

## Exercise No. 6: Quantum Fields

1. The equations of motion of a complex scalar field  $\phi(x)$  are

$$\eta^{\mu\nu} \frac{\partial^2 \phi}{\partial x^\mu \partial x^\nu} = -\frac{m^2 c^2}{\hbar^2} \phi, \quad \eta^{\mu\nu} \frac{\partial^2 \phi^*}{\partial x^\mu \partial x^\nu} = -\frac{m^2 c^2}{\hbar^2} \phi^*,$$

where  $\eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)$  is the Minkowky metric.

- (a) Write the Lagrangian density that reproduces these equations of motion.
  - (b) Perform the quantization of the field.
  - (c) What is the classical Hamiltonian of this field?
  - (d) Find the quantum Hamiltonian.
2. Compute the following commutators for the electromagnetic field:
- (a)  $[A_i(\mathbf{r}, t), \pi_i(\mathbf{r}', t)]$ ,
  - (b)  $[B_i(\mathbf{r}, t), E_j(\mathbf{r}', t)]$ .
3. Show that the propagator of a driven harmonic oscillator whose Lagrangian is  $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2 + f(t)x$  is

$$U(x_b, t_b; x_a, t_a) = \left( \frac{m\omega}{2\pi i \hbar \sin \omega T} \right)^{1/2} e^{iS_{\text{cl}}/\hbar},$$

where

$$\begin{aligned} S_{\text{cl}} = & \frac{m\omega}{2\pi i \hbar \sin \omega T} \left[ (x_a^2 + x_b^2) \cos \omega T - 2x_a x_b \right. \\ & + \frac{2x_b}{m\omega} \int_{t_a}^{t_b} f(t) \sin \omega(t - t_a) dt + \frac{2x_a}{m\omega} \int_{t_a}^{t_b} f(t) \sin \omega(t_b - t) dt \\ & \left. - \frac{2}{m^2 \omega^2} \int_{t_a}^{t_b} dt \int_{t_a}^t ds f(t) f(s) \sin \omega(t_b - t) \sin \omega(s - t_a) \right], \end{aligned}$$

and  $T = t_b - t_a$ .