Exercise No. 7: The Quantized Electromagnetic Field

1. Compute the quantum Hamiltonian operator of the electromagnetic field. Show that it is given by

$$H = \sum_{\mathbf{k}\lambda} \hbar\omega_{\mathbf{k}} \left(a^\dagger_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda} + \frac{1}{2} \right) .$$

2. (a) The angular momentum density of the electromagnetic field is $\mathbf{r}\times\mathbf{g}$ where $\mathbf{g}$ is the field's momentum density. Show that classically the total angular momentum of the field in the absence of charges is

$$\mathbf{J} = \frac{1}{4\pi c} \int d^3r E_i (\mathbf{r} \times \nabla) A_i + \frac{1}{8\pi c} \int d^3r \left(\mathbf{E} \times \mathbf{A} - \mathbf{A} \times \mathbf{E} \right) .$$

   (b) Note that the second term has no explicit $\mathbf{r}$-dependence (it has no "arm") and can be interpreted as the intrinsic angular momentum of the field (its spin). Denote the last term by $\mathbf{S}$. Show that

$$\mathbf{S} = \frac{i\hbar}{2} \sum_{\mathbf{k}} \left(\mathbf{a}_{\mathbf{k}} \times \mathbf{a}^\dagger_{\mathbf{k}} - \mathbf{a}^\dagger_{\mathbf{k}} \times \mathbf{a}_{\mathbf{k}} \right) = \hbar \sum_{\mathbf{k}\lambda} \lambda \hat{\mathbf{k}} a^\dagger_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda} ,$$

   where $\mathbf{a}_{\mathbf{k}} = \sum_i a_{\mathbf{k}i} \hat{\mathbf{e}}_{\mathbf{k}i}$ and $\lambda = \pm 1$ denotes the circular polarization states $\hat{\mathbf{e}}_{\mathbf{k}\pm} = \mp \frac{1}{\sqrt{2}} (\hat{\mathbf{e}}_{\mathbf{k}1} \pm i \hat{\mathbf{e}}_{\mathbf{k}2})$ ($\hat{\mathbf{e}}_{\mathbf{k}1}$ and $\hat{\mathbf{e}}_{\mathbf{k}2}$ are the linear polarization vectors).

   (c) Demonstrate that the only allowable projections of the spin along the photon's momentum are $\hbar\lambda = \pm\hbar$. λ is called the *helicity* of the photon.

3. Find the following commutators:
   (a) $[N_{\mathbf{k}\lambda}, \mathbf{A}]$,
   (b) $[N_{\mathbf{k}\lambda}, \mathbf{E}]$,
   (c) $[N_{\mathbf{k}\lambda}, \mathbf{B}]$,

   where $N_{\mathbf{k}\lambda}$ is the number of photons with momentum $\mathbf{k}$ and polarization λ. What do these results mean?

4. Compute the vacuum expectation values of the fields $\langle \mathbf{E} \rangle$ and $\langle \mathbf{B} \rangle$ and their uncertainties ΔE and ΔB in the vacuum. These are the so-called *vacuum fluctuations* of the electromagnetic field.