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# Coherent beliefs are not always types

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**Abstract**

For two interacting agents, we construct a space of nature states $S$ and a coherent hierarchy of beliefs ($\sigma$-additive probability measures) of one agent about $S$, about $S$ and the beliefs of the other agent about $S$, and so on—a hierarchy that has no $\sigma$-additive coherent extension over $S$ and the hierarchies of the other agent. Thus, this hierarchy of beliefs cannot be the description of the beliefs of some type in some Harsanyi [Harsanyi, J.C., 1967–1968. Games with incomplete information played by Bayesian players, parts I, II, and III. Man. Sc. 14, 159–182, 320–334, 486–502] type space. Therefore, the space $C$ of coherent hierarchies over $S$ *properly contains* the universal space $T^*$ of 'all possible types' over $S$. We show how to extract $T^*$ out of $C$ in a transfinite process. 

*Keywords:* Coherent beliefs; Types; Probabilistic uncertainty

## 1. Introduction

Two approaches coexist in the economic literature for modeling probabilistic uncertainty of several interacting agents. According to the *explicit* approach, describing a state of affairs requires, first, to specify the state of nature—detailing the objective parameters which are relevant to the economic or strategic interaction; and second, to specify for each agent a hierarchy of beliefs—probability measures—about nature, about nature and the other agents' beliefs about nature, etc. On its face, the resulting model is complicated and cumbersome to handle.

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According to the *implicit* approach, introduced by Harsanyi (1967), a state of affairs, or *state of the world* comprises of the state of nature and the *type* of each agent. A type is a probability measure on nature and the other agents' types. Such a space of states of the world is called a type space. The Harsanyi approach is implicit, in the sense that it does not describe directly the mutual beliefs of the agents. Nevertheless, it is easy to unfold from the implicit structure the explicit hierarchy of beliefs of each type about nature, about nature and the others' beliefs about nature, and so on. This hierarchy is *coherent*, in the sense that the marginals of the higher-order beliefs coincide with the corresponding lower-level beliefs. [1]

Mertens and Zamir (1985) showed how to reconcile the two approaches. Assuming the space of nature states is compact, they showed that every coherent hierarchy of beliefs of an agent admits a unique coherent limit extension to a probability measure over nature and the other agents' coherent hierarchies. Calling this extension the type of the agent, the space of nature states together with the agents' coherent hierarchies becomes a Harsanyi type space. This means that the Harsanyi implicit approach is comprehensive—every explicit description of beliefs can be identified with a type in some type space.

Mertens and Zamir (1985) also showed that their space is *universal*, in the sense that it 'contains all possible types': Every type space, with the same space of nature states and the same set of agents, admits a unique belief morphism into the Mertens–Zamir space, i.e., a morphism that preserves nature and the beliefs of the agents. This implies that there is no loss of generality in analyzing an economic interaction in any type space, possibly finite, that the modeler finds convenient, because the analysis could always be transferred intact to the universal space, in which every imaginable state of mind of each agent is represented in some state.

In works that ensued (Brandenburger and Dekel, 1993; Heifetz, 1993; Mertens et al., 1994), it was shown that the Mertens–Zamir enterprise may be carried out under various other topological assumptions about the space of nature states. In this work we show, however, that the Mertens–Zamir program breaks down if this space is allowed to be a general measurable space. With two agents, we exhibit in Section 4 a measurable space of states of nature with a coherent hierarchy of beliefs of one agent that has no coherent limit extension. This coherent hierarchy cannot be made a type in any type space. Thus, the Harsanyi implicit approach does not exhaust all the states of affairs that can be described explicitly in the general measure-theoretic case.

In a companion paper (Heifetz and Samet, 1998), we proved the existence of a universal Harsanyi type space even in this general measure-theoretic framework. In light of the above described discrepancy between the explicit and implicit approaches, this universal space is not the space of all coherent hierarchies—in-

[1] Coherence is usually called 'consistency' in the literature. In (Harsanyi, 1967), however, consistency refers to a different property, and therefore we use the former term to avoid confusion.

deed, the construction must be carried out in a completely different fashion. Nevertheless, as in every type space, one can always unfold the explicit beliefs of all orders of each type in the universal space, and these beliefs do constitute a coherent hierarchy. This means that the universal space is always a *subset* of the space of coherent hierarchies, and generally it may be a *proper subset*. In Section 5 we show how the universal space can be 'carved out' of the space of belief hierarchies.

## 2. Preliminaries

Let $X$ be a measurable space with a $\sigma$-field $\Sigma$. We refer to the measurable sets in $\Sigma$ as *events* in $X$. The set of all $\sigma$-additive probability measures on $X$ is denoted by $\Delta(X)$. We consider $\Delta(X)$ as a measurable space with the $\sigma$-field $\Sigma_\Delta$ that is generated by all sets of the form $\beta^p(E) = \{\mu | \mu(E) \geq p\}$, for an event $E$ in $X$ and $0 \leq p \leq 1$. We use the following property of this $\sigma$-field in the sequel.

**Lemma 2.1.** *Let $\mathscr{F}$ be a field on X that generates the $\sigma$-field $\Sigma$,* and $\mathscr{F}_\Delta$ *the $\sigma$-field on $\Delta(X)$ generated by sets of the form*

$$\{\beta^p(E) | E \in \mathscr{F}, \quad 0 \leq p \leq 1\}.$$

*Then* $\mathscr{F}_\Delta = \Sigma_\Delta$.

**Proof:** Denote by $\mathscr{F}'$ be the set of all events $F$ in $X$, such that $\beta^p(F) \in \mathscr{F}_\Delta$ for all $0 \leq p \leq 1$. We prove that $\mathscr{F}'$ contains $\Sigma$, which shows that $\mathscr{F}_\Delta$ contains all the generators of $\Sigma_\Delta$. Since $\mathscr{F}'$ contains the field $\mathscr{F}$ that generates $\Sigma$, it is enough to show that $\mathscr{F}'$ is a monotone class (see, e.g., Theorem 4.4.2 of Dudley, 1989). That is, we have to show that if $(E_n)_{n=1}^\infty$ is a decreasing (increasing) sequence of events in $\mathscr{F}'$ then $\cap_{n=1}^\infty E_n \in \mathscr{F}'$ $(\cup_{n=1}^\infty E_n \in \mathscr{F}')$.

If $(E_n)_{n=1}^\infty$ is decreasing, then for any $\mu \in \Delta(X)$, $(\mu(E_n))_{n=1}^\infty$ is a decreasing sequence converging to $\mu(\cap_{n=1}^\infty E_n)$. Therefore, by $\sigma$-additivity,

$$\beta^p(\cap_{n=1}^\infty E_n) = \cap_{n=1}^\infty \beta^p(E_n) \in \mathscr{F}_\Delta.$$

If $(E_n)_{n=1}^\infty$ is increasing then for any $\mu \in \Delta(X)$, $(\mu(E_n))_{n=1}^\infty$ is an increasing sequence converging to $\mu(\cup_{n=1}^\infty E_n)$. In this case, $\sigma$-additivity implies

$$\beta^p(\cup_{n=1}^\infty E_n) = \cap_{m=1}^\infty \cup_{n=1}^\infty \beta^{p-(1/m)}(E_n) \in \mathscr{F}_\Delta.$$

■

For a subset $A$ (not necessarily a measurable one) of $X$, we denote by $\tilde{\Delta}(A)$ the set of all measures $\mu$ in $\Delta(X)$ for which $\mu(E) = 1$ for all events $E$ in $X$ such that $A \subseteq E$. When $A = \{x\}$ is a measurable set, $\tilde{\Delta}(A)$ consists of a single measure which we denote by $\delta_x$. When all the singletons are measurable in $X$, we denote by $\delta\colon X \to \Delta(X)$ the map defined by $\delta(x) = \delta_x$.

The set $A$ can be considered as a measurable space with the $\sigma$-field $\Sigma \cap A$. Viewing $A$ as such, $\Delta(A)$ is a well defined measurable space with the $\sigma$-field $(\Sigma \cap A)_\Delta$. The set of measures $\tilde{\Delta}(A)$ can be also considered as a measurable space with the $\sigma$-field $\Sigma_\Delta \cap \tilde{\Delta}(A)$. The relation between $\Delta(A)$ and $\tilde{\Delta}(A)$ is given in the following lemma.

**Lemma 2.2.** *For each $\mu \in \Delta(A)$, $\mu(\cdot \cap A)$ defines a measure on $X$. The map $\mu \to \mu(\cdot \cap A)$ is a measure theoretic isomorphism from $\Delta(A)$ onto $\tilde{\Delta}(A)$. The inverse of this isomorphism maps each measure $\nu \in \tilde{\Delta}(A)$ to the outer measure* [2] *$\nu^*$ restricted to $\Sigma \cap A$.*

**Proof:** It is straightforward to show that $\mu(\cdot \cap A)$ is a probability measure in $\tilde{\Delta}(A)$ and that the map $\mu \to \mu(\cdot \cap A)$ is one-to-one. To see that the map is onto consider $\nu \in \tilde{\Delta}(A)$. By Theorem 3.3.6 in (Dudley, 1989), the outer measure $\nu^*$ is a probability measure on $(A, \ \Sigma \cap A)$, i.e., it is in $\Delta(A)$, and it satisfies $\nu(\cdot) = \nu^*(\cdot \cap A)$. Thus, $\nu \in \tilde{\Delta}(A)$ corresponds to $\nu^*(\cdot \cap A) \in \Delta(A)$.

■

In light of this lemma, when we have a measurable function to $\Delta(X)$, the image of which is in $\tilde{\Delta}(A)$, we will consider it also as a measurable function to $\Delta(A)$.

**Lemma 2.3.** *Suppose the $\sigma$-field $\Sigma$ in $X$ has a countable sub-field $\Sigma_0$ that separates the points of $X$. Then the map $\delta\colon X \to \Delta(X)$ is a measure-theoretic embedding* [3] *of $X$ into $\Delta(X)$.*

**Proof.** $\delta$ is measurable, since for every $E \in \Sigma$ and $0 < p \leq 1$ we have

$$\delta^{-1}(\beta^p(E)) = E,$$

and for $p = 0$

$$\delta^{-1}(\beta^0(E)) = X.$$

Furthermore, for every $E \in \Sigma$ we have

$$\delta(E) = \bigcap_{F \in \Sigma_0} (\beta^1(E \cap F) \cup \beta^1(E \cap \neg F))$$

and the right-hand side is measurable in $\Delta(X)$.

■

[2] The outer measure $\nu^*(B)$ for $B \subseteq X$ is defined by $\nu^*(B) = \inf\{\nu(F)\colon F \in \Sigma, \ B \subseteq F\}$.

[3] A measurable map is an embedding, if it is one to one, and maps measurable sets to measurable sets.

For measurable spaces $X$ and $Y$ and a measurable function $\varphi: X \to Y$, we denote by $\hat{\varphi}$ the function $\hat{\varphi}: \Delta(X) \to \Delta(Y)$ defined by $\hat{\varphi}(\mu) = \mu \circ \varphi^{-1}$ (that is, for each event $F$ in $Y$, $\hat{\varphi}(\mu)(F) = \mu(\varphi^{-1}(F))$. It is easy to check that $\hat{\varphi}$ is a measurable function.

We fix a finite set $I$ to be the set of *agents*. The set $I_0 = I \cup \{0\}$ includes all agents and '0' which stands for 'nature'. For a family of sets $(X_i)_{i \in I_0}$ we denote by $X$ the product $\prod_{i \in I_0} X_i$, and by $X_{-i}$, for $i \in I$, the product $\prod_{j \in I_0 \setminus \{i\}} X_j$. If $(Y_i)_{i \in I_0}$ is another family of sets, and $(f_i)_{i \in I_0}$ a family of functions, $f_i: X_i \to Y_i$, then we denote by $f$ the function $f: X \to Y$ defined by $f((x_i)_{i \in I_0}) = (f_i(x_i))_{i \in I_0}$, and by $f_{-i}$ the function $f_{-i}: X_{-i} \to Y_{-i}$ defined by $f_{-i}((x_j)_{j \in I_0 \setminus \{i\}}) = (f_j(x_j))_{j \in I_0 \setminus \{i\}}$. We consider any product, finite or infinite, of measurable spaces as a measurable space with the product $\sigma$-field.

## 3. Type spaces, type morphisms and the coherent space

*Type spaces*. Fix a measurable space $S$ the elements of which are called *states of nature*. A *type space* on $S$ is a pair $\langle (T_i)_{i \in I_0}, (m_i)_{i \in I} \rangle$, where
1. $T_0 = S$, and $T_i$, for $i \in I$, is a measurable space.
2. For each $i \in I$, $m_i$ is a measurable function $m_i: T_i \to \Delta(T_{-i})$.

The space $T_i$ is called the space of types of agent $i$, and $m_i$ specifies the belief of each type over nature and the other agents' types.

*Type morphisms*. Let $\langle (T_i)_{i \in I_0}, (m_i)_{i \in I} \rangle$ and $\langle (T'_i)_{i \in I_0}, (m'_i)_{i \in I} \rangle$ be type spaces on $S$, and $(\varphi_i)_{i \in I_0}$ and $I_0$-tuple, of measurable functions $\varphi_i: T_i - T'_i$, where $\varphi_0$ is the identity on $S$. The induced function $\varphi_{ti}: T \to T'$ is a *type morphism* if it preserves the beliefs of the agents. That is, when $t_i$ is mapped by $\varphi_{-\mathrm{i}}$ to $t'_i$, then $m_i(t_i)$ is mapped by $\hat{\varphi}_{-i}$ to $m'_i(t'_i)$. Or simply, for each $i \in I$, $m'_i \circ \varphi_i = \hat{\varphi}_{-i} \circ m_i$. The morphism is a *type isomorphism* if $\varphi$ is an isomorphism (or equivalently, if $\varphi_i$ is an isomorphism for each $i \in I_0$).

**Definition 3.1.** A type space $T^*$ is a *universal type space* on $S$, if for every type space $T$ on $S$ there is a unique type morphism from $T$ to $T^*$.

In (Heifetz and Samet, 1998), we proved that for every set of agents $I$ and a measurable space $S$ of nature states there exists a unique universal type space (up to type isomorphism).

We now turn to consider the explicit approach, in which one specifies directly the mutual beliefs of the agents. We are going to define inductively the $n$-order coherent spaces $C_i^n$, that will consist of coherent $n$-tuples $(t_i^1, \ldots, t_i^n)$ of beliefs over $C_{-i}^0, C_{-i}^1, \ldots, C_{-i}^{n-1}$—the lower-level coherent spaces of the other agents and nature. $C_i^n$ will thus be a subset of $\Delta(C_{-i}^0) \times \Delta(C_{-i}^1) \times \cdots \times \Delta(C_{-i}^{n-1}$. Moreover, $C_i^n$ will be a subset of $C_i^{n-1} \times \Delta(C_{-i}^{n-1})$. *Coherence* means that the marginal belief of $t_i^{k+1}$ on $C_{-i}^{k-1}$ is $t_i^k$ for $k \leq n-1$.

The primary object of belief of each agent $i \in I$ is nature, so let us denote [4] $C^0_{-i} = S$. The primary space of beliefs of agent $i$ will therefore be $C^1_i = \Delta(C^0_{-i})$, i.e., all the possible beliefs of agent $i$ over nature.

For $k \geq 1$, let $C^k_0 = S$. Denote by $\rho^1_{-i}$ the projection from $C^1_{-i} = S \times \prod_{j \epsilon I \setminus i} C^1_j$ to $C^0_{-i} = S$. Inductively,

(1) *Coherence*: $C^{k+1}_i$ consists of all the tuples $((t^1_i, \ldots, t^k_i), t^{k+1}_i) \in C^k_i \times \Delta(C^k_{-i})$ such that $\hat{\rho}^k_{-i}(t^{k+1}_i) = t^k_i$, that is the marginal of $t^{k+1}_i$ over $C^{k-1}_{-i}$ coincides with $t^k_i$.

(2) $\rho^{k+1}_i$ is the projection from $C^{k+1}_i$ to $C^k_i$.

In the limit, $i$'s coherent space (over $S$), $C_i$, is the set of all sequences $(t^1_i, t^2_i, \ldots)$ such that $(t^1_i, \ldots, t^k_i) \in C^k_i$ for every $k \geq 1$. These are the coherent hierarchies of beliefs of agent $i$ about nature, about nature and the other agents' beliefs about nature, etc. We denote by $\pi^k_i$ the projection from $C_i$ to $C^k_i$. Denote also $C_0 = S$, and let $\pi^k_0 \colon C_0 \to C^k_0$ be the identity map on $S$. We call $C = \prod_{i \in I_0} C_i$ the coherent space.

The next step is to consider the relationship between type spaces and the coherent space $C$. Each type space $\langle (T_i)_{i \in I_0}, (m_i)_{i \in I} \rangle$ admits a natural map, $h = (h_i)_{i \in I_0}$, that we call the hierarchy description map, into the coherent space $C$. The map unfolds the mutual beliefs of each agent $i \in I$, by assigning to each type $t_i \in T_i$ a coherent hierarchy of beliefs in $C_i$. To this end, we define the maps $h^n_i \colon T_i \to C^n_i$ as follows.

Denote by $h^0_{-i}$ the projection of $T_{-i}$ to $C^0_{-i} = T_0 = S$. For $k \geq 1$ let $h^k_0$ be the identity on $S$. For the agents $i \in I$ define inductively $h^n_i \colon T_i \to C^n_i$ by

$$h^1_i(t_i) = m_i(t_i) \circ \left(h^0_{-i}\right)^{-1}$$

$$h^{k+1}_i(t_i) = \left(h^k_i(t_i), m_i(t_i) \circ \left(h^k_{-i}\right)^{-1}\right) = \left(m_i(t_i) \circ \left(h^0_{-i}\right)^{-1}, \ldots,\right.$$

$$\left. m_i(t_i) \circ \left(h^{k-1}_{-i}\right)^{-1}, m_i(t_i) \circ \left(h^k_{-i}\right)^{-1}\right).$$

$h^{k+1}_i(t_i)$ is indeed in $C^{k+1}_i$. To show this, it is enough to prove that the marginal of the measure $m_i(t_i) \circ (h^k_{-i})^{-1} \in \Delta(C^k_{-i})$ on $C^{k-1}_{-i}$ equals $m_i(t_i) \circ (h^{k-1}_{-i})^{-1}$. This is in fact the case—for every event $F$ in $C^{k-1}_{-i}$

$$m_i(t_i) \circ \left(h^k_{-i}\right)^{-1} \circ \left(\rho^k_{-i}\right)^{-1}(F) = m_i(t_i)\left(\rho^k_{-i} \circ h^k_{-i}\right)^{-1}(F)$$

$$= m_i(t_i)\left(h^{k-1}_{-i}\right)^{-1}(F),$$

as required.

[4] Deviating slightly from our notational conventions.

Finally, define $h_0\colon T_0 \to C_0$ to be the identity on $S$, and for all $i \in I$ define $h_i\colon T_i \to C_i$ by

$$h_i(t_i) = \left(m_i(t_i) \circ \left(h^k_{-i}\right)^{-1}\right)_{k \geq 0}.$$

For spaces $S$ of nature states with appropriate topological properties, $C$ is the universal type space. This was first proved by Mertens and Zamir (1985), who showed that by the Kolmogorov extension theorem, each $c_i = (t^k_i)_{k \geq 1} \in C_i$ admits a unique $\sigma$-additive probability measure $m_i(c_i)$ over $C_{-i}$ with the appropriate corresponding marginals [5]

$$\hat{\pi}^k_{-i}(m_i(c_i)) = t^{k+1}_i \quad \forall k \geq 0$$

Thus, $\langle (C_i)_{i \in I_0}, (m_i)_{i \in I} \rangle$ is a type space, which is moreover universal: It turns out that for every type space $T$ with $T_0 = S$, the hierarchy description map $h\colon T \to C$ is a type morphism, which is the unique type morphism from $T$ to $C$. In particular, for the case $T = C$, the hierarchy description map $h\colon C \to C$ is the identity map.

In the general measure-theoretic case with which we deal here, the Kolmogorov extension theorem is not applicable. And indeed, in Section 4 we build a space of nature states $S$ and a coherent hierarchy of beliefs over it that has no $\sigma$-additive limit extension.

## 4. Coherent hierarchies which are not Harsanyi types

In this section we construct a coherent hierarchy of beliefs that cannot be extended to a belief on the space of all coherent hierarchies. Andersen and Jessen (1948) gave an example of a sequence of spaces $A_k$ with corresponding coherent probability measures $\nu_n$ on $X_n = \prod_{k=0}^{n-1} A_k$ that cannot be extended to a $\sigma$-additive probability measure on $X_\infty = \prod_{k=0}^{\infty} A_k$ (the example may also be found in (Halmos, 1950, 49.3). [6]

However, this construction does not lend itself obviously to our case. It requires some freedom in the way the components spaces $A_k$ are chosen, while the component spaces in the constructions of coherent hierarchies are determined by the basic space of states of nature, $S$.

Nevertheless, we show that the extra structure of the set of coherent beliefs does not save it from the same problem. We construct a measurable space $S$ and a hierarchy of coherent beliefs of an agent that cannot be extended to a measure

[5] By our convention, $\hat{\pi}^k_{-i}$ is the projection from $\Delta(C_{-i})$ to $\Delta(C^k_{-i})$.

[6] $A_k$ may be taken to be any decreasing sequence of subsets of [0,1] (endowed with the relative Borel $\sigma$-field), where $\cap_{k=0}^{\infty} A_k = \emptyset$, and each $A_k$ has Lebesgue outer measure 1. $\nu_n$ is concentrated on the diagonal of $X_n = \prod_{k=0}^{n-1} A_k$, which is isomorphic to $A_{n-1}$. $\nu_n$ is obtained by this isomorphism from the Lebesgue outer measure on $A_{n-1}$ (this outer measure turns to be $\sigma$-additive when restricted to the relative Borel $\sigma$-field on $A_{n-1}$).

over the coherent space based on $S$. This is done in two steps. In the *first step* we show that a construction with the properties above can be carried out with all the spaces in the product being identical with the same space $S$. In the *second step* we show how, with two agents, the spaces $S^n$ can be identified as a product of nature and an agent's coherent beliefs of orders $1, \ldots, n-1$. Thus, the measures built in the first step will be identified with coherent beliefs of the other agent about nature and the belief hierarchies of the first agent.

### *4.1. The first step*

For each $0 \leq m < n \leq \infty$ denote by $X_{m,n}$ the product $\prod_{m \leq k < n} A_k$. For $0 \leq m < n < \infty$ we write $\nu_{m,n}$ for the marginal of $\nu_n$ on $X_{m,n}$. In particular $X_{0,n} = X_n$, and $\nu_{0,n} = \nu_n$. For every $m \geq 0$, the coherent sequence $(\nu_{m,n})_{n>m}$ on $(X_{m,n})_{n>m}$ does not have a limit extension, because this sequence of measures has exactly the same properties as those of the basic sequence $\nu_n$, which precluded it from having a limit extension. [7]

We take $S$ to be $X_\infty$, and regroup the factors of the product $S^n$ as indicated in the following diagram—taking first the products along columns (from bottom to top in the diagram), and then products of these products:

$$\begin{array}{rcccccccccccccc}
n\text{th copy} & S & = & A_0 & \times & A_1 & \times & \ldots & \times & A_{n-1} & \times & A_n & \times & \ldots \\
 & \times & & & & & & & & & & & & \\
(n-1)\text{th copy} & S & = & & & A_0 & \times & \ldots & \times & A_{n-2} & \times & A_{n-1} & \times & \ldots \\
 & \times & & & & & & & & & & & & \\
 & \vdots & & & & & & & & \vdots & & \vdots & & \\
 & \times & & & & & & & & & & & & \\
1\text{st copy} & S & = & & & & & & & A_0 & \times & A_1 & \times & \ldots
\end{array}$$

We can then write:

$$S^n = \prod_{1 \leq k < n} X_{0,k} \times \prod_{0 \leq k < \infty} X_{k,k+n}.$$

We define the measure $\mu_n$ on $S^n$ to be the product measure

$$\mu_n = \prod_{1 \leq k < n} \nu_{0,k} \times \prod_{0 \leq k < \infty} \nu_{k,k+n}.$$

By the coherence of the sequences $(\nu_{m,n})_{n>m}$ it follows that the marginal of $\mu_n$ on the first $(n-1)$ copies of $S^n$ is $\mu_{n-1}$ and thus the sequence $(\mu_n)_{n \geq 1}$ is coherent. The space $S^\infty$ can be written as:

$$S^\infty = (X_{0,\infty})^\infty \times \prod_{0 \leq k < \infty} X_{k,\infty}.$$

Clearly, the sequence $(\mu_n)$ does not have an extension to $S^\infty$, as the marginal of such an extension on any of the factors $X_{0,\infty}$ would be an extension of $(\nu_{0,j})_{j \geq 1}$ on this factor.

[7] See [6] above.

### 4.2. The second step

We assume, now, that the set of agents is $I=\{1,2\}$. We will show how to embed $S^n$ in $S\times C_i^{n-1}$, the space of nature states and agent $i$'s coherent hierarchies of beliefs of length $n-1$. Using this embedding, we will be able to identify the measures $\mu_n$ as measures on $S\times C_i^{n-1}$, and thus to interpret them as a hierarchy of beliefs of the other agent $j$ about nature and the hierarchies of agent $i$.

The embedding $g_i^n$ of $S^n$ to $C_{-j}^{n-1}=S\times C_i^{n-1}$ will map $(s_0,\dots,s_{n-1})$ to the point in $S\times C_i^{n-1}$ where (0) the nature state is $s_0$, (1) agent $i$ is certain that $s_1$ occurs, (2) agent $i$ is certain that agent $j$ is certain that $s_2$ occurs, (3) agent $i$ is certain that agent $j$ is certain that agent $i$ is certain that $s_3$ occurs, and so on.

Formally, for $n=1$ the embedding $g_i^1$: $S\to S$ will be the identity. Inductively, $g_i^{n+1}$: $S^{n+1}\to S\times C_i^n$ is defined by

$$\begin{aligned}
g_i^{n+1}(s_0,\dots,s_{n-1},s_n) &= \left(g_i^n(s_0,\dots,s_{n-1}),\delta_{g_j^n(s_1,\dots,s_n)}\right)\\
&= \left(s_0,\delta_{s1},\delta_{(s_1,\delta_{s2})},\dots,\delta_{g_j^{n-1}(s_1,\dots,s_{n-1})},\delta_{g_j^n(s_1,\dots,s_n)}\right)\\
&= \left(s_0,\delta_{g_j^1(s_1)},\delta_{g_j^2(s_1,s_2)},\dots,\delta_{g_j^{n-1}(s_1,\dots,s_{n-1})},\right.\\
&\qquad \left.\delta_{g_j^n(s_1,\dots,s_n)}\right) \qquad (4.1)
\end{aligned}$$

where $j=2-i$ is the other agent.

Indeed, $g_i^n$ is into $C_{-j}^{n-1}=S\times C_i^{n-1}$. For $n=1,2$ this is immediate. If this is true for $n$, then $g_j^{n-1}(s_1,\dots,s_{n-1})$ is the projection of $g_j^n(s_1,\dots,s_{n-1},\ s_n)$ from $C_{-i}^{n-1}$ to $C_{-i}^{n-2}$. From the second line of (4.1) we therefore conclude that the marginal of $\delta_{g_j^n(s1,\dots,s_{n-1},s_n)}$ on $C_{-i}^{n-2}$ is $\delta_{g_j^{n-1}(s_1,\dots,s_{n-1})}$, as required.

In the limit, $g_i$: $S^\infty\to S\times C_i$ is defined by

$$g_i(s_0,s_1,s_2,s_3,\dots)=\left(s_0,\delta_{g_j^1(s_1)},\delta_{g_j^2(s_1,s_2)},\delta_{g_j^3(s_1,s_2,s_3)},\dots\right). \qquad (4.2)$$

**Proposition 4.3.** *For all $n\ge 1$, $g_i^n$, as well as $g_i$, are embeddings for $i=1,2$.*

**Proof.** For $n=1$ this is immediate. Suppose the claim holds for $m\le n$. Let us label the copies of $S$ in $S^{n+1}$ and write $S^{n+1}=S_0\times S_1\times\cdots\times S_n$. Since the spaces $A_k$ of Andersen and Jessen (1948) are all separable metric spaces (subsets of the interval [0,1]) with the Borel $\sigma$-field, so are $S=\prod_{k=0}^\infty A_k$ and $S^m$. Hence, by Lemma 2.3 and the induction hypothesis, $\delta(g_j^m(S_1\times\ldots\times S_m))$ is isomorphic to $S_1\times\ldots\times S_m$ for $m\le n$ with the map

$$\delta\left(g_j^m(s_1,\dots,s_m)\right)\overset{(\delta\circ g_j^m)^{-1}}{\rightarrow}(s_1,\dots,s_m)$$

Observing the third line of (4.1), $g_i^{n+1}(S_0 \times \cdots \times S_n)$ is therefore isomorphic to

$$\{(s_0, s_1, (s_1, s_2), \ldots, (s_1, \ldots, s_n)) : s_m \in S_m, \quad m \le n\}$$
$$= S_0 \times \mathrm{Diag}(S_1^n) \times \mathrm{Diag}(S_2^{n-1}) \times \ldots \times \mathrm{Diag}(S_{n-1}^2) \times S_n \tag{4.3}$$

Since $S$ is a separable metric space with the Borel $\sigma$-field, the product $\sigma$-field of $S^m$ is its Borel $\sigma$-field. Furthermore, the diagonal in $S^m$ is closed, and hence isomorphic to $S$ by the map $(s, \ldots, s) \to s$. Thus, the diagonals $\mathrm{Diag}(S_1^n)$, $\mathrm{Diag}(S_2^{n-1}), \ldots, \mathrm{Diag}(S_{n-1}^2)$ in (4.3) are isomorphic to $S_1, S_2, \ldots, S_{n-1}$, respectively. We conclude that $g_i^{n+1}(S_0 \times \cdots \times S_n)$ is isomorphic to $S_0 \times \cdots \times S_n$.

By the same argument, in the limit $g_i(S_0 \times S_1 \times S_2 \times \cdots)$ is isomorphic to

$$\{(s_0, s_1, (s_1, s_2), \ldots, (s_1, \ldots, s_n), \ldots) : s_m \in S_m, \quad m \ge 0\}$$
$$= S_0 \times \mathrm{Diag}(S_1^\infty) \times \mathrm{Diag}(S_2^\infty) \times \cdots$$

which is isomorphic to $S_0 \times S_1 \times S_2 \times \cdots$, as required.

■

Consider for $n \ge 1$, the image measures $\kappa_n = \mu_n (g_i^n)^{-1} \in \Delta(C_{-j}^{n-1})$. It is easy to show by induction that the sequence $(\kappa_n)_{n \ge 1}$ is a coherent hierarchy in $C_j$. Indeed, the crucial inductive step is to show that the marginal of $\kappa_{n+1}$ on $C_{-j}^{n-1}$ is $\kappa_n$. But this follows since, by (4.1), the projection of $g_i^{n+1}(s_0, \ldots s_n)$ on $C_{-j}^{n-1}$ is $g_i^n(s_0, \ldots s_{n-1})$.

The coherent sequence $(\kappa_n)_{n \ge 1}$ does not have a $\sigma$-additive coherent extension to $C_{-j}$. Suppose, to the contrary, that $\kappa$ were such an extension. Then, since $g_i$ is an embedding, it would follow that the set function $\mu$ defined by $\mu(E) = \kappa(g_i(E))$, for each measurable set $E$, is a well defined $\sigma$-additive probability measure on $S^\infty$. Moreover, $\mu$ would be a coherent extension of $\mu_n$. Indeed, Let $E$ be a measurable set in $S^n$, and $F = E \times (\prod_{k=n+1}^{\infty} S_k)$. Then, $\mu(F) = \kappa(g_i(F)) = \kappa_n(\pi_{-j}^{n-1}(g_i(F)))$, where the latter equality holds since $\kappa$ is a coherent extension of $\kappa_n$. But, by (4.2), $\kappa_n(\pi_{-j}^{n-1}(g_i(F))) = \kappa_n(g_i^n(E))$. Finally, since $g_i^n$ is an one-to-one,

$$k_n(g_i^n(E)) = \mu_n\left((g_i^n)^{-1}(g_i^n(E))\right) = \mu_n(E)$$

Together this implies that $\mu(F) = \mu_n(E)$, and this contradicts the non-existence of a coherent extension of $(\mu_n)_{n \ge 1}$.

## 5. Which hierarchies belong to the universal space?

In a companion paper (Heifetz and Samet, 1998), we proved that there exists, for every space of nature states $S$, a unique universal type space $\langle (T_i^*)_{i \in I_0}, (m_i^*)_{i \in I} \rangle$. Each $T_i^*$, $i \in I$ consists of hierarchies of beliefs—all those hierarchies that result by applying the hierarchy description map $h_i$ to some type space over $S$. In that paper it was proved that the hierarchy description maps $h_i : T_i^* \to C_i$ are the identity maps. In particular, this means that

$$T_i^* \subseteq C_i, \quad i \in I. \tag{5.1}$$

Furthermore, it was proved that for every $t_i^* = ((t_i^*)^k)_{k \geq 1} \in T_i^*$, the marginal belief of $m_i^*(t_i^*)$ over $\pi_{-i}^k(T_{-i}^*)$ is $(t_i^*)^{k+1}$, i.e.,

$$m_i^*(t_i^*)\left(\pi_{-i}^k\right)^{-1} = (t_i^*)^{k+1}. \tag{5.2}$$

In other words, $m_i^*(t_i^*)$ is the limit extension of $((t_i^*)^k)_{k \geq 1}$.

However, in Section 4 we found a space of nature states $S$ and a hierarchy of beliefs $(\kappa_n)_{n \geq 1}$ on it with no coherent limit extension whose corresponding marginals are the $\kappa_n$. This means that the universal space $T^*$ is in general a proper subset of the coherent space $C$. In this section we show how the universal space $T^*$ may be 'extracted' from $C$.

The coherence condition (2) in the definition of $C$ makes it possible to associate with each $t_i \in C_i$ an additive set function (with total mass 1) on $C_{-i}$, which is not necessarily $\sigma$-additive, denoted by $n_i(t_i)$. This set function is defined on the field

$$\left\{\left(\pi_{-i}^k\right)^{-1}(E) | k \geq 0,\ E \text{ is measurable in } C_{-i}^k\right\},$$

which generates the $\sigma$-field on $C_{-i}$. For an event $E$ in $C_{-i}^k$

$$n_i(t_i)\left(\left(\pi_{-i}^k\right)^{-1}(E)\right) = t_i^{k+1}(E). \tag{5.3}$$

It is easy to see that coherence guarantees that if $(\pi_{-i}^k)^{-1}(E) = (\pi_{-i}^l)^{-1}(F)$ for some events $E$ in $C_{-i}^k$ and $F$ in $C_{-i}^l$, then $t_i^{k+1}(E) = t_i^{l+1}(F)$ and therefore (5.3) defines an additive set function unambiguously. We can state now a condition under which the space of coherent beliefs coincides with the universal one. The following proposition is a special case of Proposition 5.5 and therefore is not proved now.

**Proposition 5.4.** *If for each $i \in I$ and $t_i \in C_i$, the additive set function $n_i(t_i)$ can be extended to a $\sigma$-additive probability measure in $\Delta(C_{-i})$ then $C = T^*$.*

Under reasonable topological assumptions on $S$ this condition on $n_i(t_i)$ is met. But, as we saw, this is not necessarily the case when no topological assumptions are made.

We now show how in general $T^*$ can be 'carved out' from $C$ in an inductive process. Denote by $G_i^0$ the set of all $t_i \in C_i$ such that $n_i(t_i)$ can be extended to a $\sigma$-additive measure on $C_{-i}$. We denote, with some abuse of notation, this $\sigma$-additive measure by $n_i(t_i)$ (when the extension exists it is necessarily unique and belongs to $\tilde{\Delta}(C)$). $G^0$ may fail now to be $T^*$ for one reason: for some $t_i \in G_i^0$, $n_i(t_i)$ may not be in $\tilde{\Delta}(G_{-i}^0)$. To fix this we define, by transfinite induction, [8] a decreasing chain of spaces $G_i^\alpha$ for each ordinal $\alpha$. For nature, i.e., for $i = 0$, let $G_0^\alpha = S$ for all $\alpha$. For $i \in I$, we denote by $G_i^{<\alpha}$ and $G^{<\alpha}$ the spaces

[8] For an exposition of ordinals and transfinite induction, see e.g., (Devlin, 1993).

$\cap_{\beta<\alpha} G_i^\beta$ and $\cap_{\beta<\alpha} G^\beta$ correspondingly. Since the chain is decreasing, $\cap_{\beta<\alpha} G_i^\beta = G_i^{\alpha-1}$ and $\cap_{\beta<\alpha} G^\beta = G^{\alpha-1}$ for non-limit ordinals $\alpha$. We define

$$G_i^\alpha = G_i^{<\alpha} \cap (n_i)^{-1}\left(\tilde{\Delta}(G_{-i}^{<\alpha})\right).$$

That is, $G_i^\alpha$ contains $t_i$ from the previously defined $G_i^\beta$ only if the probability measure associated with it, $n_i(t_i)$, has its mass concentrated on the previously defined $G_{-i}^\beta$.

**Proposition 5.5.** *For some ordinal $\alpha$, $G_\alpha G^{\alpha+1}$, and for this ordinal, $G^\alpha = T^*$.*

**Proof:** First we prove by transfinite induction that $T^* \subseteq G^\alpha$ for all $\alpha$. By comparing (5.2) and (5.3) we observe that for $t_i^* \in T^*$, $n_i(t_i^*)$ and $m_i^*(t_i^*)$ coincide on a generating field of $C_{-i}$ and hence $n_i(t_i^*)$ can be extended to a $\sigma$-additive measure on $C_{-i}$ which is $m_i^*(t_i^*)$. But $m_i^*(t_i^*) \in \tilde{\Delta}(T_{-i}^*)$, and by (5.1), $\tilde{\Delta}(T_{-i}^*) \subseteq \tilde{\Delta}(C_{-i})$. Therefore, $m_i^*(t_i^*) \in \tilde{\Delta}(C_{-i})$. This shows that $T^* \subseteq G^0$. If $T^* \subseteq G^\beta$ for all $\beta < \alpha$, then $T^* \subseteq G^{<\alpha}$. Thus, $m_i^*(t_i^*) \in \tilde{\Delta}(T_{-i}^*) \subseteq \tilde{\Delta}(G_{-i}^{<\alpha})$, and therefore $T^* \subseteq G^\alpha$.

Since $G^\alpha$ is decreasing there must be some $\alpha$ for which $G^\alpha = G^{\alpha+1}$—the ordinal up to which the chain of $G-s$ is strictly decreasing may not exceed the cardinality of $C$. By what we have shown, $T^* \subseteq G^\alpha$. For this $\alpha$, $n_i$ is a map from $G_i^\alpha$ to $\tilde{\Delta}(G_{-i}^\alpha)$, or according to Lemma 2.2, $n_i\colon G_i^\alpha \to \Delta(G_{-i}^\alpha)$.

**Lemma 5.6.** *$\langle (G_i^\alpha)_{i\in I_0}, (n_i)_{i\in I}\rangle$ is a type space over $S$.*

**Proof.** We have to prove that $n_i$ is measurable for each $i \in I$. For each measurable $E$ in $\pi_{-i}^k(G_{-i}^\alpha)$

$$n_i^{-1}\left(\beta^p\left(\left(\pi_{-i}^k\right)^{-1}(E)\right)\right) = \left\{t_i \in G_i^\alpha | n_i(t_i)\left(\left(\pi_{-i}^k\right)^{-1}(E)\right) \ge p\right\}$$
$$= \left\{t_i \in G_i^\alpha | t_i^{k+1}(E) \ge p\right\}. \tag{5.7}$$

The last set is a measurable subset of $G_i^\alpha$, as it is defined by an event in $G_i^\alpha$. The field of events $(\pi_{-i}^k)^{-1}(E)$ generates the $\sigma$-field on $G_{-i}^\alpha$ and hence, by Lemma 2.1, sets of the form $\beta^p((\pi_{-i}^k)^{-1}(E))$ generate the $\sigma$-field of $\Delta(G_{-i}^\alpha)$. Thus, by (5.7), $n_i$ is measurable.

■

**Lemma 5.7.** *For all $i \in I$, the hierarchy description map*

$$h_i\colon G_i^\alpha \to C_i$$

*is the identity on $G_i^\alpha$.*

**Proof.** We have to prove that for all $k \geq 1$ and $t_i = (t_i^k)_{k \geq 1} \in G_i^\alpha$,

$$h_i^k(t_i) = \pi_i^k(t_i) = \left(t_i^1, \ldots, t_i^k\right).$$

For every event $E \subseteq C_{-i}^0 = S$,

$$n_i(t_i)\left(\pi_{-i}^0\right)^{-1}(E) = t_i^1(E),$$

because $n_i(t_i)$ is a coherent extension of $t_i^1$. Inductively, if the claim holds for $k-1$, then for every event $E \in C_{-i}^{k-1}$

$$n_i(t_i)\left(\pi_{-i}^{k-1}\right)^{-1}(E) = t_i^k(E),$$

because $n_i(t_i)$ is a coherent extension of $t_i^k$. Therefore, by the induction hypothesis

$$h_i^k(t_i) = \left(h_i^{k-1}(t_i), n_i(t_i)\left(\pi_{-i}^{k-1}\right)^{-1}\right) = \left(t_i^1, \ldots, t_i^{k-1}, t_i^k\right),$$

as required.

■

**End of Proof of Proposition 5.5.** As constructed in (Heifetz and Samet, 1998), $T^*$ is the set of all hierarchies that result by applying the description map $h$ on type spaces over $S$. In particular, by Lemma 5.6, $h$ sends $G^\alpha$ into $T^*$. By Lemma 5.8, $h$ is the identity on $G^\alpha$. Therefore $G^\alpha \subseteq T^*$, as required.

**Corollary 5.8.** *Let $F$ be the operator on subsets $(X_i)_{i \in I_0}$ of $(G_i^0)_{i \in I_0}$ defined by*

$$\begin{aligned} F_0\left((X_i)_{i \in I_0}\right) &= X_0 \\ F_i\left((X_i)_{i \in I_0}\right) &= X_i \cap n_i^{-1}\left(\tilde{\Delta}(X_{-i})\right), \quad i \in I \end{aligned}$$

*Then the universal space $T^*$ is a fixed point of $F$, which is maximal in the sense that every other fixed point $Y$ of $F$ is contained in $T^*$.*

**Proof.** $T^*$ is obviously a fixed point of $F$. If $Y$ is a fixed point of $F$, Lemmas 5.6 and 5.7 and the end of proof of proposition 5.5 apply word by word when $G^\alpha$ is replaced by $Y$. Hence the conclusion that $Y \subseteq T^*$.

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