Games and Economic Behavior 93 (2015) 108–116

Contents lists available at ScienceDirect

Games and Economic Behavior

www.elsevier.com/locate/geb

Agreeing to agree and Dutch books ☆

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ARTICLE INFO

Article history:
Received 7 August 2014
Available online 13 August 2015

JEL classification:
C70
D82

Keywords:
Agreement theorem
Common knowledge
Common prior
Dutch book
No trade theorem

ABSTRACT

We say that agreeing to agree is possible for an event E if there exist posterior beliefs of the agents with a common prior such that it is common knowledge that the agents' posteriors for E coincide. We propose a notion called Dutch book which is a profile of interim contracts between an outsider and the agents based on the occurrence of E, such that the outsider makes positive profit in all states. We show that in a finite state space, when the agents cannot tell whether E occurred or not, agreeing to agree is possible for E if and only if there is no Dutch book on E. This characterization also holds in countable state spaces with two agents. We weaken the notion of Dutch book to characterize agreeing to agree in a countable state space with multiple agents, when each set in each agent's information partition is finite.

1. Introduction

Aumann (1976) demonstrates the impossibility of agreeing to disagree: for any posteriors with a common prior, if the agents' posteriors for an event E are different (they disagree), then the agents cannot have common knowledge (agreement) of these posteriors. Here we ask what are the properties of an event E that make agreeing to *agree* possible. That is, under what conditions there are posteriors of the agents that are derived from a common prior, such that the agents have common knowledge (agreement) that their posteriors of E are the same (agree). Lehrer and Samet (2011) obtained a characterization of the possibility of agreeing to agree for two agents. In this paper, we study the possibility of agreeing to agree for an arbitrary finite set of agents.

Clearly, in any state at which an agent's posterior for E is nontrivial, the agent cannot tell whether E occurred or not. We say in this case that the agent is ignorant of E. Lehrer and Samet (2011) observed that for two agents in a finite state space, ignorance of all agents in all states is also a sufficient condition for agreeing to agree. Consider the following example from Lehrer and Samet (2011).

☆ We are grateful to the Advisory Editor and two anonymous referees for helpful comments. This paper combines the results obtained by Chen and Li and independently by Lehrer, Samet and Shmaya. Chen and Li acknowledge the financial support of the Singapore Ministry of Education Academic Research Fund Tier 1. Lehrer, Samet and Shmaya acknowledge the financial support of BSF through Grant #0603616612. Li acknowledges the financial support of President's Graduate Fellowship. Samet acknowledges the financial support of ISF through Grant #1517/11.

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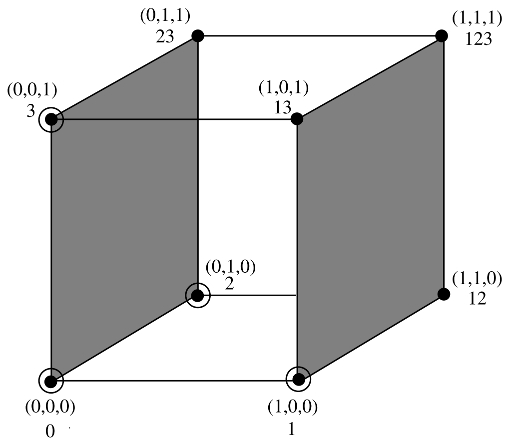
http://dx.doi.org/10.1016/j.geb.2015.08.002
0899-8256/



The eight states in Example 2 are depicted as black dots. The partition of firm 1 consists of two sets of states marked by the dark faces of the cube. The partitions of 2 and 3 are similarly defined. The four states of event E are circled. Firm 1 is ignorant of E: in each of its partition elements, it does not know whether E or $\neg E$ occurred. Similarly, the other two firms are ignorant of E. Yet, there is no common prior such that the posterior probabilities of E are the same for all firms in all states.

Fig. 1. Ignorance does not imply possibility of agreeing to agree.

Example 1. Consider two firms 1, 2. Firm i can be either profitable, denoted by $x_i = 1$, or unprofitable, $x_i = 0$. There are four possible states of the world of the form (x_1, x_2), where each x_i is 0 or 1. Firm i knows only whether it is profitable or not. We denote by 0 the state $(0, 0)$ and all other states are denoted by the names of the profitable firms.

Consider first the event E that both firms have the same financial situation. That is, E consists of the state where $x_1 = x_2 = 0$ and $x_1 = x_2 = 1$. Clearly, both agents are ignorant of E at every state. There are many posteriors that have a common prior such that the firms have common knowledge that the posterior probabilities of E coincide. For instance, if we take the uniform distribution on the four states as the common prior, then both firms have the same posterior for E namely, $1/2$.

However, the characterization of Lehrer and Samet (2011) does not hold when the number of agents exceeds two. Specifically, ignorance is no longer sufficient for agreeing to agree with more than two agents, as demonstrated by the following example.

Example 2. Consider three firms 1, 2, 3. Firm i can be either profitable, denoted by $x_i = 1$, or unprofitable, $x_i = 0$. There are eight possible states of the world of the form (x_1, x_2, x_3), where each x_i is 0 or 1. Firm i knows only whether it is profitable or not. We denote by 0 the state $(0, 0, 0)$ and all other states are denoted by the names of the profitable firms. For example, the state $(1, 0, 1)$ is denoted by 13.

The states of the world are the vertices of the unit cube in $\mathbb{R}^3$. The partition for firm i consists of the face of the cube where $x_i = 0$ and the face where $x_i = 1$ (see Fig. 1).

Let E be the event that no more than one firm is profitable. That is, $E = \{0, 1, 2, 3\}$. Since each face contains at least one point from E and one point from $\neg E$, each firm is ignorant of E.

Suppose that agreeing to agree is possible for E, with posteriors of E being constantly p, and P is a common prior of the types. Consider the face $x_1 = 1$. If the probability of this face is positive, then the requirement that the posterior of E in this face is p says that $P(1)/p = P(12, 13, 123)/(1 - p)$. If the probability of the face is 0, then this equality trivially holds. Writing the similar equations for the faces $x_2 = 1$ and $x_3 = 1$ and summing the equations, we have: $P(1, 2, 3)/p = [2P(12, 13, 23, 123) + P(123)]/(1 - p)$. The right hand side is at least $2P(\neg E)/(1 - p) = 2$ and the left hand side is at most $P(E)/p = 1$.[1] We reach a contradiction. Thus, agreeing to agree is impossible although all firms are ignorant in every state.

This example calls for an alternative characterization of agreeing to agree. For this purpose, we introduce the notion of Dutch book. Consider the following contract between an outsider and some agent i. The contract specifies an amount f_i to be transferred from i to the outsider if E is the case and in the opposite direction if E is not the case. The transfer is made ex post, that is, it requires the knowledge of the state, or at least the knowledge whether E is true or not true in the state. However, the contract is an interim contract, in the sense that the amount transferred, f_i, is known to i.[2] A *Dutch book* is a profile of such interim contracts, one for each agent, under which the outsider profits regardless of the true state.[3] Note

[1] Since the posteriors of E are all p, it follows that $P(E) = p$ and thus $P(\neg E) = 1 - p$.
[2] That is, $f_i : \Omega \to \mathbb{R}$ is measurable with respect to the information partition of each agent; see Section 3.1.
[3] Thanks to ignorance, posterior probabilities exist such that by accepting a Dutch book, each agent obtains a positive expected payoff in every states.

that, like ignorance, the notion of Dutch book involves no probabilistic terms such as prior or posterior. We show that in a finite state space, the possibility of agreeing to agree is equivalent to the absence of a Dutch book (Theorem 1).

Example 2 (continued). Since agreeing to agree for E is impossible, we can find a Dutch book on E. Indeed, consider an ex ante contract f_i which is 1 at the states in the face $x_i = 0$ and -1 at the states in the face $x_i = 1$. The outsider gets 3 dollars at $\omega = 0$ and $\omega = 123$ and gets 1 dollar at the other six states.

Example 2 is a case where agreeing to agree is impossible and there exists a Dutch book. We now present another example where agreeing to agree is possible and there is no Dutch book.

Example 3. In Example 2, consider the event E that all firms have the same financial situation. That is $E = \{0, 123\}$. Consider the prior P defined by $P(0) = P(123) = 1/4$ and $P(\omega) = 1/12$ for all the other six states. Then, the posterior of E on each face is 1/2 and agreeing to agree is possible for E.

Since agreeing to agree for E is possible, there cannot be a Dutch book on E. To see this, suppose that there exists a Dutch book (f_1, f_2, f_3). As $\omega = 0 \in E$, it follows that the sum of the contracts is positive. That is, $\sum_i f_i(x_i = 0) > 0$. Similarly, at $\omega = 123 \in E$, $\sum_i f_i(x_i = 1) > 0$. For all the other 6 states, event E does not happen. Therefore, for each such ω, either $f_i(x_i = 0) + f_j(x_j = 0) + f_k(x_k = 1) < 0$ or $f_i(x_i = 1) + f_j(x_j = 1) + f_k(x_k = 0) < 0$ where i, j, and k are all different. Summing up these 6 inequalities results in $3\sum_i f_i(x_i = 0) + 3\sum_i f_i(x_i = 1) < 0$ which is a contradiction.

This characterization has a spirit similar to well known equivalence results between the existence of a common prior and the absence of agreeable bets.[4] Indeed, both results hold in a finite state space. However, while the transferred amounts in agreeable bets are allowed to vary arbitrarily across different *states*, in a Dutch book they can only vary across different *information sets* and are constant on each one of them.

Next, we study agreeing to agree in countable state spaces. For such state spaces with *two* agents, Lehrer and Samet (2011) showed that agreeing to agree is possible if and only if there exists a nonempty finite event F at which it is common knowledge that the agents cannot tell whether or not E occurred and this still holds true after F becomes common knowledge. This characterization also implies that whenever agreeing to agree is possible for two agents, it is possible with a prior that has a finite support.

In a countable state space with two agents, we show that our characterization remains true; moreover, a Dutch book exists if and only if a bounded Dutch book exists. Hence, with only two agents, agreeing to agree is possible if and only if there is no (bounded) Dutch book.

When there are more than two agents, we need to weaken the notion of Dutch book to characterize agreeing to agree. In Example 4, we construct a three-agent space in which agreeing to agree is impossible for E and yet there exists no Dutch book on E. Furthermore, in Example 5, we construct another three-agent space in which agreeing to agree is possible for E but only with a prior that has an infinite support. Hence, Lehrer and Samet's characterization does not hold, either. Moreover, in the latter example, there is an unbounded Dutch book on E. Hence, the boundedness of a Dutch book becomes essential for the case with more than two agents.

To characterize agreeing to agree in a countable space with more than two agents, consider interim contracts under which the outsider never loses more than ε. We say that there is no *weak Dutch book* if at some state, the outsider's profit in these interim contracts vanishes with ε. Absence of a weak Dutch book is similar to the notion of strong trade consistency in Lehrer and Samet (2014). We show that when each information set of each agent is finite, agreeing to agree is possible if and only if there is no bounded weak Dutch book (Theorem 3). Unlike the equivalence between the existence of a common prior and strong trade consistency proved in Lehrer and Samet (2014, Theorem 4), the equivalence between the possibility of agreeing to agree and the absence of a bounded weak Dutch book does not hold without this finiteness assumption.

The rest of the paper is organized as follows. Section 2 presents the basics of the model of knowledge and beliefs. In Section 3, we characterize agreeing to agree in terms of the existence of Dutch book in a finite state space. Section 4 studies the case with a countable state space. Some of the proofs are in Appendix A.

2. Preliminaries

Knowledge spaces. A *knowledge space* for a set N of n agents, is a tuple $(\Omega, (\Pi_i)_{i\in N})$, where Ω is a countable state space, and for each i, Π_i is a partition of Ω. We denote by $\Pi_i(\omega)$ the element of Π_i that contains ω. Say a knowledge space is finite when Ω is finite. Subsets of Ω are called *events*. We say that agent i *knows* event E at ω if $\Pi_i(\omega) \subseteq E$.

Types. The set of all probability distributions on Ω is denoted by $\Delta(\Omega)$. We consider $\Delta(\Omega)$ as a subset of the space $l_1(\Omega)$ of absolutely summable sequences, in which $\Delta(\Omega)$ is closed. A *type function* of agent i for the knowledge space $(\Omega, (\Pi_i)_{i\in N})$ is a function $t_i : \Omega \to \Delta(\Omega)$. We write t_i^ω instead of $t_i(\omega)$. The *type of i at ω* is t_i^ω. We require of the type function t_i that for each $\pi \in \Pi_i$, it takes the same value in all the states in π, which we denote by t_i^π, and that $t_i^\pi(\pi) = 1$.

[4] See, for example, Morris (1994), Samet (1998), Bonanno and Nehring (1999), Feinberg (2000), Ng (2003), Heifetz (2006), and Lehrer and Samet (2014).

Priors. A prior for i's type function t_i is a probability distribution $P \in \Delta(\Omega)$, such that for each $\pi \in \Pi_i$ with $P(\pi) > 0$, $t_i^{\pi}(\cdot) = P(\cdot \mid \pi)$. A probability distribution P which is a prior for all agents is called a *common prior*. A type profile that has a common prior is *consistent*. For any probability $P \in \Delta(\Omega)$, one can define types of agents by taking conditional probability of P on each π when $P(\pi) > 0$ and define the types arbitrarily on π when $P(\pi) = 0$. The probability P is obviously a common prior for these types.

Expectations. For a real valued function f on Ω and $P \in \Delta(\Omega)$, we denote by $\mathbf{E}^P(f)$ the expectation of f with respect to P, and by $\mathbf{E}^P(f \mid \Pi_i)$ the conditional expectation of f with respect to Π_i, which is a Π_i-measurable function on Ω. If $\mathbf{E}^{t_i^{\omega}}(f)$ exists for each ω, we define the function $\mathbf{E}_i(f)$ on Ω by, $\mathbf{E}_i(f)(\omega) = \mathbf{E}^{t_i^{\omega}}(f)$. Obviously, $\mathbf{E}_i(f)$ is Π_i-measurable, and when P is a prior of the type function t_i, then $\mathbf{E}^P(f \mid \Pi_i) = \mathbf{E}_i(f)$ almost surely. For an event E we denote by $\mathbf{1}_E$ the indicator function on E, that is, $\mathbf{1}_E(\omega) = 1$ for all $\omega \in E$ and $\mathbf{1}_E(\omega) = 0$ for all $\omega \in \neg E$, the complement of E. Clearly, $\mathbf{E}_i(\mathbf{1}_E)(\omega) = t_i^{\omega}(E)$.

3. Agreeing to agree

Definition 1. Let E be an event in a knowledge space. We say that *agreeing to agree is possible for* E if the space has a consistent type profile $(t_i)_{i \in N}$ such that for some $p \in (0,1)$, $t_i^{\pi}(E) = p$ for each i and $\pi \in \Pi_i$, or equivalently, $\mathbf{E}_i(\mathbf{1}_E) = p$ for all i.[5]

The possibility of agreeing to agree is closely related to the notion of *ignorance*. We say that agent i is *ignorant of* E *at* ω, if she does not know E and does not know $\neg E$ at ω. That is, if $\Pi_i(\omega) \cap E \neq \emptyset$ and $\Pi_i(\omega) \cap \neg E \neq \emptyset$. We say that i is *ignorant of* E if i is ignorant of E in all states. If agreeing to agree is possible for E, then for each i and $\pi \in \Pi_i$, $0 < t_i^{\pi}(E) < 1$. Therefore, $\pi \cap E \neq \emptyset$ and $\pi \cap \neg E \neq \emptyset$. Thus we conclude:

Claim 1. *If agreeing to agree is possible for E, then all agents are ignorant of E.*

Ignorance of all agents is also sufficient for the possibility of agreeing to agree for E in the following case.

Proposition 1. *In a finite knowledge space with two agents, if both agents are ignorant of E, then agreeing to agree is possible for E.*[6]

This follows from the fact that in this case we can construct a loop. A *loop* is an event which consists of $2n$ distinct points $\omega_1, \omega_1', \dots \omega_n, \omega_n'$ for some integer $n \geq 1$, such that $\{\omega_1, \dots, \omega_n\} \subseteq E$, $\{\omega_1', \dots, \omega_n'\} \subseteq \neg E$ and for each $k = 1, \dots, n$ $\omega_k' \in \Pi_1(\omega_k)$ and $\omega_k \in \Pi_2(\omega_{k-1}')$, where $\omega_0' = \omega_n'$. Each π that contains points of the loop has the same number of points of the loop from E and from $\neg E$. Thus the uniform distribution on the loop induces types on such π's such that $t_i^{\pi}(E) = 1/2$ for $i = 1, 2$. The elements π that do not contain states of the loop have probability 0, and we are free to choose any type on them. Due to ignorance we can choose the types such that $t_i^{\pi}(E) = 1/2$ for $i = 1, 2$. Thus agreeing to agree is possible for E. For details, see Lehrer and Samet (2011).

For more than two players, ignorance is a necessary condition for the possibility of agreeing to agree for E but, as demonstrated by Example 2, not a sufficient condition. In order to formulate a necessary and sufficient condition, it is enough to find a sufficient condition for the case that ignorance holds. Under the assumption of ignorance, the requirement for *all* types in Definition 1 can be relaxed.

Claim 2. *If all agents are ignorant of E, then agreeing to agree is possible for E if and only if for some $P \in \Delta(\Omega)$, and $p \in (0,1)$, $\mathbf{E}^P(\mathbf{1}_E \mid \Pi_i) = p$ a.s. for all i.*

Proof. Suppose that for some profile of types $(t_i)_{i \in N}$ with a common prior P, $\mathbf{E}_i(\mathbf{1}_E) = p$ for all i, then $\mathbf{E}^P(\mathbf{1}_E \mid \Pi_i) = \mathbf{E}_i(\mathbf{1}_E)$ a.s., and we are done. Conversely, suppose that $\mathbf{E}^P(\mathbf{1}_E \mid \Pi_i) = p$ a.s. for all i. For each i and ω such that $P(\Pi_i) > 0$ define $t_i^{\omega}(\cdot) = P(\cdot \mid \Pi_i)$. Else, the ignorance of i guarantees the existence of a probability distribution with support $\Pi_i(\omega)$ such that $t_i^{\omega}(E) = p$. We define t_i^{ω} to be such a probability. Obviously, P is a common prior for this profile of types and $t_i^{\omega}(E) = p$ for all i and ω. □

Finally, we can replace the number p in Definition 1 and Claim 2 by the number $\frac{1}{2}$.

Claim 3. *If all agents are ignorant of E, then agreeing to agree is possible for E if and only if for some $P \in \Delta(\Omega)$, $\mathbf{E}^P(\mathbf{1}_E \mid \Pi_i) = \frac{1}{2}$ a.s. for all i.*

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[5] This is consistent with the definition of the possibility of agreeing to agree in Lehrer and Samet (2011); see Section 3.2. We also discuss agreeing to agree with $p = 0$ and $p = 1$ in 3.2.

[6] Refer to Lehrer and Samet (2011).

Proof. Suppose for some Q and $p \in (0,1)$, $\mathbf{E}^Q(\mathbf{1}_E \mid \Pi_i) = p$ a.s. for each i. Let $P(\cdot) = \frac{1}{2}Q(\cdot \mid E) + \frac{1}{2}Q(\cdot \mid \neg E)$. Then, $P \in \Delta(\Omega)$. Suppose that $Q(\pi) > 0$ for some i and $\pi \in \Pi_i$. Then $P(E \cap \pi) = \frac{1}{2}Q(E \cap \pi \mid E) = \frac{1}{2}[Q(E \cap \pi)/Q(\pi)][Q(\pi)/Q(E)] = \frac{1}{2}[Q(E \mid \pi)][Q(\pi)/p] = \frac{1}{2}Q(\pi)$. Similarly, $P(\neg E \cap \pi) = \frac{1}{2}Q(\pi)$. Thus, $P(E \mid \Pi_i(\omega)) = \frac{1}{2}$ for each ω such that $P(\Pi_i(\omega)) > 0$. But, $P(X) = 0$ if and only if $Q(X) = 0$, and thus $\mathbf{E}^P(\mathbf{1}_E \mid \Pi_i) = \frac{1}{2}$ a.s. with respect to P. □

3.1. Dutch books

We now describe a necessary and sufficient condition for the possibility of agreeing to agree for E in finite knowledge spaces with any number of agents, where all of them are ignorant of E.

Let $C = \mathbf{1}_E - \mathbf{1}_{\neg E}$. An *interim contract* between an *outsider* and agent i is given by a real-valued function $f_i : \Omega \to \mathbb{R}$ which is measurable with respect to Π_i, that specifies a transfer of $f_i C$ from the agent to the outsider. Due to the measurability requirement, agent i knows the value of f_i, which is traded, and hence the description of this contract as interim.

Definition 2. A *Dutch book* on E is a profile of interim contracts $(f_i)_{i\in N}$ such that: $\sum_i f_i(\omega)C(\omega) > 0$ for all ω. That is, the net transfer to the outsider is positive in all states. Equivalently, $(f_i)_{i\in N}$ is a Dutch book if $\sum_i f_i(\omega) > 0$ for all $\omega \in E$ and $\sum_i f_i(\omega) < 0$ for all $\omega \notin E$.

Note that by ignorance, we can define a profile of types $(t_i)_{i\in N}$, such that for each i and $\pi \in \Pi_i$, $t_i^\omega(E)$ is close to 1 if f_i is negative on π, and close to 0 if f_i is positive on π. Thus, $\mathbf{E}^{t_i^\pi}(f_i C) < 0$ for each i; that is, the expected payoff is positive in all states for every agent and the agents are always willing to trade. Obviously, the existence of a Dutch book on E, like the possibility of agreeing to agree on E, is a property that depends only on the knowledge space and the event E.

Theorem 1. *In a finite knowledge space where all agents are ignorant of E, agreeing to agree is possible for E if and only if there is no Dutch book on E.*

3.2. Remarks

Variation on Definition 1: Lehrer and Samet (2011) have a slightly different definition of the possibility of agreeing to agree for E. According to their definition, it is required that common knowledge of agreement has a positive prior probability. Formally, it is required that for some p and for a common prior P of the type profile $(t_i)_{i\in N}$, $P(K^\infty(A)) > 0$, where $A = \{\omega \mid t_i^\omega(E) = p,\ \text{for all } i \in N\}$. Obviously, this condition is implied by Definition 1. Moreover, this condition holds when Definition 1 holds for a subspace which consists of some elements of the meet. Thus, all our results can be easily restated with slight variations using the definition of Lehrer and Samet (2011).

Agreeing to agree with $p = 0$ and $p = 1$: We require in Definition 1 that in all states the posterior of all agents is a non-trivial p, that is, $p \in (0,1)$. The cases $p = 0$ and $p = 1$ are much simpler. A necessary and sufficient condition that $t_i^\pi(E) = 0$ for all i and $\pi \in \Pi_i$ is that all i do not know E in all states. That is, for each i and $\pi \in \Pi_i$, $\pi \cap \neg E \neq \emptyset$. Necessity is obvious. For sufficiency, let P be any probability for which $P(E) = 0$. The posteriors for $\pi \in \Pi_i$ such that $P(\pi) > 0$ must satisfy $t_i^\pi(E) = 0$ and in all other π's we can define arbitrarily the types such that this equality holds, in virtue of the condition. Similarly, a necessary and sufficient condition that $t_i^\pi(E) = 1$ for all i and $\pi \in \Pi_i$ is that all i do not know $\neg E$ in all states.

Dutch books and bets: A bet $(b_i)_{i\in N}$, consists of real valued functions b_i on Ω such that $\sum_i b_i = 0$. The bet is agreeable if for each i, $\mathbf{E}^{t_i^\omega}(b_i) > 0$. Unlike Dutch books, there is no measurability requirements on bets. Dutch books and bets are related in the following way. For a given Dutch book $(f_i)_{i\in N}$, add an agent 0 to the set of agents, with partition $\Pi_0 = \{\Omega\}$ and arbitrary type function t_0^Ω. Define for $i \in N$, $b_i = -f_i C$, and $b_0 = \sum_{i\in N} f_i C$. Then $\sum_{i\in N\cup\{0\}} b_i = 0$. Since $(f_i)_{i\in N}$ is a Dutch book, agent 0 has positive expectation of her bet; moreover, with ignorance, there are type functions $(t_i)_{i\in N}$ under which each agent $i \in N$ has positive expectation of b_i. Thus, $(b_i)_{i\in N\cup\{0\}}$ is an agreeable bet. Note, that although the b_i's for $i \in N$ are measurable, b_0 may not be measurable with respect to Π_0. The relation of Dutch books to bets shows that the type functions $(t_i)_{i\in N}$ cannot have a common prior. Indeed, if P is a common prior of $(t_i)_{i\in N}$, then define $t_0^\Omega = P$. Then all the agents in $N \cup \{0\}$ have a common prior and an agreeable bet which is impossible (see Morris, 1994; Samet, 1998, and Feinberg, 2000).

4. Countable knowledge space

For countable spaces we need to distinguish between bounded and unbounded Dutch books. A Dutch book $(f_i)_{i\in N}$ is *bounded* if $f_i \in l_\infty(\Omega)$ for each $i \in N$. However, for the case of two agents, the difference between the two does not matter.

Proposition 2. *In a countable knowledge space with two agents, if there is a Dutch book on E, there is a bounded Dutch book on E.*

Proof. Suppose that (f_1, f_2) is a Dutch book. That is, $f_1(\omega) + f_2(\omega)$ is positive for $\omega \in E$ and negative for $\omega \in \neg E$. Let $g : \mathbb{R} \to \mathbb{R}$ be a strictly increasing and bounded function and define $g_1 = g \circ f_1$ and $g_2 = (-g) \circ (-f_2)$. Then, g_1 and g_2 are bounded. Moreover, as $f_1 > -f_2$ on E, and $f_1 < -f_2$ on $\neg E$, it follows by the monotonicity of g, that $g_1 > -g_2$ on E, and $g_1 < -g_2$ on $\neg E$. Thus, (g_1, g_2) is a bounded Dutch book on E. □

We show in Example 5 that Proposition 2 does not hold for more than two agents. The case of two agents is also special because the equivalence in Theorem 1 for finite knowledge spaces extends to countable spaces with two agents.

Theorem 2. *In a countable knowledge space with two agents where all agents are ignorant of E, agreeing to agree is possible for E if and only if there is no Dutch book on E.*

With more than two agents the "only if" part of Theorem 2 still holds.

Proposition 3. *In a countable knowledge space, if there exists a bounded Dutch book on E, then agreeing to agree is impossible for E.*

The proof can be taken verbatim from the proof of Theorem 2, since the expectations in the latter can be taken also in the infinite case when the Dutch book is bounded. However, the following example shows that the converse does not hold. In this example agreeing to agree is impossible, yet, there is no Dutch book, not even unbounded.

Example 4. Consider three firms $\{1, 2, 3\}$. Each firm is involved in two projects that yield profits x_i and y_i in $\mathbb{Q}_{[-1,1]}$, the set of rational numbers in the interval $[-1, 1]$. Thus, the set of states Ω consists of tuples $(x_1, y_1, x_1, y_1, x_1, y_1) \in \mathbb{Q}^6_{[-1,1]}$. Let $X_i, Y_i : \Omega \to \mathbb{Q}_{[-1,1]}$ be the natural projections. We assume that each firm knows only its own profits, and thus the partition of firm i, Π_i, is generated by (X_i, Y_i).

Let E be the event that $(x_1 + x_2 + x_3, y_1 + y_2 + y_3) > (0, 0)$ in the lexicographic order, that is,

$$E = \{X_1 + X_2 + X_3 > 0\} \cup \{X_1 + X_2 + X_3 = 0 \text{ and } Y_1 + Y_2 + Y_3 > 0\}.$$

We first show that agreeing to agree is impossible for E. Assume by contradiction that for some $P \in \Delta(\Omega)$, $\mathbf{E}^P(C|\Pi_i) = 0$ a.s., then:

$$\mathbf{E}^P X_i C = \mathbf{E}^P \mathbf{E}^P (X_i C|\Pi_i) = \mathbf{E}^P X_i \mathbf{E}^P (C|\Pi_i) = 0$$

for every i. Therefore,

$$\mathbf{E}^P (X_1 + X_2 + X_3)C = 0. \tag{1}$$

Similarly,

$$\mathbf{E}^P (Y_1 + Y_2 + Y_3)C = 0. \tag{2}$$

But the definition of C implies that $(X_1 + X_2 + X_3)C \geq 0$. From this inequality and (1), it follows that $(X_1 + X_2 + X_3)C = 0$ a.s. which implies that $X_1 + X_2 + X_3 = 0$ a.s. Thus, by the definition of E, $E = \{Y_1 + Y_2 + Y_3 > 0\}$ a.s. By this equality and the definitions of C, it follows that $(Y_1 + Y_2 + Y_3)C \geq 0$ a.s. Moreover, since $P(\{Y_1 + Y_2 + Y_3 > 0\}) = P(E) > 0$, it follows that $P((Y_1 + Y_2 + Y_3)C > 0$ which contradicts (2).

We now show that there exists no Dutch book on E. The existence of a Dutch book means that there are functions $g_i : \mathbb{Q}_{[-1,1]} \times \mathbb{Q}_{[-1,1]} \to \mathbb{R}$ such that $\sum_i g_i(x_i, y_i) > 0$ if $(x_1, y_1, x_2, y_2, x_3, y_3) \in E$ and $\sum_i g_i(x_i, y_i) < 0$ otherwise. Consider a sequence (a_k) of positive numbers in $\mathbb{Q}_{[-1,1]}$ such that $2a_{k+1} < a_k$ for each k. Then, as $(a_k, 0, -a_{k+1}, 0, -a_{k+1}, 0) \in E$

$$g_1(a_k, 0) + g_2(-a_{k+1}, 0) + g_3(-a_{k+1}, 0) > 0.$$

Two similar inequalities hold when we start with $g_2(a_k, 0)$ and $g_3(a_k, 0)$. Summing the three inequalities, and denoting $g(x, y) = g_1(x, y) + g_2(x, y) + g_3(x, y)$, we conclude that $g(a_k, 0) > -2g(-a_{k+1}, 0)$. Similarly, as $(-a_{k+1}, 0, a_{k+2}, 0, a_{k+2}, 0) \in \neg E$ it follows that $g(-a_{k+1}, 0) + 2g(a_{k+2}, 0) < 0$. The two inequalities imply that $g(a_k, 0) > 4g(a_{k+2}, 0)$ for all k. As $(a_k, 0, a_k, 0, a_k, 0) \in E$, it follows that $g(a_k, 0) > 0$ for each k. This with the last inequality implies that

$$\lim_{k \to \infty} g(a_k, 0) = 0. \tag{3}$$

Now, for each k, $(a_k, 0, 0, -1, 0, -1) \in E$, and therefore $g(a_k, 0) + 2g(0, -1) > 0$. From (3) it follows that $g(0, -1) \geq 0$. However, this is a contradiction since $(0, -1, 0, -1, 0, -1) \in \neg E$.

The next example shows that Proposition 3 cannot be strengthened by dropping the boundedness of the Dutch book. In this example there exists an unbounded Dutch book, however, agreeing to agree is possible for E.

Example 5. The set of agents is $\{1, 2, 3\}$, and $\Omega = \mathbb{N} \times \{1, 2, 3\}$, where $\mathbb{N}$ is the set of nonnegative integers. The partition of player i, Π_i, consists of sets $\{(n, i), (n+1, i'), (n+1, i'')\}$ for $n > 0$, and the set $\{(0, 1), (0, 2), (0, 3), (1, i'), (1, i'')\}$, where i' and i'' are the two agents other than i. Consider the event $E = \{(n, i) | n \text{ is even}, i \in \{1, 2, 3\}\}$.

To see that agreeing to agree is possible for E, consider $P \in \Delta(\mathbb{N})$ that satisfies for each i and $n > 0$, $P(n, i) = \frac{1}{2^{n+2}}$, and $P(0, i) = \frac{1}{12}$. Then for each i and π in Π_i, $P(E \mid \pi) = \frac{1}{2}$. To see that there exists an unbounded Dutch book, consider for each i the Π_i-measurable function f_i which takes the value $(-3)^n$ on $\{(n, i), (n+1, i'), (n+1, i'')\}$ for $n > 0$ and the value 1 on $\{(0, 1), (0, 2), (0, 3), (1, i'), (1, i'')\}$.

In Theorem 1, we show that the possibility of agreeing to agree implies that there is no Dutch book. The proof can be carried out for countable spaces if the existence of the expectations is guaranteed. In case the common prior which makes possible agreeing to a agree has a finite support, the expectations exist even for unbounded functions. Hence, this proof implies the following.

Proposition 4. *In a countable knowledge space, if there exists a Dutch book on E, then agreeing to agree with a finite support common prior is impossible for E.*

Example 4 shows that the converse implication does not hold. Lehrer and Samet (2011) showed that for two agents, agreeing to agree is possible if and only if it is possible with a common prior that has a finite support. Example 5 shows that this does not hold for countable spaces. In this example, it follows by Proposition 4 that agreeing to agree with a common prior with a finite support is impossible. Yet, in this example agreeing to agree is possible.

The previous examples suggest that we need to weaken the notion of Dutch book to characterize agreeing to agree in a countable state space with multiple agents. Following the idea in Lehrer and Samet (2014), we obtain such a characterization under a mild restriction on the structure of the knowledge space. In what follows, we adopt the convention that the supremum over an empty set is $-\infty$.

Definition 3. Let $\mathcal{F}_\varepsilon$ be the set of profiles of interim contracts $(f_i)_{i\in N}$ such that $f_i \in l_\infty(\Omega)$ for each $i \in N$ and $\sum_{i\in N} f_i C > -\varepsilon$. We say that there is a bounded weak Dutch book on E if

$$\inf_{\varepsilon>0} \sup_{(f_i)\in\mathcal{F}_\varepsilon} \sum_{i\in N} f_i C(\omega) = \infty, \forall \omega \in \Omega.$$

If there is a bounded Dutch book $(f_i)_{i\in N}$ on E, then there is a bounded weak Dutch book on E. Indeed, since $\sum_{i\in N} f_i C > 0$, it follows that $(M f_i)_{i\in N} \in \mathcal{F}_\varepsilon$ for all $\varepsilon > 0$ and $M > 0$. Thus, for each ω,

$$\inf_{\varepsilon>0} \sup_{\left(\bar{f}_i\right)\in\mathcal{F}_\varepsilon} \sum_{i\in N} \bar{f}_i C(\omega) \geq \sup_{M>0} \sum_{i\in N} M f_i C(\omega) = \infty.$$

Remark 1. In Lehrer and Samet (2014, p. 171), a bet $(b_i)_{i\in N}$ is said to be ε-agreeable if for each i, $\mathbf{E}_i(b_i)(\omega) > -\varepsilon$. The set of all ε-agreeable bets is denoted by $\mathcal{B}_\varepsilon$. Lehrer and Samet (2014, p. 171) say that a profile of type functions $(t_i)_{i\in N}$ is not strongly trade consistent if $\inf_{\varepsilon>0} \sup_{(b_i)_{i\in N}\in\mathcal{B}_\varepsilon} \mathbf{E}_i(b_i)(\omega) = \infty$ for every i and every ω. Analogous to Section 3.2, existence of a bounded weak Dutch book and strong trade consistency are related in the following way. Add an agent 0 to the set of agents, with partition $\Pi_0 = \{\Omega\}$ and arbitrary type function t_0^Ω. For any interim contract $(f_i)_{i\in N} \in \mathcal{F}_\varepsilon$, define for $i \in N$, $b_i = -f_i C$, and $b_0 = \sum_{i\in N} f_i C$. Then $\sum_{i\in N\cup\{0\}} b_i = 0$. Moreover, with ignorance, there are type functions $(t_i)_{i\in N}$ (independently of f_i) under which each agent $i \in N$ has positive expectation of b_i. Hence, $(b_i)_{i\in N} \in \mathcal{B}_\varepsilon$. Suppose that there is a bounded weak Dutch book. Then, $\inf_{\varepsilon>0} \sup_{(f_i)\in\mathcal{F}_\varepsilon} \sum_{i\in N} f_i C(\omega) = \infty$ for every $\omega \in \Omega$. It follows that $\inf_{\varepsilon>0} \sup_{(b_i)_{i\in N\cup\{0\}}\in\mathcal{B}_\varepsilon} \mathbf{E}_0(b_0)(\omega) = \infty$. Thus, $(t_i)_{i\in N\cup\{0\}}$ is not strongly trade consistent.

We say that a knowledge space $\left(\Omega, (\Pi_i)_{i\in N}\right)$ is *locally finite* if for each i and $\pi \in \Pi_i$, π is a finite set. We now show that in a countable and locally finite knowledge space, the possibility of agreeing to agree for E is equivalent to the absence of bounded weak Dutch books on E.[7] The proof is similar to Theorem 4 in Lehrer and Samet (2014) and we provide a sketch of the proof in Appendix A.

Theorem 3. *In a countable and locally finite knowledge space where all agents are ignorant of E, agreeing to agree is possible for E if and only if there is no bounded weak Dutch book on E.*

[7] It is straightforward to see that the result still holds if for all $\omega \in \Omega$ and for all $i \in N$, $\Pi_i(\omega) \cap E$ and $\Pi_i(\omega) \cap \neg E$ is a finite union of sets in the join (coarsest common refinement) of the agents' information partitions and the partition $\{E, \neg E\}$.

The "only if" part holds even without the assumption of local finiteness, by Proposition 3. But, unlike Theorem 4 in Lehrer and Samet (2014), the "if" part of Theorem 3 fails without the local finiteness restriction. This can be seen in Example 4, where local finiteness does not hold. We will show here that $\sum_{i\in N} f_i C(\omega_0) < 3\varepsilon$, where $\omega_0 = (0,-1,0,-1,0,-1)$ for any $(f_i)_{i\in N} \in \mathcal{F}_\varepsilon$, implying that there is no bounded weak Dutch book on E. Yet, as we have shown, agreeing to agree is impossible for E, in this example.

Let $(g_i)_{i\in N}$ be the functions in Example 4 that describe a profile of interim contracts $(f_i)_{i\in N}$. Assume that this profile is in $\mathcal{F}_\varepsilon$. For the sequence (a_k) in this example it follows that $g(a_k,0) > -2g(-a_{k+1},0) - 3\varepsilon$ and $-g(-a_{k+1},0) > 2g(a_{k+2},0) - 3\varepsilon$. Hence, $g(a_k,0) > 4g(a_{k+2},0) - 9\varepsilon$. This implies that $\limsup_k g(a_k,0) < 3\varepsilon$. In particular, for some k, $g(a_k,0) < 3\varepsilon$. Moreover, $g(a_k,0) + 2g(0,-1) > -3\varepsilon$. Hence, $g(0,-1) > -3\varepsilon$. Since $\sum_{i\in N} f_i C(\omega_0) = -g(0,-1)$, it follows that $\sum_{i\in N} f_i C(\omega_0) < 3\varepsilon$.

Appendix A

Proof of Theorem 1. Let

$$A = \left\{ \sum_{i\in N} f_i C : (f_i)_{i\in N} \text{ is a profile of interim contracts} \right\}.$$

Obviously, there is no Dutch book on E if and only if A is disjoint from $\mathbb{R}^\Omega_{++}$, the strict positive orthant of $\mathbb{R}^\Omega$. We show that A and $\mathbb{R}^\Omega_{++}$ are disjoint if and only if agreeing to agree is possible for E.

Suppose that agreeing to agree is possible for E. Then, by Claim 3 there exists P such that $\mathbf{E}^P(C|\Pi_i) = 0$ a.s. for every $i \in N$. Let $(f_i)_{i\in N}$ be interim contracts. Then,

$$\mathbf{E}^P(\sum_i f_i C) = \sum_i \mathbf{E}^P(f_i C) = \sum_i \mathbf{E}^P(\mathbf{E}^P(f_i C|\Pi_i)) = \sum_i \mathbf{E}^P(f_i \mathbf{E}^P(C|\Pi_i)) = \sum_i \mathbf{E}^P(f_i \cdot 0) = 0. \quad (4)$$

Thus, $\sum_i f_i C$ cannot be positive everywhere, i.e., $\mathbb{R}^\Omega_{++} \cap A \neq \emptyset$.

Conversely, suppose that $\mathbb{R}^\Omega_{++}$ and A are disjoint. Since both sets are convex, there is a non-zero vector P in $\mathbb{R}^\Omega_{++}$ that separates them[8]: $\langle P, f\rangle \le 0 \le \langle P, r\rangle$ for every $f \in A$ and $r \in \mathbb{R}^\Omega_{++}$. The reason for the constant 0 in these inequalities is that $0 \in A$ and 0 is in the closure of $\mathbb{R}^\Omega_{++}$. Now, vectors of the form $\varepsilon \mathbf{1}_\Omega + \mathbf{1}_{\{\omega\}}$ are in $\mathbb{R}^\Omega_{++}$ for every $\omega \in \Omega$ and $\varepsilon > 0$. Thus, $0 \le \langle P, (\varepsilon \mathbf{1}_\Omega + \mathbf{1}_\omega)\rangle = \varepsilon + P(\{\omega\})$, and we conclude that P is non-negative. Since it is non-zero we can assume that P is a probability distribution.

Fix i and $\pi \in \Pi_i$. By setting $f_i = \pm \mathbf{1}_\pi$ and $f_j = 0$ for $j \neq i$ one may infer that $\pm \mathbf{1}_\pi C \in A$. Thus, $\langle P, \mathbf{1}_\pi C\rangle \le 0$ and $\langle P, \mathbf{1}_\pi C\rangle \ge 0$, which implies that $\langle P, \mathbf{1}_\pi C\rangle = \mathbf{E}^P(\mathbf{1}_\pi C) = 0$. This implies that when $P(\pi) > 0$, $t_i^\pi(E) = 1/2$. Since, all agents are ignorant of E, Claim 2 implies that agreeing to agree is possible for E. □

Proof of Theorem 2. A *chain* is a finite sequence $\pi^1, \ldots, \pi^k$ of elements of $\Pi_1 \cup \Pi_2$, such for each $i \ge 1$, π_i and π_{i+1} belong to different partitions, and if $\pi_i \in \Pi_1$, then $\pi_i \cap \pi_{i+1} \cap E \neq \emptyset$, and when $\pi_i \in \Pi_2$, then $\pi_i \cap \pi_{i+1} \cap \neg E \neq \emptyset$. If there is a chain in which $\pi_1 = \pi_k$, then we can construct a loop, as in Claim 1, and therefore agreeing to agree is possible for E.

Suppose that agreeing to agree is impossible for E. Define a binary relation $\succ$ on $\Pi_1 \cup \Pi_2$, where $\pi \succ \pi'$ when there is a chain that starts at π and ends at π'. Clearly, $\succ$ is transitive. Moreover, since we assumed that agreeing to agree is impossible for E, there is no loop on E and thus $\succ$ is irreflexive. Hence, $\succ$ is a strict partial order. By Fishburn (1970) [Theorem 2.5], there is a function $\gamma : \Pi_1 \cup \Pi_2 \to \mathbb{R}$ such that $\pi \succ \pi'$ implies $\gamma(\pi) > \gamma(\pi')$. Now define an interim contract (f_1, f_2) by

$$f_1(\pi) = \gamma(\pi) \text{ for } \pi \in \Pi_1; \; f_2(\pi) = -\gamma(\pi) \text{ for } \pi \in \Pi_2.$$

To see that (f_1, f_2) is a Dutch book note, that for each $\omega \in E$, $\Pi_1(\omega) \cap \Pi_2(\omega) \cap E \neq \varnothing$. Hence, $\Pi_1(\omega) \succ \Pi_2(\omega)$. Thus, $\gamma(\Pi_1(\omega)) > \gamma(\Pi_2(\omega))$ and hence $f_1(\Pi_1(\omega)) + f_2(\Pi_2(\omega)) > 0$. Similarly, for each $\omega \in \neg E$, $\Pi_1(\omega) \cap \Pi_2(\omega) \cap \neg E \neq \varnothing$, and thus $f_1(\Pi_1(\omega)) + f_2(\Pi_2(\omega)) < 0$.

Now, assume that there is a Dutch book $(f_i)_{i\in N}$ on E, which by Proposition 2, we may assume is bounded. Suppose to the contrary, agreeing to agree is possible for E, then, by the boundedness of $(f_i)_{i\in N}$, we can show, as in 4, that there exists P such that $\mathbf{E}^P\left(\sum_{i\in N} f_i C\right) = 0$. Since $\sum_{i\in N} f_i C(\omega) > 0$ for all ω, it follows that $\mathbf{E}^P\left(\sum_{i\in N} f_i C\right) > 0$. We have a contradiction. □

Proof of Theorem 3. The "only if" part is similar to the proof of the "only if" part in Theorem 2. For the "if" part, define, as in the Proof of Theorem 1, $A = \left\{\sum_{i\in N} f_i C : (f_i)_{i\in N} \text{ is a profile of interim contracts}\right\}$.

[8] $\langle a, b\rangle$ denotes the inner product of a and b. In particular, if a is a probability vector, then $\langle a, b\rangle$ is the expectation of b with respect to the probability a, $\mathbf{E}^a(b)$.

Let L_- be the strict negative orthant of $l_\infty(\Omega)$ and B be the unit ball in $l_\infty(\Omega)$. Then, similar to the proof of Lehrer and Samet (2014, Theorem 4), by the nonexistence of a bounded weak Dutch book on E, we can show that for some $\varepsilon > 0$ and ω_0, $\text{conv}\left(A, \mathbf{1}_{\omega_0} + \varepsilon B\right) \cap L_- = \varnothing$. Since both $\text{conv}\left(A, \mathbf{1}_{\omega_0} + \varepsilon B\right)$ and L_- are convex and have non-empty interior, there is a nonzero continuous functional φ on $l_\infty(\Omega)$ (namely, a finitely additive measure on Ω) such that $0 < \sum_{\omega\in\Omega} \varphi(\mathbf{1}_\omega) < \infty$ and $\varphi(g) \geq 0 \geq \varphi(h)$ for all $g \in \text{conv}\left(A, \mathbf{1}_{\omega_0} + \varepsilon B\right)$ and $h \in L_-$. Define a probability function P on Ω by $P(\omega) = \varphi(\mathbf{1}_\omega) / \sum_{\omega'\in\Omega} \varphi(\mathbf{1}_{\omega'})$.

To see that agreeing to agree is possible for E under P, set, for each $\pi \in \Pi_i$, $f_i = \mathbf{1}_\pi$ and $f_j = 0$ for $j \neq i$. Then, $f_i C \in A$, Similarly, $-f_i C \in A$. Thus, $\varphi(\mathbf{1}_\pi C) = 0$. That is, $\varphi(\mathbf{1}_{\pi\cap E}) - \varphi(\mathbf{1}_{\pi\cap\neg E}) = 0$. Since the knowledge space is locally finite, each π is finite. Since φ is finitely additive, it follows that $\varphi(\mathbf{1}_{\pi\cap E}) = \sum_{\omega\in\pi\cap E} \varphi(\mathbf{1}_\omega)$, and similarly $\varphi(\mathbf{1}_{\pi\cap\neg E}) = \sum_{\omega\in\pi\cap\neg E} \varphi(\mathbf{1}_\omega)$. Dividing $\varphi(\mathbf{1}_{\pi\cap E}) - \varphi(\mathbf{1}_{\pi\cap\neg E}) = 0$ by $\sum_{\omega'\in\Omega} \varphi(\mathbf{1}_{\omega'})$, we get

$$P(\pi \cap E) - P(\pi \cap \neg E) = 0.$$

Hence, agreeing to agree is possible for E under P. □

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