Conditional belief types

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Abstract

Decision making requires that agents have beliefs about what happens given events that are believed or known not to happen. Such beliefs can be modeled by conditional probability functions which allow conditioning on unconditionally null events. Players with such beliefs must have conditional beliefs about conditional beliefs. We model this using a type space where a player's type at a state is a conditional probability on the space. We axiomatize type spaces using conditional belief operators, and examine additional three axioms of increasing strength: *introspection* that requires that the agent is unconditionally certain of her beliefs; *echo*, according to which the *unconditional* beliefs that are implied by the condition must be held *given the condition*; and *determination*, which says that the conditional beliefs are the unconditional beliefs that are conditionally certain. The echo axiom implies that conditioning events must be unconditionally certain. Thus, conditioning on an event is conditioning on the agent being certain of the event. This formalizes the meaning frequently given to conditioning in probability theory. The echo axiom also implies that the probability given an event is a prior of the unconditional probability. Type spaces are closely related to the sphere models of counterfactual conditionals and to models of hypothetical knowledge.

1 Introduction

1.1 Probability and conditional probability

Beliefs of players are standardly modeled in game theory and economics as probabilistic beliefs. Such beliefs fail to express counterfactual thinking of players which requires the specification of beliefs given events that are certain not to occur. Thus, for example, in a game in extensive form a player may justify her choice of action by specifying her beliefs concerning the continuation of the game following actions that she is certain not to choose. Probabilistic beliefs are inadequate for describing such beliefs, as conditional probabilities are not defined for conditioning events of probability zero. The straightforward modeling of such counterfactual beliefs is by considering conditional probability as a primitive notion, following Rényi (1955).[1] Here, a conditional probability space consists of a finite set with a field of events $\mathscr{F}$ and a subfield of conditions $\mathscr{C}$. A *conditional probability function* on this space is a binary function $P(E\,|\,C)$, where E is an event and C a non-null condition in $\mathscr{C}$, such that for each condition C , $P(\cdot\,|\,C)$ is a probability function concentrated at C. The probability associated with the sure event is the *unconditional* part of the conditional probability. The various probabilities thus defined are related to each other by the Bayes rule whenever the conditioning event has positive unconditional probability. The conditional probability P induces a *hierarchy* of conditions $(S_1,\ldots,S_k)$. Each condition S_i is the most probable condition given $\cup_{j\geq i}S_j$. That is, S_i is the support in $\mathscr{C}$ of $P(\cdot\,|\,\cup_{j\geq i}S_j)$.

1.2 Probabilistic beliefs in interaction

Strategic interaction naturally involves epistemic interaction, namely beliefs about beliefs. Here, we study such interaction when beliefs are described by conditional probability.[2] the way we do it is analogous to the way unconditional probabilistic beliefs are studied. We start by delineating the modeling of such interactive beliefs.

Probabilistic beliefs about probabilistic beliefs can be described in two ways. In the *model theoretic* approach we consider a state space which is a probability space, that is, a

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[1] The idea of taking conditional probability as the primitive notion dates back to Keynes (1921), Popper (1934, 1968) and de Finetti (1936). Rényi (1955) was the first to provide a rigorous measure-theoretic treatment.

[2] The game theoretic literature offers two models of conditional thinking, due to Samet (1996) and Battigalli and Siniscalchi (1999). We discuss the relationship between our work and theirs in Subsections 5.2 and 5.3.

set of states with a field of events. Each state is associated with probability measures over the state space itself, one for each agent, called the agents' *types* at the state. This way, probabilistic beliefs about events can in turn be expressed as events, and hence themselves become objects of belief. The type spaces of Harsanyi (1967-68) and the more general models of Aumann (1976) and Mertens and Zamir (1985) are models of this kind.

In the *axiomatic* approach we formalize sentences of the form "for player i the probability of x is at least p," in such a way that x itself may describe probabilistic beliefs. We then formulate the requirements that probabilistic belief should satisfy using such sentences. In Heifetz and Samet (1998) and Samet (2000), "for agent i the probability of ... is at least p" is formalized by an operator on events, denoted by $B_i^p(\cdot)$. For an event E, $B_i^p(E)$ is the event that for i the probability of E is at least p.[3] In every type space the operators B_i^p can be derived by taking $B_i^p(E)$ to be the set of all states in which i's type assigns a probability of at least p to E. The axioms on these operators guarantee that on any state space where they are defined, there are types on the state space from which the operators are derived. Further axioms can characterize subfamilies of type spaces.[4] Since each axiom involves the operators of only one agent, we can omit the subscript of the agent and study operators B^p.

1.3 Conditional probabilistic beliefs in interaction

We model interactive beliefs expressed by conditional probabilities analogously to the modeling of interactive beliefs expressed by probabilities that was described in the previous subsection. A *conditional probability type space* (a *type space*, for short) consists, as before, of a state space with a field of events $\mathscr{F}$, but now, in addition, we specify a *condition field* $\mathscr{C}$

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[3] Gaifman (1986) also defined belief spaces (which he named *high order probability spaces*) and characterized them in terms of axioms imposed on an operator that maps each event E and closed interval I into another event, described as "the probability of E lies in I."

[4] The phrase "for agent i the probability of ... is at least p" can be formalized as an operator in a formal language, rather than a set-theoretic operator. This gives rise to a modal logic of probabilistic beliefs for which type spaces serve as semantical models. The most notable examples are Fagin, Halpern, and Megiddo (1990), Fagin and Halpern (1994) and Heifetz and Mongin (2001). The axioms in such languages are analogous to the axioms on set-theoretic operators. However, the modal logic approach has to overcome problems that arise because the field of real numbers is Archimedean. These problems are circumvented either by using a richer language (Fagin and Halpern, 1994) that allows the description of expectations, or by introducing a strong inference rule (Heifetz and Mongin, 2001). The set-theoretic axiomatic approach which is free of the finitary nature of the formal language, avoids these problem while preserving the appeal of the axioms. Halpern (1999b), compares the syntactic and set-theoretic axiomatizations for the logics of knowledge, belief, and counterfactuals.

which is a subfield of $\mathscr{F}$ and serves as the field from which conditioning events are drawn. As in the case of probabilistic belief type spaces, we assume only one agent. The generalization to several agents is straightforward. The agent's *type function* t assigns to each state ω *the agent's type at* ω, which is a conditional probability $t^{\omega}(\cdot\,|\,\cdot)$ over the state space. A conditional probability type space is in particular a probability type space when we consider the assignment of the unconditional probabilities $t^{\omega}(\cdot)$ to the states. The partition of the state space into events of the same type is denoted by Π. Since there is a one to one correspondence between the types and the elements of Π, we also refer to these elements as the types of the agent. The *belief field* $\mathscr{E}$ is the field generated by the types. We require that all belief events can serve as conditions, that is, $\mathscr{E} \subseteq \mathscr{C}$.

The axiomatic approach formalizes sentences of the form "the conditional probability of x given y is at least p" by considering binary set-theoretic operators B^p, where for an event E and a conditioning event C, $B^p(E\,|\,C)$ is the event that the probability of E given C is at least p. When the condition is the whole space we omit it and write $B^p(E)$. For a given type space the operators B^p are easily derived: $B^p(E\,|\,C)$ consists of all the states at which the conditional probability of E given C is at least p, that is, all the states ω for which $t^{\omega}(E\,|\,C) \geq p$.

We first characterize type spaces axiomatically. We list axioms on the operators B^p such that a state space with operators that satisfy these axioms is necessarily a type space that gives rise to these operators. The axioms are of two kinds. The first are axioms similar to the ones used in Samet (2000), which guarantee that for each condition C, $B^p(\cdot\,|\,C)$ is an operator derived from a probabilistic type function on the state space. The second are two axioms that guarantee that the types functions for the different conditioning events C form a type function that assigns to each state a conditional probability. The axioms we consider next imply some structure on the type space.

1.4 Introspection

The assumption of introspection is common in the modeling of knowledge and belief. It roughly says that one is fully aware of one's own mental state. In the context of interactive probabilistic beliefs introspection means that one is certain of (that is, assigns probability 1 to) one's probabilistic beliefs. For interactive conditional probability beliefs, introspection says that one is *unconditionally* certain of one's beliefs. This idea in terms of the operators takes the form of the axiom:

$$B^p(E\,|\,C) \subseteq B^1(B^p(E\,|\,C)).$$

That is, if the probability of E given C is at least p then this event is unconditionally certain.

We show, similarly to Samet (2000), that introspection can be equivalently expressed in terms of the type function: The introspection axiom holds in a type space if and only if for each element π in Π the type π is unconditionally certain at π. Introspection also implies that for each belief event E, $E = B^1(E)$. A corollary of this is that introspection also implies negative introspection.

1.5 Conditional certainty and counterfactual conditionals

Type spaces for conditional probability can be considered as an extension of the sphere model for counterfactual conditionals suggested by Lewis (1973). To see this we consider a type space that satisfies introspection and for which the condition field and the belief field coincide, that is, $\mathscr{E} = \mathscr{C}$. If we consider the types, namely the elements of Π, as states, then the hierarchies associated with each state can be viewed as the spheres in Lewis (1973); the closest types in the event C to a given type π are the most probable types in the hierarchy of the type π. Moreover, when we restrict the certainty operator B^1 to subsets of types, and write $C \hookrightarrow E$ for $B^1(E\,|\,C)$, then the truth condition for $\hookrightarrow$ is the one in Lewis (1973). That is, $C \hookrightarrow E$ is true for type π when E holds true at those types in C that are the most probable (or closest to π) according to the type π. We discuss in detail the relation to counterfactual conditionals in Subsection 5.1.

1.6 Echo

The Echo axiom describes how the *unconditional* beliefs *implied by C* are echoed in the *conditional* beliefs *given C*. Formally, the axiom is:

$$\text{If } C \subseteq B^p(E) \text{ then } B^p(E\,|\,C) = \Omega.$$

That is, if C implies the unconditional belief $B^p(E)$ then the conditional belief $B^p(E\,|\,C)$ is sure.[5]

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[5] Various versions of axioms that relate conditional probabilities to unconditional probability have been studied. Some authors call such axioms *Miller's principle* after Miller (1966), who claimed that a certain version of this axiom is paradoxical. See Samet (1999) and the discussion and references therein.

The echo axiom is equivalently expressed in terms of types:

> The echo axiom holds if and only if the *conditional* probability *given* C at a state ω, $t^{\omega}(\cdot\,|\,C)$, is the expectation of the *unconditional* probabilities $t^{\omega'}(\cdot)$ *at* states ω' in that part of C which is the most probable according to t^{ω}. Moreover, the expectation is taken with respect to $t^{\omega}(\cdot\,|\,C)$ itself.

The echo axiom implies the introspection axiom. In particular, the belief type space of the unconditional types $t^{\omega}(\cdot)$ satisfies introspection. By the equivalence above, for each ω and C, $t^{\omega}(\cdot\,|\,C)$ is a convex combination of types in the belief type space. Hence, $t^{\omega}(\cdot\,|\,C)$ is a prior on this space (Samet (1998a)). Moreover, since the expectation is taken with respect to $t^{\omega}(\cdot\,|\,C)$, it is an invariant probability function for the Markov chain defined by the unconditional types $t^{\omega}(\cdot)$ (Samet (1998b)).

1.7 Conditioning on being informed

The echo axiom also implies that the field of condition $\mathscr{C}$ coincides with the belief field $\mathscr{E}$. Together with the introspection axioms this implies that for each condition C, $C = B^1(C)$. This makes it possible to formalize a standard interpretation of conditioning in probability theory as the agent's probability when she is *informed* of the conditioning event. For example, Billingsley (1995, p. 427) says:

> *It is helpful to consider conditional probability in terms of an observer in possession of partial information. As always, observer, information, know, and so on are informal, nonmathematical terms.*

In the standard model of a probability space, being informed of an event is not itself an event, and therefore, this interpretation must remain informal. In our framework we can consider the event $B^1(C)$ as a formalization of the claim that the agent is being informed of C.[6] As for each C, $C = B^1(C)$, conditioning on an event C under the echo assumption means conditioning on the event that the agent is being informed of C.

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[6] The partition Π can be used to define a knowledge operator K on the type space. With this operator, for every condition C, $C = K(C)$. Thus, being informed can be formalized as having knowledge of. See Halpern, Samet, and Segev (2009) for a discussion of the definition of knowledge in terms of belief.

1.8 Determination

Like echo, the axiom of determination relates conditional beliefs to unconditional ones. It says that the *conditional* beliefs are conditionally certain to be the *unconditional* beliefs. Formally,

$$B^p(E \mid C) \subseteq B^1(B^p(E) \mid C).$$

Determination implies that the hierarchy of conditions defined by each type t^ω is most particulary a hierarchy of types. That is, the conditions in the hierarchy are elements of the type partition Π. As a result, determination implies that for each C and ω, $t^\omega(\cdot \mid C)$ is a determined unconditional type. Moreover, $t^\omega(\cdot \mid C) = t^{\omega'}(\cdot)$, where π' is the most probable type in C. In particular, by the equivalence stated for echo, determination implies echo.

When we interpret $B^1(E \mid C)$ as a counterfactual conditional $C \hookrightarrow E$ and we restrict it to belief events, then type spaces that satisfy the axiom of determination are the models suggested by Stalnaker (1968) for counterfactual conditionals. In such models, for each condition C and state ω there exists a single state in C which is closest to ω.

2 Preliminaries

Throughout the paper we fix a triple $(\Omega, \mathscr{F}, \mathscr{C})$, where Ω is a finite set of *states*, $\mathscr{F}$ is a field of subsets of Ω called *events*, and $\mathscr{C}$ is a subfield of $\mathscr{F}$ called the *condition field*. A *condition* is a non-null event in $\mathscr{C}$, that is an event in $\mathscr{C}^+ = \mathscr{C} \setminus \{\emptyset\}$.

2.1 Probability and Conditional Probability

A *probability function* on $(\Omega, \mathscr{F})$ is a function $P\colon \mathscr{F} \to [0,1]$ satisfying *normality*, that is, $P(\Omega) = 1$, and *additivity*, that is, for all $E, F \in \mathscr{F}$, if $E \cap F = \emptyset$ then $P(E \cup F) = P(E) + P(F)$. The set of all probability functions on $(\Omega, \mathscr{F})$ is denoted $\Delta(\Omega, \mathscr{F})$. Given two events E, C with $P(C) > 0$, we define $P(E \mid C) = P(E \cap C)/P(C)$ and call it the *probability of E given C*. It is easy to see that the function $P(\cdot \mid C)$ so defined is a probability function on $(\Omega, \mathscr{F})$. In order to define $P(\cdot \mid C)$ without the requirement that $P(C) > 0$, we take conditional probability as primitive, rather than deriving it from probability.

A *conditional probability function* on $(\Omega, \mathscr{F}, \mathscr{C})$ is a function $P\colon \mathscr{F} \times \mathscr{C}^+ \to [0,1]$, where we write $P(E \mid C)$ for $P(E, C)$, satisfying the following properties for all $E, F \in \mathscr{F}$ and $C, D \in$

$\mathscr{C}^+$:

(N) $P(C \mid C) = 1;$

(A) $P(E \cup F \mid C) = P(E \mid C) + P(F \mid C)$ if $E \cap F = \emptyset;$

(C) $P(E \mid C) = P(E \mid D) P(D \mid C)$ if $E \subseteq D \subseteq C.$

The set of all conditional probability functions on $(\Omega, \mathscr{F}, \mathscr{C})$ is denoted $\Delta(\Omega, \mathscr{F}, \mathscr{C})$. The (conditional) normality and additivity properties, (N) and (A), ensure that for each condition C the function $P(\cdot \mid C)$ is a probability function in $\Delta(\Omega, \mathscr{F})$, one that puts probability 1 on C. The probability function $P(\cdot \mid \Omega)$ is called the *unconditional part* of P, and we often omit the condition, and write $P(\cdot)$. Property (C), the *chain rule*, imposes some relations between the probability functions. In particular, it follows from (N) and (C) that if $P(C) > 0$ for a condition C, then $P(E \mid C) = P(E \cap C)/P(C)$ for each event E.[7]

2.2 The hierarchy induced by a conditional probability

A conditional probability function P induces a hierarchy $(S_1, \ldots, S_k)$ of events in the field of conditions $\mathscr{C}$, that form a partition of Ω. The condition S_i is infinitely more probable than the conditions that follow it, in the sense that it is the support of $P(\cdot \mid (S_i \cup \cdots \cup S_k))$ in $\mathscr{C}$. That is, S_i is the smallest condition that is certain for this probability.

The hierarchy is constructed by induction, starting with $S_0 = \emptyset$, and defining S_i, for $i > 0$, to be the support in $\mathscr{C}$ of $P(\cdot \mid \Omega \setminus (S_1 \cup \cdots \cup S_{i-1}))$. Obviously, the sets S_i are disjoint and nonempty, and therefore for some k, $(S_1, \ldots, S_k)$ is a partition of Ω. We call $(S_1, \ldots, S_k)$ the *hierarchy* associated with P.[8]

The *P-positive part* of a condition C, denoted C^+, is defined as follows. Let $i_C = \min\{1 \leq i \leq k : C \cap S_i \neq \emptyset\}$. Then, $C^+ = C \cap S_{i_C}$. It is easy to see that the only part of a condition that matters for conditioning is the P-positive part of the condition. That is,

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[7] Myerson (1986, pp. 336–337) defines a conditional probability function for the case $\mathscr{C} = \mathscr{F}$. Variants of conditional probabilities are also studied by Hammond (1994) and Halpern (2010).

[8] Rényi (1956) describes an equivalence relation on conditions which in the finite case results in the hierarchy described here. He further defines *dimensionally ordered measures* which in the finite case are the probability functions $P(\cdot \mid (S_i \cup \cdots \cup S_k))$. The hierarchical description of conditional probability, for $\mathscr{C} = \mathscr{F}$, is studied in Blume, Brandenburger, and Dekel (1991, pp. 71–72) under the name *lexicographic probability systems*. A proof of the equivalence of the hierarchical description to the axiomatic one, for $\mathscr{C} = \mathscr{F}$, appears in Monderer, Samet, and Shapley (1992). Here, conditional probabilities are presented for the more general case where $\mathscr{C}$ is any subfield of $\mathscr{F}$.

Claim 1. *For each condition C, $P(\cdot \,|\, C) = P(\cdot \,|\, C^+)$.*

3 Conditional belief types

3.1 Type Functions and Belief Operators

In order to expressstatements about conditional beliefs as events, we consider a state space where each state is associated with conditional beliefs on the state space, much as in the standard model of belief spaces, unconditional beliefs are associated with states. Here, a *type function* is a function $t\colon \Omega \to \Delta(\Omega, \mathscr{F}, \mathscr{C})$ which assigns to each state a conditional probability function on $(\Omega, \mathscr{F}, \mathscr{C})$. For each event E and condition C, the function $t(\cdot)(E \,|\, C)$ is required to be measurable with respect to $\mathscr{C}$. That is, for each $p \in [0,1]$,

$$\left\{\omega \in \Omega \,:\, t(\omega)(E \,|\, C) \geq p\right\} \in \mathscr{C}.$$

This measurability condition, which is stronger than measurability with respect to $\mathscr{F}$, enables conditioning on the events concerning the agent's conditional beliefs themselves. In what follows, for each state ω we write t^{ω} for $t(\omega)$, and call it the *type at* ω. We also write $t^{\omega}(\cdot)$ instead of $t^{\omega}(\cdot \,|\, \Omega)$ and we call it the *unconditional type at* ω. Obviously, $(\Omega, \mathscr{F})$ with the function $\omega \to t^{\omega}(\cdot)$ form an unconditional probability type space.

A *family of conditional belief operators* (a *family of operators*, for short) is a collection of operators $(B^p)_{p \in [0,1]}$ where $B^p\colon \mathscr{F} \times \mathscr{C}^+ \to \mathscr{C}$ for each $p \in [0,1]$. For an event E and condition C, we write $B^p(E \,|\, C)$ rather than $B^p(E, C)$. It is the event that the *belief in E given C is at least p*. If $C = \Omega$, we omit the condition and write just $B^p(E)$. This is the event that the *unconditional* belief in E is at least p. The requirement that the images of the operators B^p is in $\mathscr{C}$, rather than $\mathscr{F}$, is imposed in order to enable conditioning on events concerning beliefs. This is analogous to the measurability condition on the type function t.

A type function t corresponds in a natural way to a family of operators: the event that the belief in E given C is at least p consists of all the states where the type assigns a probability of at least p to E given C. Formally, for all $E \in \mathscr{F}$, $C \in \mathscr{C}^+$, and $p \in [0,1]$, we let

$$B^p(E \,|\, C) = \left\{\omega \in \Omega \,:\, t^{\omega}(E \,|\, C) \geq p\right\}. \tag{1}$$

Now we introduce axioms that characterize the families of operators that correspond to type functions. For all $E, F \in \mathscr{F}$, $C, D \in \mathscr{C}^+$, and $p, q, p_n \in [0,1]$:

(P1) $B^0(E \mid C) = \Omega$;

(P2) $B^p(E \cap F \mid C) \cap B^q(E \cap \neg F \mid C) \subseteq B^{p+q}(E \mid C)$ for $p+q \leq 1$;

(P3) $\neg B^p(E \cap F \mid C) \cap \neg B^q(E \cap \neg F \mid C) \subseteq \neg B^{p+q}(E \mid C)$ for $p+q \leq 1$;

(P4) $B^p(E \mid C) \cap B^q(\neg E \mid C) = \emptyset$ for $p+q > 1$;

(P5) $\cap_n B^{p_n}(E \mid C) \subseteq B^p(E \mid C)$ for $p_n \uparrow p$;[9]

(PN) $B^1(C \mid C) = \Omega$;

(PC) $B^p(E \mid D) \cap B^q(D \mid C) \subseteq B^{pq}(E \mid C)$ for $E \subseteq D \subseteq C$.

Axioms (P1)–(P5) and (PN) correspond to the requirement that for each condition C, the function $t(\omega)(\cdot \mid C)$ is a probability function for each ω. Analogous axioms were introduced by Samet (2000, p. 174) for (unconditional) belief operators, and we refer the reader to that article for a discussion. Axiom (PN) corresponds to the axiom of conditional normality, and axiom (PC) is the counterpart of the chain rule of conditional probability functions. These axioms characterize the families of operators that correspond to type functions.

Theorem 1. *A family of operators corresponds to a type function if and only if it satisfies* (P1)–(P5), (PN), *and* (PC). *In this case, the type function is unique.*

The proof of this theorem is in the Appendix.

In the remainder of the paper we fix a conditional type function t and its corresponding family of conditional belief operators (B^p) defined by (1). By Theorem 1, $(B^p)_{p\in[0,1]}$ must satisfy (P1)–(P5), (PN), and (PC). For each state ω, we denote by $(S_1^\omega, \dots, S_{k^\omega}^\omega)$ the hierarchy associated with t^ω.

Example 1. Consider the type space where $\Omega = \{1,2,3,4\}$, $\mathscr{F} = 2^\Omega$, and $\mathscr{C}$ is the field generated by the partition $\{\{1\},\{2\},\{3,4\}\}$. The table below describes the type function t. Columns correspond to states. The type at ω, t^ω, is described by the column corresponding to ω. Rows correspond to conditioning events in $\mathscr{C}^+$. Thus, the entry at column ω and row C is the probability function $t^\omega(\cdot \mid C)$, described by a vector of the probabilities of the four states.

The last row is the unconditional part of the types at each state. The hierarchies associated with each state are: $(\{1\},\{2\},\{3,4\})$ with state 1; $(\{2\},\{1,3,4\})$ with state 2; and $(\{3.4\},\{1,2\})$ with states 3 and 4.

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[9] Here $p_n \uparrow p$ means that the sequence $p_1, p_2, \dots$ converges to p from below.

To demonstrate the use of the belief operators, note that $B^{1/3}(\{3,4\} \,|\, \{1,3,4\}) = \{2,3,4\}$, and
$B^{1/2}(\{3\} \,|\, \{3,4\}) = \Omega$.

	1	2	3	4
$\{1\}$	$(1,0,0,0)$	$(1,0,0,0)$	$(1,0,0,0)$	$(1,0,0,0)$
$\{2\}$	$(0,1,0,0)$	$(0,1,0,0)$	$(0,1,0,0)$	$(0,1,0,0)$
$\{3,4\}$	$(0,0,1/2,1/2)$	$(0,0,1/2,1/2)$	$(0,0,1/2,1/2)$	$(0,0,1/2,1/2)$
$\{1,2\}$	$(1,0,0,0)$	$(0,1,0,0)$	$(1/5,4/5,0,0)$	$(1/5,4/5,0,0)$
$\{1,3,4\}$	$(1,0,0,0)$	$(2/3,0,1/6,1/6)$	$(0,0,1/2,1/2)$	$(0,0,1/2,1/2)$
$\{2,3,4\}$	$(0,1,0,0)$	$(0,1,0,0)$	$(0,0,1/2,1/2)$	$(0,0,1/2,1/2)$
Ω	$(1,0,0,0)$	$(0,1,0,0)$	$(0,0,1/2,1/2)$	$(0,0,1/2,1/2)$

3.2 The belief field

The *belief field*—the field of events that express beliefs—can be described in terms of the belief operators or the type function. We show the equivalence of these two descriptions.

For the first description, let Π denote the partition of Ω into states with the same type, so that for each state ω, the element of the partition containing ω is

$$\Pi(\omega) = \left\{\omega' \in \Omega : t^{\omega'} = t^{\omega}\right\}.$$

For the second description, let $\mathscr{B}$ denote the range of the belief operators. That is,

$$\mathscr{B} = \left\{B^p(E\,|\,C) : p \in [0,1], E \in \mathscr{F}, C \in \mathscr{C}^+\right\}.$$

Proposition 1. *$\mathscr{B}$ and Π generate the same field of events, denoted $\mathscr{E}$, and called the* belief field.

We conclude that the for each state ω, $\Pi(\omega)$ is in $\mathscr{F}$, i.e., it is an event—the event that the agent's type is t^{ω}. Note, also, that since $\mathscr{B} \subseteq \mathscr{C}$ and $\mathscr{C}$ is a field, $\mathscr{E} \subseteq \mathscr{C}$. Thus, every nonempty event in the belief field is a condition, and in particular, all the elements of the partition are.

The following two lemmas imply Proposition 1 by the finiteness of Ω.

Lemma 1. *For each state ω, let $\mathscr{B}_\omega$ be the family of all $B \in \mathscr{B}$ such that $\omega \in B$. Then, $\Pi(\omega) = \cap_{B\in\mathscr{B}_\omega} B$.*

Proof. The inclusion of $\Pi(\omega)$ in $\cap_{B\in\mathscr{B}_\omega} B$ follows immediately from (1) and the definition of Π. For the opposite inclusion, fix a state $\omega' \in \cap_{B\in\mathscr{B}_\omega} B$. Since the type function t induces the family (B^p), it follows that for all $p \in [0,1]$, $E \in \mathscr{F}$, and $C \in \mathscr{C}^+$, if $t^\omega(E\,|\,C) \geq p$, then $t^{\omega'}(E\,|\,C) \geq p$. By normality and additivity, the probability functions $t^\omega(\cdot\,|\,C)$ and $t^{\omega'}(\cdot\,|\,C)$ must coincide. Thus, $\omega' \in \Pi(\omega)$. ■

Lemma 2. *For each $B \in \mathscr{B}$, $B = \cup_{\omega\in B} \Pi(\omega)$.*

Proof. For all $\omega \in \Omega$, $E \in \mathscr{F}$, $C \in \mathscr{C}^+$ and $p \in [0,1]$, if $\omega \in B^p(E\,|\,C)$ then $\Pi(\omega) \subseteq B^p(E\,|\,C)$, by definition of Π. Thus, each $B \in \mathscr{B}$ is the union of the elements of Π contained in B. ■

We concluded from Proposition 1 that every non-null event in the belief field is a condition, namely $\mathscr{E} \subseteq \mathscr{C}$. In Example 1, there are three different types; one in state 1, one in state 2, and one in states 3 and 4. Thus the partition Π into types is $\{\{1\},\{2\},\{3,4\}\}$. In this example we have the equality $\mathscr{E} = \mathscr{C}$.

4 Introspection, Echo, and Determination

Although the family of operators is able to express conditional beliefs about conditional beliefs, the axioms considered so far, (P1)–(P5), (PC), and (PN), make no special provision regarding iterations of the operators, that is, consideration of events $B^p(E\,|\,C)$ where E or C are themselves events that describe beliefs. This is reflected in the fact that, except for measurability of the type function, no restriction is imposed on how types in different states are related to each other. In this section we introduce three such requirements, expressed in terms of axioms on the family of operators; we study how these axioms are related to each other, and investigate their impact on the relationship between types at different states.

4.1 Introspection

Beliefs, conditional or unconditional, are in the agent's mind. The agent satisfies *introspection* if she is unconditionally certain of her beliefs. We formalize this in terms of the belief operators by the following axiom. For all $E \in \mathscr{F}$, $C \in \mathscr{C}^+$, and $p \in [0,1]$,

$$\text{(Int)} \qquad B^p(E\,|\,C) \subseteq B^1(B^p(E\,|\,C)).$$

Introspection can be equivalently expressed in terms of properties of the type function, as follows.

Proposition 2. *Axiom* (Int) *holds if and only if for each* ω*,* $t^{\omega}(\Pi(\omega)) = 1$*.*

Proof. Since $\mathscr{B}$ is finite, Lemma 1 implies that (b) holds if and only if, for each $\omega \in \Omega$ and $B \in \mathscr{B}$, if $\omega \in B$ then $t^{\omega}(B) = 1$. This is true if and only if $\omega \in B^1(B)$ for each $B \in \mathscr{B}$ and $\omega \in B$, that is, if and only if (Int) holds. ■

Since $\Pi \subseteq \mathscr{E} \subseteq \mathscr{C}$, and as S_1^{ω} is the support of $t^{\omega}(\cdot)$, i.e., the minimal event in $\mathscr{C}$ which is certain for this probability function, we conclude from Proposition 2:

Corollary 1. *Axiom* (Int) *holds if and only if for each* ω*,* $S_1^{\omega} \subseteq \Pi(\omega)$*.*

Introspection can also be expressed in terms of the belief field.

Proposition 3. *Axiom* (Int) *holds if and only if for each* E *in the belief field* $\mathscr{E}$*,* $E = B^1(E)$*.*

Proof. Axiom (Int) follows form the condition in the proposition by substituting $B^p(E\,|\,C)$ for E. Suppose that (Int) holds and let $E \in \mathscr{E}$. Consider a state $\omega \in E$. Since Π generates $\mathscr{E}$, by Proposition 1, E is a union of elements of Π, and thus $\Pi(\omega) \subseteq E$. Therefore, $t^{\omega}(E) \geq t^{\omega}(\Pi(\omega)) = 1$, and thus $\omega \in B^1(E)$. If $\omega \in \neg E$, then by the same argument, $t^{\omega}(\neg E) = 1$. Thus, $t^{\omega}(E) = 0$, and $\omega \in \neg B^1(E)$. ■

When belief and knowledge are studied, axioms of the type (Int) are said to capture *positive* introspection. In contrast, *negative* introspection refers to knowing that one does *not* know and believing that one does not believe. For knowledge in belief, negative introspection does not follow from positive introspection.[10] But when probabilistic beliefs are involved negative introspection is implied by positive introspection. Indeed, since events of the form $\neg B^p(E\,|\,C)$ are in $\mathscr{E}$ negative introspection follows immediately by Proposition 3.

Corollary 2. *If* (Int) *holds, then for all* $E \in \mathscr{F}$*,* $C \in \mathscr{C}^+$*, and* $p \in [0,1]$*,*

$$\neg B^p(E\,|\,C) = B^1(\neg B^p(E\,|\,C)).$$

In Example 1, (Int) holds, by Proposition 2, since in each state ω, the unconditional probability at ω, which is the last element in the column ω, assigns probability 1 to $\Pi(\omega)$. That is, $t^1(\{1\}) = 1$, $t^2(\{2\}) = 1$, and $t^3(\{3,4\}) = t^4(\{3,4\}) = 1$.

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[10] In modal logic, the axiom of positive introspection is known as axiom (4) and negative introspection as axiom (5).

4.2 Echo

The next axiom says that the *unconditional* probabilities at the states in a condition C determine the *conditional* probabilities given C. More specifically, if the agent *unconditionally* believes an event E with probability at least p when the condition C holds, then the agent must believe E with probability at least p *given* C. Thus, the *conditional* beliefs *given* C "echo" the *unconditional* beliefs held *at* C. Formally, for each event E, condition C, and p,

$$\text{(Echo)} \qquad \text{if} \quad C \subseteq B^p(E) \quad \text{then} \quad B^p(E \,|\, C) = \Omega.$$

We explore first the case where C is an element π of the partition Π. That is, when the condition is a single type. In this case (Echo) implies that the conditional probability given the type is the unconditional probability of that type. By the definition of Π we can write for $\pi \in \Pi$, t^π for the type which is the same at all the states in π.

Proposition 4. *If* (Echo) *holds, then for all* $\omega \in \Omega$ *and* $\pi \in \Pi$,

$$t^\omega(\cdot \,|\, \pi) = t^\pi(\cdot) \tag{2}$$

Proof. Suppose (Echo) holds. If $t^\pi(E) \geq p$ then $\pi \subseteq B^p(E)$. Thus, by (Echo), $t^\omega(E \,|\, \pi) \geq p$. As this is true for all p and E, the probability functions $t^\omega(\cdot \,|\, \pi)$ and $t^\pi(\cdot)$ coincide. ■

Axiom (Echo) has two important implications.

Proposition 5. *Axiom* (Echo), *implies,*

(a) axiom (Int);

(b) $\mathscr{C} = \mathscr{E}$, *that is, the condition field and the belief field coincide.*

Proof. For $\pi = \Pi(\omega)$, we obtain by (2) and (N), $t^\omega(\Pi(\omega)) = t^\omega(\Pi(\omega) \,|\, \Pi(\omega)) = 1$. Thus (Int) follows from Proposition 2.

Suppose that (Echo) holds, but $\mathscr{C} \neq \mathscr{E}$. Since $\mathscr{E} \subseteq \mathscr{C}$ and Π generates $\mathscr{E}$, there must exist $\pi \in \Pi$ and C' and C'' in $\mathscr{C}^+$ such that $C' \cup C'' = \pi$. Now, if $t^\pi(E) \geq p$ then $C' \subseteq B^p(E)$ and $C'' \subseteq B^p(E)$ and hence, by (Echo), $t^\pi(E \,|\, C') \geq p$ and $t^\pi(E \,|\, C'') \geq p$. Since this is true for all E and p, it follows that $t^\pi(\cdot \,|\, C') = t^\pi(\cdot \,|\, C'') = t^\pi(\cdot)$. Thus, $t^\pi(C') = t^\pi(C' \,|\, C') = 1$ and $t^\pi(C'') = t^\pi(C'' \,|\, C'') = 1$, which is a contradiction, since $C' \cap C'' = \emptyset$. ■

When (Echo) holds, then (Int) holds by Proposition 5. This implies, by Proposition 3, that for each $E \in \mathscr{E}$, $E = B^1(E)$. Finally, each condition C is in $\mathscr{E}$, again by Proposition 5. Thus we conclude:

Corollary 3. *When* (Echo) *holds, then for each condition* C*,* $C = B^1(C)$*.*

Thus, with (Echo), conditioning on C means conditioning on C being unconditionally certain. This is a formalization of the common informal idea that the probability given a condition is the probability that obtains when the agent is informed of the condition.

The following equivalence theorem extends Proposition 4 for conditioning events that are not a single type, and provides a necessary and sufficient condition for (Echo) in terms of the type function. The condition is that the probability given C at a state ω is a convex combination of the unconditional types at C with weights that are given by the conditional probability of the types. We discuss this condition in more detail below.

Theorem 2. *Axiom* (Echo) *holds if and only if for each state* ω *and condition* C*,*

$$t^{\omega}(\cdot\,|\,C) = \sum_{\pi \subseteq C} t^{\omega}(\pi\,|\,C)\,t^{\pi}(\cdot). \tag{3}$$

Proof. Suppose (Echo) holds. By part (b) of Proposition 5, each $C \in \mathscr{C}^+$ is a union of elements of Π. Thus, by normality and additivity, for each ω, E, and C, $t^{\omega}(E\,|\,C) = \sum_{\pi\subseteq C} t^{\omega}(E \cap \pi\,|\,C)$. Applying the chain rule to each of the summands, then normality, and finally (2) we obtain:

$$t^{\omega}(E\cap\pi\,|\,C) = t^{\omega}(E\cap\pi\,|\,\pi)t^{\omega}(\pi\,|\,C) = t^{\omega}(E\,|\,\pi)t^{\omega}(\pi\,|\,C) = t^{\pi}(E)t^{\omega}(\pi\,|\,C).$$

Since this holds for each E, (3) follows.

Conversely, suppose that (3) holds. Then $\mathscr{C} = \mathscr{E}$. Indeed, if this were not the case, then, as in the proof of part (b) in Proposition 5, there is a condition C which is a nontrivial subset of some $\pi \in \Pi$. For such C the sum righthand side of (3) has no summands and the equation cannot hold. Thus, each condition is the union of elements of Π. Assume that for a condition C, $C \subseteq B^p(E)$. For $\pi \subseteq C$, $\pi \subseteq B^p(E)$ and hence $t^{\pi}(E) \geq p$. Thus, by (3), for any ω, $t^{\omega}(E\,|\,C) = \sum_{\pi\subseteq C} t^{\pi}(E)t^{\omega}(\pi\,|\,C) \geq p\sum_{\pi\subseteq C} t^{\omega}(\pi\,|\,C) = p$. Thus for each ω, $\omega \in B^p(E\,|\,C)$, which shows that axiom (Echo) holds. ■

The summation in (3) can be done for elements π in a subset of C, as we state next.

Corollary 4. *Axiom* (Echo) *holds if and only if for each state* ω *and condition* C*,*

$$t^{\omega}(\cdot\,|\,C) = \sum_{\pi \subseteq C^+} t^{\omega}(\pi\,|\,C)\,t^{\pi}(\cdot), \tag{4}$$

where C^+ *is the* t^{ω}*-positive part of* C*.*

Indeed, plug C^+ for C in (3), and then replace $t^\omega(\cdot\,|\,C^+)$ with $t^\omega(\cdot\,|\,C)$, by Claim 1.

In Example 1, (Echo) holds. Conditioning on a single type, namely, on $\{1\}$, $\{2\}$ or $\{3,4\}$, in any state, results in the unconditional probability for this type, as claimed in Proposition 4. Thus in each of the first three lines the probability function is the same in all the columns. Consider now the conditioning on $C=\{1,3,4\}$ which is a union of the two types $\{1\}$ and $\{3,4\}$. At state 2, the probabilities of these types are $t^2(\{1\})=2/3$, and $t^2(\{3,4\})=1/3$. The conditional probability at 2 is $t^2=(2/3)t^{\{1\}}+(1/3)t^{\{3,4\}}=(2/3,\,0,\,1/3,\,1/3)$, according to Theorem 2. Note that in this example, (Int) holds and $\mathscr{E}=\mathscr{C}$, according to Proposition (5).

4.2.1 Conditional probabilities as priors

It is well known that in (unconditional) belief spaces that satisfy introspection, a probability function on the space is a prior for an agent if and only if it is a convex combination of the agent's types (see Samet (1998a)). Moreover, in such spaces, a probability function is a prior if and only if it is an invariant probability of the Markov chain on the type space where the types at the states are the transition probability functions (see Samet (1998b)). Now, equation (3) shows that the conditional probability given C is a convex combination of the unconditional types. Moreover, this equation can be equivalently written as,

$$t^\omega(\cdot\,|\,C)=\sum_{\pi\subseteq C}t^\omega(\pi\,|\,C)\,t^\pi(\cdot)=\sum_{\pi\subseteq C}\sum_{\omega'\in\pi}t^\omega(\omega'\,|\,C)\,t^{\omega'}(\cdot)=\sum_{\omega'\in C}t^\omega(\omega'\,|\,C)\,t^{\omega'}(\cdot).$$

Thus, the conditional probability $t^\omega(\cdot\,|\,C)$ is an invariant probability of the Markov chain with transition probability functions that are the unconditional probability functions $t^{\omega'}(\cdot)$. We conclude:

Corollary 5. *When (Echo) holds, then for each ω and C, the probability function $t^\omega(\cdot\,|\,C)$ is a prior of the unconditional type space, and in particular it is an invariant probability of the Markov chain with transition probability functions that are the unconditional types $t^{\omega'}(\cdot)$.*

4.3 Determination

Like (Echo), the next axiom, which we call *determination*, relates conditional beliefs to unconditional beliefs: for all E, C, and p,

(Det) $$B^p(E\,|\,C)\subseteq B^1(B^p(E)\,|\,C).$$

Unlike (Echo), the unconditional beliefs here are not those that are held *at* the condition, but rather the unconditional beliefs that are *conditionally* certain. When the agent assigns to E a probability of at least p *given* C, she is certain that this would be her unconditional belief, *given* C.

Axiom (Det) turns out to be stronger than (Echo). Therefore, when (Det) holds, for each ω and C, $t^{\omega}(\cdot \,|\, C)$ is a convex combination of the unconditional types at C. However, with (Det), this convex combination is trivial, and consists of a single type at C. Thus, beliefs given C are the beliefs of a *determined* type in C. As the following theorem states, this determined type is the most probable one in C with respect to the type t^{ω}.

Theorem 3. *Axiom* (Det) *holds if and only if the following two conditions hold:*

- *Axiom* (Echo) *is satisfied;*
- *for each state ω, the hierarchy $(S_1^{\omega}, \dots, S_{k^{\omega}}^{\omega})$ consists of elements of the partition Π.*

Thus, when (Det) *holds, then for each ω and C, $t^{\omega}(\cdot \,|\, C) = t^{\pi}(\cdot)$, where π is the t^{ω}-positive part of C.*

Proof. Suppose that (Det) holds, and let $C \subseteq B^p(E)$. Assume that contrary to (Echo) there exists $\omega \notin B^p(E \,|\, C)$. This implies that for some $q > 1 - p$, $\omega \in B^q(\neg E \,|\, C)$. Thus, by (Det), $\omega \in B^1(B^q(\neg E) \,|\, C)$. Therefore $C \cap B^q(\neg E) \neq \emptyset$, and as $C \subseteq B^p(E)$, $B^p(E) \cap B^q(\neg E) \neq \emptyset$ which is impossible.

Suppose that (Det) holds, and consider an element S_i^{ω} of the hierarchy associated with t^{ω}. Assume that $\pi \subseteq S_i^{\omega}$. By the definition of S_i^{ω}, $t^{\omega}(\pi \,|\, \Omega \setminus (S_1^{\omega} \cup \cdots \cup S_{i-1}^{\omega}))$, and therefore, $t^{\omega}(\pi \,|\, S_i^{\omega}) > 0$. Now, if $t^{\omega}(E \,|\, S_i^{\omega}) \geq p$, then $t^{\omega}(B^p(E) \,|\, S_i^{\omega}) = 1$, by (Det). Thus, $\pi \cap B^p(E) \neq \emptyset$, which implies by the definition of Π, $\pi \subseteq B^p(E)$. Thus, $t^{\pi}(E) \geq p$. Since this is true for each E and p it follows that $t^{\omega}(\cdot \,|\, S_i^{\omega}) = t^{\pi}(\cdot)$. If $\pi \neq \pi$, then $t^{\omega} \neq t^{\pi'}$, and thus, π' is not a subset of S_i^{ω}.

Suppose now that the two conditions in the theorem hold. Then, for each ω and C, the t^{ω}-positive part of C is an element of Π. Therefore, by Corollary 4, for each ω, E, and C, $t^{\omega}(E \,|\, C) = t^{\pi}(E)$, where π is the t^{ω}-positive part of C. Note also that by Proposition 5 and Proposition 2, $t^{\pi}(\pi) = 1$. Now, to prove that (Det) holds, assume that $t^{\omega}(E \,|\, C) \geq p$. Then, $t^{\pi}(E) \geq p$. Hence, $\pi \subseteq B^p(E)$. But, $t^{\omega}(\pi \,|\, C) = t^{\pi}(\pi) = 1$ and therefore $t^{\omega}(B^p(E) \,|\, C) = 1$. ■

Axiom (Det) is not satisfied in Example 1. Indeed, $t^2(\cdot\,|\,\{1,3,4\})$ is a non-trivial combination of $t^{\{1\}}$ and $t^{\{3,4\}}$ and the type is not determined. Axiom (Det) requires that $t^2(\cdot\,|\,\{1,3,4\})$ be either $(0, 0, 1/2, 1/2)$ or $(1, 0, 0, 0)$.

5 Related models

5.1 Counterfactual conditionals

Type spaces for conditional probability can be considered as an extension of models of counterfactual conditionals suggested by Lewis (1973). To see this we consider a type space that satisfies (Int) and for which the condition field and the belief field coincide, that is, $\mathscr{E} = \mathscr{C}$. These two assumptions imply, by Corollary 1, that for each π the hierarchy $(S^{\pi}_1, \ldots, S^{\pi}_{k^{\pi}})$ satisfies $S^{\pi}_1 = \pi$.

Consider now the restriction of the operator $B^1(\cdot\,|\,\cdot)$, which is defined on $\mathscr{F} \times \mathscr{E}$, to $\mathscr{E} \times \mathscr{E}$. With this restriction both the domain and range of B^1 are measurable with respect to the belief field $\mathscr{E}$ which is generated by Π. Thus, we can view B^1 as an operator on the state space with elements that are the members of Π. In this context we refer to these elements as *epistemic states*. The condition for $\pi \in \Pi$ to be in $B^1(E\,|\,C)$ is that $t^{\pi}(E\,|\,C) = 1$. By Claim 1, this is equivalent to $t^{\pi}(E\,|\,C^{+}) = 1$, where C^{+} is the t^{ω}-positive part of C. Thus, π is in $B^1(E\,|\,C)$ when $S^{\pi}_{i_C} \subseteq E$, where i_C is the smallest index i for which $S^{\pi}_i \subseteq C$.

We now describe the structure delineated in the previous paragraph using the terminology of counterfactual conditionals. We change the graphical notation of $B^1(E\,|\,C)$ and write it as $C \hookrightarrow E$ with the intended reading of "if C then E". We think of the hierarchy $(S^{\pi}_1, \ldots, S^{\pi}_{k^{\pi}})$ as a partial order of the epistemic states, expressing closeness to π. Thus, the first element in the hierarchy, which is π, is the closest to π and the types in S^{π}_i are closer to π than those in S^{π}_k with $k > i$. We call a union of the form $\cup^{i}_{j=1} S^{\pi}_j$, a *sphere*. The family of spheres centered at π is denoted $\$^{\pi}$. Using this terminology, the truth condition for the conditional $C \hookrightarrow E$, described in the previous paragraph, is as follows. The conditional holds true for the epistemic state π (that is π is in $C \hookrightarrow E$) when E holds in all the epistemic states in the intersection of C with the smallest sphere in $\$^{\pi}$ that intersects C non-vacuously. The description of the sphere system model, and the truth condition for the counterfactual conditional operator $\hookrightarrow$ are those given in Lewis (1973).

When the type space satisfies (Det) then each hierarchy consists of single types, or, using the term adopted in this subsection, of single epistemic states. The hierarchy at π is a simple ordering of epistemic states with π being the first. In this case π is in $C \hookrightarrow E$ when E contains the closest epistemic state to π, in the ordering associated with π. This model was proposed for counterfactual conditionals by Stalnaker (1968).

Conditional probability can be viewed as an extension of counterfactual conditionals. It provides us with a family of conditional operators that can be denoted by $\hookrightarrow_p$, where $C \hookrightarrow_p E$ is $B^p(E \,|\, C)$. We have shown that the restriction of the operator $\hookrightarrow_1$, denoted above as $\hookrightarrow$, to epistemic states is a counterfactual conditional. The axioms of (Echo) and (Det) extend the principle of truth condition of $\hookrightarrow_1$ to the family of probabilistic conditional operators $\hookrightarrow_p$ as follows. Whether the conditional $C \hookrightarrow_1 E$ is true in some state is answered by asking whether E is true, where C is used to select the states at which we check the truth of E. These are the states in C that are closest to the given state or, in the terminology of conditional probability, the most probable states in C. Analogously, whether the probabilistic conditional $C \hookrightarrow_p E$ is true in some state is answered by asking whether a probabilistic statement about E is true, where C is used to select the states at which we check the truth of the probabilistic statement about E, in the same manner that these states are selected for $\hookrightarrow_1$.

5.2 Hypothetical knowledge

A non-probabilistic version of epistemic conditioning is studied in Samet (1996). Conditional knowledge is described in this paper by a *hypothetical knowledge* operator on a state space that associates with each pair of events $H \neq \emptyset$ and E the event $K^H(E)$. To ease the comparison to our paper we denote $K^H(E)$ by $K(E \,|\, H)$ and the unconditional knowledge $K(\cdot \,|\, \Omega)$ by $K(\cdot)$. In what follows we compare the conditional knowledge operator K to the conditional certainty operator B^1 in this paper.

Seven axioms, (K1*)-(K7*), characterize a structure of the state space. Except for the truth axiom (K7*), $K(E) \subseteq E$, they all either correspond to special cases of our axioms on conditional belief or follow from these axioms. In particular, axiom (K1*), $K(E \,|\, H) = K(K(E \,|\, H))$, which reflects introspection, is an instance of the equality in Proposition 3. Axioms (K*2) and (K*3) are $K(E \,|\, H) = K(K(E) \,|\, H)$ and $\neg K(E \,|\, H) = K(\neg K(E) \,|\, H)$. The first axiom corresponds to the instance of (Det) for $p = 1$. The second, follows from (Det).[11]

The structure defined by these axioms has two elements. The first is a partition Π of

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[11] Indeed, (Det) implies its negative version, $\neg B^p(E \,|\, C) \subseteq B^1(\neg B^p(E) \,|\, C)$. To see the latter, note that $\omega \notin B^p(E \,|\, C)$ implies $\omega \in B^q(\neg E \,|\, C)$ for some $q > 1 - p$. Hence $\omega \in B^1(B^q(\neg E) \,|\, C)$ by (Det), and therefore $\omega \in B^1(\neg B^p(E) \,|\, C)$ because $B^q(\neg E) \subseteq \neg B^p(E)$.

the state space. The second is a *hypothesis transformation function* τ which assigns to each $\pi \in \Pi$ and hypothesis H an element of Π, $\pi' = \tau(\pi, H)$, such that $\pi' \cap H \neq \emptyset$, and $\pi = \pi'$ whenever $\pi \cap H \neq \emptyset$. The conditional $K(E \mid H)$ is true at π, that is, $\pi \subseteq K(E \mid H)$, when $K(E)$ is true at $\tau(\pi, H)$. The partition Π turns out to be a partition into types. Thus, in all the states in an element $\pi \in \Pi$ the conditional knowledge is the same. Thus, we may refer to π in Π as a type, as in the case of beliefs.

Because of axioms (K2*) and (K*3), which correspond to (Det), the structure of the type space in Samet (1996) has similar features to the one studied here under (Det). Consider the restriction of K to the epistemic field, namely the field generated by the partition Π. In this case, $K(E \mid H)$ is true in π if $\tau(\pi, H) \subseteq H$, which follows from the requirement that $\tau(\pi, H) \cap H \neq \emptyset$, and $\tau(\pi, H) \subseteq E$. Thus, the events known *given* H for some given type π are those events that are known *unconditionally* for the type $\tau(\pi, H)$ in H.

Compare this to a type space as defined in this paper that satisfies (Det), and consider the restriction of B^1 to the epistemic field. In this case, certainties given C for π are the unconditional certainties for the most probable type, according the type π, in C. This condition is similar to the one described in the previous paragraph, in that the conditional epistemic attitude is *determined* by the unconditional attitude of a *single* type.[12]

However the truth conditions for K and B^1 differ in that for the first, the single type in the condition is determined by some order on types, while in the second it is selected arbitrarily by τ. This is due to the fact that the axioms (PN) and (PC), which guarantee the hierarchy of types, have no counterpart in Samet (1996). Moreover, these two axioms are the reason why (Det) implies that the field of conditions is the epistemic field. In Samet (1996) this is no longer true. The field of condition there is the whole power set.

5.3 Conditioning with persistent beliefs

Battigalli and Siniscalchi (1999, 2002) studied conditional probability in a product type space à la Harsanyi (1967-68). Each agent's type is associated with a family of probability functions over basic states and the types of the *other* agents. Formally, in the model there is a set S of *external states*, a family $\mathscr{H}$ of nonempty subsets of S called *relevant hypotheses*, and a set of types T_i for each agent i. For each type t_i of agent i, there is a family $(\mu_i(t_i)(\cdot \mid H))_{H \in \mathscr{H}}$ of probability functions over $S \times T_{-i}$ satisfying conditional normality and

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[12] Halpern (1999a) considers relaxations of (K3*) which makes τ a correspondence rather than a function, which implies that unconditional knowledge is not determined by a single type.

the chain rule (when each hypothesis is viewed as a subset of $S \times T_{-i}$ in the obvious way). The authors note, that since in their model an agent's beliefs about her own type are not formalized, their analysis is based on the implicit assumption that the agent is certain of her own type for any given hypothesis. That is, for every agent i, type t_i and hypothesis H, the measure $\mu_i(t_i)(\cdot\,|\,H)$ is implicitly viewed as a measure on $S \times T_i \times T_{-i}$ that puts probability one on $S \times \{t_i\} \times T_{-i}$.

Formalizing the belief of each agent about his own type allows a formal statement of the said assumption and a comparison of their model with ours. For simplicity, assume that there are only two agents, 1 and 2, and that S, T_1 and T_2 are finite sets. Let $\Omega = S \times T_1 \times T_2$ and let $\mathscr{F}$ denote the product algebra of events on Ω. Let $\mathscr{C}$ be the family of events of the form $H \times T_1 \times T_2$, where $H \in \mathscr{H}$. Finally, for each agent i, assume a family of operators $B_i^p(\cdot\,|\,\cdot)$ mapping each event and condition into an event of the form $S \times E_i \times T_{-i}$, where $E_i \subseteq T_i$. By a straightforward modification of our proofs, we can verify that, for each agent i, the family $(\mu_i(t_i)(\cdot\,|\,H))_{H\in\mathscr{H}}$ corresponds to the family of operators $B_i^p(\cdot\,|\,\cdot)$ if and only if the latter satisfy axioms (P1)–(P5), (PN), (PC), and the axiom $B_i^p(E\,|\,C) \subseteq B_i^1(B_i^p(E\,|\,C)\,|\,D)$ for all $p \in [0,1]$, $E \in \mathscr{F}$, and $C, D \in \mathscr{C}$. We refer to this axiom, which divorces beliefs from basic facts, as *belief persistency*: the agent is certain that her beliefs are the same for any given non-epistemic hypothesis. As an example, suppose that $\neg E$ is a relevant hypothesis, then the following is an instance of the belief persistency axiom: $B_i^1(E) \subseteq B_i^1(B_i^1(E)\,|\,\neg E)$. That is, if the player is certain of E, then she is certain that given that E is false, she is certain of E.

The use of conditions in Battigalli and Siniscalchi (1999) and in this paper are diametrically opposed. In our paper not only is conditioning on epistemic event allowed. We *require* that all epistemic events be conditions. Moreover, the axiom of echo implies that *all* conditions are epistemic events. Thus, we capture the idea that conditional statements mean conditioning on the agent being informed of the condition (see Corollary 3). This is a standard form of counterfactual reasoning that underlies rational analysis of decision making. In Battigalli and Siniscalchi (1999), conditions are defined as events that concern external, non-epistemic states. Indeed, the belief persistency axiom requires that one cannot condition on one's beliefs. To see this, suppose that $B^1(\neg E)$ was allowed as a relevant hypothesis. Then $B_i^1(E) \subseteq B_i^1(B_i^1(E1)\,|\,B^1(\neg E))$ would be an instance of the axiom. But the righthand side of this inclusion is the empty set and therefore the axiom cannot hold. Thus the restriction of relevant hypothesis to non-epistemic events is inherent in their paper.

Appendix

Proof of Theorem 1. It is easy to check that if a family of operators corresponds to a type function then it must satisfy (P1)–(P5), (PN), and (PC). Indeed, (P1)–(P5) follow from property (A) of conditional probability functions and the fact that these range in $[0,1]$, whereas (PN) and (PC) follow from (N) and (C), respectively. To show the converse, we need the following preliminary result.

Lemma 3. *Let* $(B^p)_{p\in[0,1]}$ *be a family of operators satisfying* (P1)–(P4) *and* (PN)*. Fix* $C \in \mathscr{C}^+$*. For every* $p \in [0,1]$ *and* $E,F \in \mathscr{F}$ *such that* $E \subseteq F$*,* $B^p(E\,|\,C) \subseteq B^p(F\,|\,C)$*. Moreover, for every* $E \in \mathscr{F}$ *and* $p,r \in [0,1]$ *such that* $r > p$*,* $B^r(E\,|\,C) \subseteq B^p(E\,|\,C)$*.*

Proof. The first claim follows from (P2), setting $q = 0$ and using (P1). From the first claim and (PN) it follows that $B^1(\Omega\,|\,C) = \Omega$. Thus, setting $p = 1$ and $E = \Omega$ in (P4), we have $B^q(\emptyset\,|\,C) = \emptyset$ for all $q \in (0,1]$. Letting $F = \Omega$ and $q = r - p$ in (P3), the second claim follows. ■

Fix a family of operators $(B^p)_{p\in[0,1]}$ satisfying (P1)–(P4), (PN), and (PC). We now define a function t on Ω that assigns to each $\omega \in \Omega$ a function $t^\omega(\cdot\,|\,\cdot) : \mathscr{F} \times \mathscr{C}^+ \to [0,1]$. Then we show that t is a type function. Finally, we prove that t is the unique type function to which $(B^p)_{p\in[0,1]}$ corresponds. For every $\omega \in \Omega$, $E \in \mathscr{F}$, and $C \in \mathscr{C}^+$ define $I(\omega,E,C) = \{p \in [0,1] : \omega \in B^p(E\,|\,C)\}$. By (P1), this set is nonempty, as 0 belongs to it. By (P5), it has a maximum. Thus, for each $\omega \in \Omega$ define $t^\omega(\cdot\,|\,\cdot)$ by letting $t^\omega(E\,|\,C) = \max I(\omega,E,C)$ for every $E \in \mathscr{F}$ and $C \in \mathscr{C}^+$. To prove that t is a type function, we now fix $\omega \in \Omega$ and prove that t^ω satisfies (N), (A), and (C).

For every $C \in \mathscr{C}^+$, since $1 \in I(\omega,C,C)$ by (PN), we have $t^\omega(C\,|\,C) = 1$. Thus, t^ω satisfies (N). To prove that it satisfies (A), fix any $C \in \mathscr{C}^+$ and $A,B \in \mathscr{F}$ such that $A \cap B = \emptyset$. Let $p = t^\omega(A\,|\,C)$ and $q = t^\omega(B\,|\,C)$. Then, by the first claim in Lemma 3, $\omega \in B^p(A\,|\,C) \cap B^q(B\,|\,C) \subseteq B^p(A\,|\,C) \cap B^q(\neg A\,|\,C)$. Since the righthand side is nonempty, we conclude by (P4) that $p + q \leq 1$. By letting $E = A \cup B$ and $F = A$ in (P2), we obtain $B^p(A\,|\,C) \cap B^q(B\,|\,C) \subseteq B^{p+q}(A \cup B\,|\,C)$. As the lefthand side contains ω, it follows that $t^\omega(A \cup B\,|\,C) \geq p + q$. Thus, $t^\omega(\cdot\,|\,C)$ is superadditive. Moreover, if $p + q = 1$ we must have $t^\omega(A \cup B\,|\,C) = p + q$. Therefore, it suffices to prove subadditivity for the case $p + q < 1$. Fix any $p' > p$ and $q' > q$ with $p' + q' \leq 1$. Then $\omega \in \neg B^{p'}(A\,|\,C) \cap \neg B^{q'}(B\,|\,C)$ and hence, by (P3), $t^\omega(A \cup B\,|\,C) < p' + q'$. As this is true for all such p', q', we conclude that $t^\omega(A \cup B\,|\,C) \leq p + q$. This concludes the proof that t^ω satisfies (A). To prove that t^ω satisfies (C), fix any $E \in \mathscr{F}$ and $C,D \in \mathscr{C}^+$ with $E \subseteq D \subseteq C$.

Let $p = t^{\omega}(E\,|\,D)$ and $q = t^{\omega}(D|C)$. Then $\omega \in B^p(E\,|\,D) \cap B^q(D|C)$ and hence, by (PC), $\omega \in B^{pq}(E\,|\,C)$. Thus, $t^{\omega}(E\,|\,C) \geq pq = t^{\omega}(E\,|\,D)t^{\omega}(D|C)$. This also holds if we replace E by $\neg E \cap D$, so $t^{\omega}(\neg E \cap D|C) \geq t^{\omega}(\neg E \cap D|D)t^{\omega}(D|C)$. Neither of these inequalities can be strict, because by adding them we would obtain, by the normality of $t^{\omega}(\cdot\,|D)$ and the additivity of $t^{\omega}(\cdot\,|C)$ and $t^{\omega}(\cdot\,|D)$, the contradiction $t^{\omega}(D|C) > t^{\omega}(D|D)t^{\omega}(D|C) = t^{\omega}(D|C)$. Thus, $t^{\omega}(E\,|\,C) = t^{\omega}(E\,|\,D)t^{\omega}(D|C)$. This shows that t^{ω} satisfies (C). The proof that t is a type function is complete.

To see that $(B^p)_{p\in[0,1]}$ corresponds to t, just note that for all $p \in [0,1]$, $E \in \mathscr{F}$, and $C \in \mathscr{C}^+$, we have $B^p(E\,|\,C) \subseteq \{\omega \in \Omega : t^{\omega}(E\,|\,C) \geq p\}$ by the definition of t, while the opposite inclusion holds by the second claim in Lemma 3. To establish uniqueness, let $\tilde{t}$ be a type function, and suppose that $(B^p)_{p\in[0,1]}$ corresponds to $\tilde{t}$. Fix $E \in \mathscr{F}$ and $C \in \mathscr{C}^+$, and let $p = \tilde{t}^{\omega}(E\,|\,C)$. Then $\omega \in B^p(E\,|\,C)$, and hence $p \in I(\omega, E, C)$. But for $q > p$, we have $\tilde{t}^{\omega}(E\,|\,C) < q$ and hence $q > \max I(\omega, E, C)$. Thus, $\tilde{t}^{\omega}(E\,|\,C) = \max I(\omega, E, C) = t^{\omega}(E\,|\,C)$. ■

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