresponding to TU games is denoted by Γ_{TU}. For a coalition T and a real number c the unanimity game $U_{T,c}$ is the game in Γ_{TU} corresponding to the TU game $u_{T,c}$ defined by $u_{T,c}(S)=1$ if $S\supseteq T$ and $U_{T,c}(S)=0$ otherwise.

A *payoff configuration (PC)* $\boldsymbol{x}$ is an element in $\prod_{S\subseteq N}\mathbb{R}^S$ which we denote by $\boldsymbol{x}=(x_S)_{S\subseteq N}$. It assigns to each coalition S the *payoff vector* x_S. In particular let $\mathbf{0}$ denote the *PC* $\boldsymbol{x}$ for which $x_S=0_S$ for each S. For $\lambda\in\mathbb{R}^N$ and a *PC* $\boldsymbol{x}$, $\lambda\boldsymbol{x}$ is the *PC* $\boldsymbol{y}=(y_S)_{S\subseteq N}$ for which $y_S=\lambda x_S$.

A *solution function* on a space of games Γ is a set valued function F that assigns to each game V in Γ a set (possibly empty) $F(V)$ of payoff configurations. Each point in $F(V)$ is a *solution* of V.

3. Egalitarian solutions and the Harsanyi solution

A payoff configuration $\boldsymbol{x}=(x_S)_{S\subseteq N}$ is a *symmetric egalitarian solution* of the game V in Γ_0 if for each coalition T there exists a number ξ_T such that for each $S\subseteq N$:

(E1) $\qquad x_S^i=\sum_{T\subseteq S,\, i\in T}\xi_T,$

(E2) $\qquad x_S\in\partial V(S).$

Note that V has no symmetric egalitarian solution if and only if for some S either $V(S)$ is empty or $V(S)$ contains the vector $(c,c,\ldots,c)$ for every positive real number c. Otherwise V has a unique symmetric egalitarian solution. (See, for example, Kalai and Samet, 1983.) The number ξ_T can be interpreted as the dividend allocated by the coalition T to each one of its members, and thus the symmetric egalitarian solution seems to follow the principle of sharing equally. For $\omega=(\omega^i)_{i\in N}$ in $\mathbb{R}^N_{++}$ the payoff configuration $\boldsymbol{x}=(x_S)_{S\subseteq N}$ is an *ω-egalitarian solution* of the game V in Γ_0 if for each coaltion T there exists a number ξ_T such that for each coalition S:

(E_ω) $\qquad x_S^i=\omega^i\sum_{T\subseteq S,\, i\in T}\xi_T,$

($\mathrm{E}_\omega 2$) $\qquad x_S\in\partial V(S).$

Here again V has no ω-egalitarian solution if and only if for some S either $V(S)$ is empty or $V(S)$ contains the vector $c\omega$ for each real c.

We remark that the ω-egalitarian solution is related to the symmetric one by rescaling of utilities. Namely, $\boldsymbol{x}$ is the ω-egalitarian solution of V if and only if $\omega^{-1}\boldsymbol{x}$ is the symmetric egalitarian solution of $\omega^{-1}V$, where $\omega^{-1}=((\omega^i)^{-1})_{i\in N}$. The asymmetry of the players in the ω-egalitarian solution is thus only apparent and it can be corrected by choosing the 'right' scale of utilities.

For further discussion of the egalitarian solution the reader is referred to Kalai and Samet (1983).

A payoff configuration $x=(x_S)_{S\subseteq N}$ is a *Harsanyi solution* for the game V in Γ_1 if for some λ in $\mathbb{R}^N_{++}$:

(H1) λx is the symmetric egalitarian solution for λV,

(H2) λx_N maximizes total utility in λV, i.e.

$$\sum_{i\in N} \lambda^i x_N^i \geq \sum_{i\in N} \lambda^i y^i \text{ for each } y=(y^i)_{i\in N} \text{ in } V(N).$$

A Harsanyi solution for the game V satisfies simultaneously an egalitarian requirement (H1) and a utilitarian requirement (H2). This simultaneity is achieved by choosing appropriately rescaling factors for the game. These factors are endogenously determined for each game V and are not given in advance as are the weights in the ω-egalitarian solution. Observe that this definition excludes the possibility of λ which is not positive since in such a case λV is not at all a game. There are other definitions of Harsanyi solution in which some λ's may be zero. (See Harsanyi, 1959, 1963.) Since we are inerested in a Harsanyi solution mainly for games in Γ_1, there is no loss of generality since condition H2 may hold for such games only with $\lambda > 0$.

The *symmetric egalitarian solution function E* is the function which assigns to each V in Γ_0 the set which contains the symmetric egalitarian solution of V when it exists and is empty otherwise. *The ω-egalitarian solution function E_ω* is similarly defined. *The Harsanyi solution function H* assigns to each game V in Γ_1 the set of all the Harsanyi solutions of V.

4. The axioms

The following axioms imposed on a possible solution function F defined on a space of games Γ are used to characterize the Harsanyi solution function on Γ_1 by Hart (1983). The games V and W in these axioms are arbitrary games (in Γ).

(A1) *Scale covariance. $F(\lambda V)=\lambda F(V)$ for each $\lambda > 0$ in $\mathbb{R}^N$.*

(A2) *Efficiency. For each $x\in F(V)$, $x_S \in \partial V(S)$ for each S.*

(A3) *Conditional additivity. If $U=V+W$, $x\in F(V)$, $y\in F(W)$ and $x_S+y_S\in \partial U(S)$ for each S, then $x+y\in F(U)$.*

(A4) *Independence of irrelevant alternatives* (IIA). *If* $x \in F(W)$ *and for each* S, $V(S) \subseteq W(S)$ *and* $x_S \in V(S)$, *then* $x \in F(V)$.

(A5) *Unanimity games. For each unanimity game* $U_{T,c}$, $F(U_{T,c}) = \{z\}$, *where* $z_S^i = c/|T|$ *if* $i \in T \subseteq S$ *and* $z_S^i = 0$ *otherwise.*

(A6) *Zero inessential games. If for each* S, $0 \in \partial V(S)$, *then* $\mathbf{0} \in F(V)$.

Theorem 4.1. (Hart, 1983). *The Harsanyi solution function is the only solution function on* Γ_1 *which satisfies Axioms* A1–A6.

5. The main results

The smoothness of $V(N)$ required of games in Γ_1 is essential for the characterization of both the Harsanyi solution and the NTU value; neither one satisfies conditional additivity on Γ_0. (See Aumann, 1983, for an example.)

The failure of Axioms A1–A6 to characterize a solution on Γ_0 is not due to the conditional additivity axiom alone, but is rather a result of inconsistency of several axioms which we name in the following proposition.

Proposition 5.1. *There is no solution function on* Γ_0 *satisfying the following axioms: scale covariance* (A1), *conditional addditivity* (A3), *IIA* (A4), *and unanimity games* (A5).

The smoothness of $V(N)$ is closely related to the utilitarian part of the Harsanyi solution (H2) which requires maximization of total utility. Smoothness enables us to determine local transfer rates of utility between the players unambigously. Since admitting games which are not smooth in $V(N)$ makes the axioms inconsistent, one may interpret it as the inconsistency of utilitarianism with the axioms. If we do not require utilitarianism, we are still left with equity or egalitarianism. But this requirement is not consistent with scale covariance. It is thus reasonable to delete one of the axioms causing the inconsistency: the scale covariance axiom. This intuitive argument is justified by the following theorem.

Theorem 5.2. *The symmetric egalitarian solution function is the unique solution function on* Γ_0 *satisfying Axioms* A2–A6.

It is possible to require that an egalitarian solution maximizes total welfare using the same exogenously given weights that are used for equating gains. We denote the solution function thus defined by G. More precisely, a payoff configuration $x = (x_S)_{S \subseteq N}$ is in $G(V)$ if and only if:

(i) $\quad x \in E(V)$.

(ii) $\sum_{i \in S} x_S^i \geq \sum_{i \in S} y^i$ *for each* S *and each* $y \in V(S)$.

The solution functions E and G are related to each other in the following theorem.

Theorem 5.3. *The minimal and maximal (relative to set inclusion) solution functions on* Γ_0 *satisfying Axioms* A2–A5 *are* G *and* E*, respectively.*

Finally, the nonsymmetric egalitarian solutions may be obtained by changing the unanimity games axiom (A5). For a given $\omega > 0$ in $\mathbb{R}^N$ consider the axiom:

Axiom A5-ω. For each unanimity game $U_{T,c}$, $F(U_{T,c}) = \{z\}$, where $z_S^i = c\omega^i / (\sum_{j \in S} \omega^j)$ if $i \in T \subseteq S$ and $z_S^i = 0$ otherwise.

Theorem 5.4. *The* ω*-egalitarian solution function is the unique solution function on* Γ_0 *satisfying Axioms* A2–A4, A5-ω *and* A6.

Observe that this characterization differs from that of Kalai and Samet in that it does not define the whole family of egalitarian solution, but rather defines each egalitarian solution separately by changing appropriately Axiom A5. The weights that are nowhere mentioned in the axioms of Kalai and Samet are given here as part of Axiom A5-ω, although their meaning as interpersonal utility comparisons weights is a result of the combination of A5-ω with the other axioms.

6. Proofs

Proof of Proposition 5.1. Consider the games V_0, V_1 and V_2 which are defined as follows:

$$V_0(N) = \left\{ x \in \mathbb{R}^N \,\middle|\, \sum_{i \in N} x^i \leq |N| \right\},$$

$$V_1(N) = \{ x \in \mathbb{R}^N \mid x \leq 1_N \},$$

$$V_2(N) = \left\{ x \in \mathbb{R}^N \,\middle|\, \sum_{i \in N} a^i x^i \leq |N| \right\}$$

and

$$V_i(S) = -\mathbb{R}_+^S, \quad \text{for } S \neq N \text{ and } i = 1, 2, 3,$$

where $a = (a^i)_{i \in N}$ satisfies:

$$a > 0, \quad a \neq 1_N \quad \text{and} \quad \sum_{i \in N} a^i = |N|.$$

Suppose that F is a solution function on Γ_0 which satisfies Axioms A1, A3, A4 and

A5. Then by A4 and A5:

$$F(V_1) \supseteq F(V_0) = \{z\},$$

where $z_N = 1_N$ and $z_S = 0_S$ for $S \neq N$. By A1:

$$F(V_2) = F(a^{-1}V_0) = a^{-1}F(V_0) = \{a^{-1}z\},$$

where $a^{-1} = ((a^i)^{-1})_{i \in N}$ and

$$F(V_1 + V_2) = F(2V_2) = 2F(V_2) = \{2a^{-1}z\}.$$

now $w = a^{-1}z + z$ satisfies $w_S \in \partial(V_1 + V_2)(S)$ for each S and thus by A3 $a^{-1}z + z = 2a^{-1}z$, which contradicts $a \neq 1_N$. □

Proposition 6.1. *The symmetric egalitarian solution function E, satisfies Axioms* A2–A6.

Proof. Efficiency is satisfied by E2. For the conditional additivity, let $x \in E(V)$, $y \in E(W)$ and $x_S + y_S = \partial(V + W)(S)$ for each S. Clearly, $x + y$ satisfies E2 for the game $V + W$. There exist numbers ξ_T and η_T for each T such that $x_S^i = \sum_{T \subseteq S, i \in T} \xi_T$ and $y_S^i = \sum_{T \subseteq S, i \in T} \eta_T$, for each S. Therefore, $(x_S + y_S)^i = \sum_{T \subseteq S, i \in T} \xi_T + \eta_T$ and E1 is also satisfied for $x + y$.

To check IIA assume $x \in F(W)$ and for each S, $V(S) \subseteq W(S)$ and $x_S \in V(S)$. Clearly, for each S, $x_S \in \partial V(S)$ and thus x satisfies E2 for V. E1 is satisfied by x since $x \in F(W)$. (Observe that condition E1 is independent on the game.) The symmetric egalitarian solution for each unanimity game $U_{T,c}$ is seen to be the one specified by Axiom A5 by choosing $\xi_S = c/|S|$ and $\xi_T = 0$ for $T \neq S$. Finally, if $\mathbf{0} \in \partial V(S)$ for each S, then $\mathbf{0}$ satisfies E2 for V and E1 is satisfied by choosing $\xi_T = 0$ for each T. □

Proposition 6.2. *For each solution function F which satisfies* A2–A5, *$F(V) \subseteq E(V)$ for each $V \in \Gamma_0$.*

Proof. Let F be a solution function which satisfies A2–A5 and let $x = (x_S)_{S \subseteq N}$ be a solution in $F(V)$. Define the games V_1 and U_0 by:

$$V_1(S) = \{x \in \mathbb{R}^S \mid x \leq x_S\}, \qquad \text{for each } S,$$

$$U_0(S) = \left\{x \in \mathbb{R}^S \,\middle|\, \sum_{i \in S} x^i \leq 0\right\}, \qquad \text{for each } S.$$

By Proposition 6.1 in Hart (1983), for every game $U \in \Gamma_{\mathrm{TU}}$, $F(U) = H(U)$ and since for such a game $H(U) = E(U)$ we conclude that $F(U) = E(U)$. Since $U_0 \in \Gamma_{\mathrm{TU}}$ and $E(U_0) = \{\mathbf{0}\}$ we find that $F(U_0) = \{\mathbf{0}\}$. By IIA, $x \in F(V_1)$. But $x_s + 0_s \in \partial(V_1 + U_0)$ for each S and thus by the conditional additivity (A3) $x \in F(V_1 + U_0)$. Again $V_1 + U_0$ is a game in Γ_{TU} and therefore $x \in E(V_1 + U_0)$. Finally, we observe that $x_S \in \partial V(S)$ for each S and therefore $x \in E(V)$, which shows that $F(V) \subseteq E(V)$. □

Proof of Theorem 5.2. Let $x \in E(V)$. Define the game V_2 by:

$$V_2(S) = \left\{ x \in \mathbb{R}^S \;\middle|\; \sum_{i \in S} x^i \leq \sum_{i \in S} x_S^i \right\}, \quad \text{for each } S.$$

The game V_1 is the one defined in the proof of Proposition 6.2. By the definition of E, and since $V_2 \in \Gamma_{\mathrm{TU}}$, $\{x\} = E(V_1) = E(V_2) = F(V_2)$. Therefore by IIA $x \in F(V_1)$. Consider the game W defined by $W(S) = V(S) - x_S$ for each S. Then $0 \in \partial W(S)$ for each S and by A6 $\mathbf{0} \in F(W)$. But $V_1 + W = V$ and for each S $x_S + 0_S \in \partial V(S)$ which by conditional additivity A3 imply $x \in F(V)$. Therefore $E(V) \subseteq F(V)$ and together with Proposition 6.2 the proof is complete. □

Proof of Theorem 5.3. By Proposition 6.2 E is the maximal solution function which satisfies Axioms A2–A5. We now show that for each solution function F which satisfies A2–A5 $F(V) \supseteq G(V)$ for each $V \in \Gamma_0$. Let $x = (x_S)_{S \subseteq N}$ be a solution in $G(V)$. Consider the game V_1 defined by $V_1(S) = \{x \in \mathbb{R}^S \mid \sum_{i \in S} x^i \leq \sum_{i \in S} x_S^i\}$ for each S. Since $\{x\} = E(V)$ it follows that $\{x\} = E(V_1)$ and because $V_1 \in \Gamma_{\mathrm{TU}}$, $F(V_1) = E(V_1) = \{x\}$. By the definition of G, IIA is applicable to the games V and V_1 and it implies $x \in F(V)$. We finish by proving that G satisfies A2–A5. The Axioms A2, A3 and A5 are satisfied by G since $G(V) \subseteq E(V)$. To see that IIA (Axiom A4) is also satisfied, let $x \in G(W)$ and suppose that for each S $V(S) \subseteq W(S)$ and $x_S \in V(S)$. Then $x \in E(V)$ because E satisfies IIA. Moreover, for each S, $\sum_{i \in S} x_S^i$ maximizes total utility over $W(S)$ and therefore also over $V(S)$, which shows that $x \in G(V)$. □

Proof of Theorem 5.4. First we observe that if a solution function F is satisfying Axioms A2, A3 and A5-ω, then for each $V \in \Gamma_{\mathrm{TU}}$, $F(V) = E_\omega(V)$. The proof is completely analogous to that of Proposition 6.1 in Hart (1983). Using this fact the proof of Theorem 5.4 follows by replacing each E in the proof of Proposition 6.2 and Theorem 5.2 by E_ω. □

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