Quasicrystals

Homework assignment No. 2 February 29, 2000
Ron Lifshitz (ronlif@post.tau.ac.il)

1. **Normal subgroups:** We said in class that a subgroup N of a group G is called a *normal subgroup* (denoted by $N \triangleleft G$) if all its right cosets are equal to its left cosets, *i.e.*

$$\forall a \in G: \quad Na = aN. \tag{1}$$

Equivalently, by multiplying (1) by a^{-1} from the right, we see that N is a normal subgroup of G if $aNa^{-1} = N$, or

$$\forall a \in G, n \in N: \quad ana^{-1} \in N. \tag{2}$$

Because of this property, normal subgroups are also called *invariant subgroups.*

a) Show that if $N \triangleleft G$ then the cosets of N in G form a group under set multiplication. This group is called the *quotient group* or the *factor group* of G by N, and denoted by G/N.

b) Show, by giving a counterexample, that if H is a subgroup of G which is *not* normal, its (left or right) cosets do not form a group.

2. **Space groups:** The space group $\mathcal{S}$ of a periodic crystal in n dimensions contains all elements of the form $(r|\mathbf{t})$, leaving the crystal invariant, where $r \in O(n)$ is an n-dimensional, proper ($\det r = 1$) or improper ($\det r = -1$), rotation, and $\mathbf{t}$ is an n-dimensional translation. If $\mathbf{x}$ is any point in the crystal, then the operation $(r|\mathbf{t})$ on $\mathbf{x}$ is defined by

$$(r|\mathbf{t})\mathbf{x} = r\mathbf{x} + \mathbf{t}. \tag{3}$$

a) Show that the multiplication rule for elements in $\mathcal{S}$ is

$$(r_1|\mathbf{t}_1)(r_2|\mathbf{t}_2) = (r_1 r_2|\mathbf{t}_1 + r_1\mathbf{t}_2). \tag{4}$$

b) What is the inverse of $(r|\mathbf{t})$?

c) Show that all the pure translations $(e|\mathbf{t}) \in \mathcal{S}$, where e is the identity rotation, form an abelian normal subgroup of $\mathcal{S}$.

d) Show that all the pure rotations $(r|\mathbf{0})$, where $\mathbf{0}$ is the zero translation, form a subgroup of $\mathcal{S}$ which is not normal.

3. **Centralizers and Normalizers:** Let S be an arbitrary subset of a group G. The *centralizer* of S in G, denoted $C(S)$, is the subset of G containing all elements which commute with all the elements in S, namely

$$C(S) = \{a \in G | \forall s \in S: \ as = sa\}. \tag{5}$$

a) Show that $C(S)$ is a subgroup of G.

The centralizer of a single element $s \in G$, $C(s)$, is also called the *normalizer* of s in G. The centralizer of the whole group $C(G)$ is called the *center* of the group G.

b) Let $R_z(\theta)$ be a 3-dimensional rotation about the z axis by an angle θ. Find the normalizer of $R_z(\theta)$ in $O(3)$, assuming $\theta \neq \pi$.