Solution for the Quantum Physics 1 Exam from Sept. 5, 2004

1. The solution of the BCS equation has been found in an exercise to be

$$u_k^2 = 1 - v_k^2 \ , \quad v_k^2 = \frac{1}{2}\left(1 - \frac{\epsilon - \lambda}{\sqrt{(\epsilon-\lambda)^2 + \Delta^2}}\right) \ .$$

   Additionally, as the number of particles in the BCS state is not fixed, the average number of particles has to be fixed as N:

$$\langle N \rangle = 2\sum_{k>0} v_k^2 = \Omega\left(1 - \frac{\epsilon - \lambda}{\sqrt{(\epsilon-\lambda)^2 + \Delta^2}}\right) = N \ . \tag{1}$$

   From the definition of Δ

$$\Delta = G\sum_{k>0} u_k v_k = \frac{1}{2} G\Delta \sum_{k>0} \frac{1}{\sqrt{(\epsilon-\lambda)^2 + \Delta^2}}$$

   so

$$\frac{G\Omega}{\sqrt{(\epsilon-\lambda)^2 + \Delta^2}} = 2 \ . \tag{2}$$

   By substituting (2) in (1) we get

$$2\left(\epsilon - \lambda\right) = G\left(\Omega - N\right) \ , \tag{3}$$

   and by using (3) in (2)

$$\frac{G\Omega}{\sqrt{G^2\left(\Omega - N\right)^2 + 4\Delta^2}} = 1 \ .$$

   Finally we have

$$\begin{aligned} \Delta &= \frac{1}{2} G\sqrt{N(2\Omega - N)} \ , \\ v_k^2 &= \frac{N}{2\Omega} \ . \end{aligned}$$

2. The ground state energy is

$$E = \langle \mathrm{BCS}|H|\mathrm{BCS}\rangle = 2\sum_{k>0} \epsilon v_k^2 - G\left(\sum_{k>0} u_k v_k\right)^2 = N\epsilon - \frac{1}{4} GN(2\Omega - N) \ .$$

3. The lowest order kinetic correction is

$$E_0^{(1)} = \left\langle \psi \middle| \left(-\frac{\mathbf{p}^4}{8m^3c^2}\right) \middle| \psi \right\rangle \ .$$

Note that $H\,|\psi\rangle = E_0\,|\psi\rangle$ and $H = \frac{\mathbf{p}^2}{2m} - \frac{e^2}{r}$ so

$$\begin{aligned} E_0^{(1)} &= -\frac{1}{2mc^2}\left\langle \psi \middle| \left(H + \frac{e^2}{r}\right)^2 \middle| \psi \right\rangle = -\frac{1}{2mc^2}\left\langle \left(E_0 + \frac{e^2}{r}\right)^2 \right\rangle \\ &= -\frac{1}{2mc^2}\left(E_0^2 + 2e^2 E_0 \left\langle \frac{1}{r} \right\rangle + e^4 \left\langle \frac{1}{r^2} \right\rangle\right) . \end{aligned}$$

Since $\left\langle \frac{1}{r} \right\rangle = \frac{1}{a_0}$ and $\left\langle \frac{1}{r^2} \right\rangle = \frac{2}{a_0^2}$,

$$E_0^{(1)} = -\frac{1}{2mc^2}\left(E_0^2 + \frac{2e^2}{a_0}E_0 + \frac{2e^4}{a_0^2}\right) .$$

4. The field-dependent corrections are

$$E_0^{(2)} = \left\langle -\frac{e\hbar}{2m^2c^2 r}\frac{d\Phi}{dr}\mathbf{L}\cdot\mathbf{S} - \frac{e\hbar^2}{8m^2c^2}\nabla\cdot\mathbf{E} \right\rangle .$$

The ground state of the hydrogen atom is an $L = 0$ state, and hence $\mathbf{L}\cdot\mathbf{S} = 0$ — the first term vanishes. The second term involves $\nabla\cdot\mathbf{E} = -\nabla^2\Phi = 4\pi e\delta^3(\mathbf{r})$, from which follows

$$E_0^{(2)} = -\frac{\pi e^2\hbar^2}{2m^2c^2}\psi^*(0)\psi(0) = -\frac{e^2\hbar^2}{2m^2c^2a_0^3} .$$

5. (a) The final state is $\frac{1}{\sqrt{2}}(|0_a, 1_b\rangle + i\,|1_a, 0_b\rangle)$. If a single photon is found in b, then no photons are found at a.

   (b) Using second quantization formalism for bosons

$$a_1^\dagger a_2^\dagger\,|0\rangle = \frac{i}{2}\left(a_a^\dagger a_a^\dagger + a_b^\dagger a_b^\dagger\right)|0\rangle = \frac{i}{\sqrt{2}}\left(|2_a, 0_b\rangle + |0_a, 2_b\rangle\right) .$$

   Thus, a one-photon state is never observed at b in the first place.

   (c) $|\alpha = 1\rangle$ behaves like a classical state, so the final state is $\left|\alpha = \frac{i}{\sqrt{2}}\right\rangle_a \left|\alpha = \frac{1}{\sqrt{2}}\right\rangle_b$. The state found at b is immaterial as there is no correlation between the states at a and at b.

$$P = \left|\left\langle n = 1 \middle| \alpha = i/\sqrt{2} \right\rangle\right|^2 = \frac{1}{2e^{1/2}} .$$

   (d) The final state is $\left|\alpha = \frac{1+i}{\sqrt{2}}\right\rangle_a \left|\alpha = \frac{1+i}{\sqrt{2}}\right\rangle_b$. As in 5c the state in b is immaterial and

$$P = \left|\left\langle n = 1 \middle| \frac{1+i}{\sqrt{2}} \right\rangle\right|^2 = e^{-1} .$$

6. (a) The final state is $\frac{1}{\sqrt{2}}(|0_a, 1_b\rangle + i\,|1_a, 0_b\rangle)$. After the measurement at b this states collapses into $|f\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_a + i\,|1\rangle_a\right)$ since $\langle \alpha = 1|0\rangle_b = \langle \alpha = 1|1\rangle_b = e^{-1/2}$. The required probability is

$$P = |\langle \alpha = 1|f\rangle|^2 = \frac{1}{e} .$$

(b) The final state is $\frac{i}{\sqrt{2}}(|2\rangle_a |0\rangle_b + |0\rangle_a |2\rangle_b)$. We have $\langle\alpha = 1|0\rangle_b = e^{-1/2}$ and $\langle\alpha = 1|2\rangle = \frac{e^{-1/2}}{\sqrt{2}}$. Therefore the final state collapses to $|f\rangle = \frac{1}{\sqrt{3}}(\sqrt{2}\,|2\rangle_a + |0\rangle_a)$, so

$$P = |\langle\alpha = 1|f\rangle|^2 = \frac{4}{3e} \ .$$

(c) Again, there is no correlation between the states at a and at b, and using the scalar product of two coherent states from one of the exercises

$$P = \left|\left\langle\alpha = 1|i/\sqrt{2}\right\rangle\right|^2 = e^{-3/2} \ .$$

(d) Once more, because there is no correlation between the states at a and at b,

$$P = \left|\left\langle\alpha = 1\left|\frac{1+i}{\sqrt{2}}\right.\right\rangle\right|^2 = e^{-(2-\sqrt{2})} \ .$$

7. Fermions have no coherent states so only the first two cases are relevant.

(a) This is the same as in the case of bosons.

(b) Using second quantization formalism for fermions ($\{a_a^\dagger, a_b^\dagger\} = \delta_{ab}$):

$$a_1^\dagger a_2^\dagger |0\rangle = a_b^\dagger a_a^\dagger |0\rangle = |1_a, 1_b\rangle \ .$$

As a result one always finds a single fermion both at a and at b.

(c) Irrelevant for fermions.

(d) Irrelevant for fermions.