# Exercise No. 11: Second Quantization III

1. Two electrons are in plane wave states in a box. Calculate to first order in the Coulomb interaction the energy difference of parallel and anti-parallel spin alignments (the exchange interaction).
2. Consider a system of fermions with the Hamiltonian $H = H_0 + V$ where $H_0$ is a single-particle operator and $V$ is a two-particle interaction. Let $|\psi_n\rangle$ be the multi-particle eigenstates of $H$ so that $H|\psi_n\rangle = E_n|\psi_n\rangle$. The Hartree–Fock ground state is $|\Phi\rangle = \prod_{\mu \le F} a_\mu^\dagger |0\rangle$, where $F$ denotes the Fermi level.
   (a) Explain why the binding energy of a particle in a state $\lambda \le F$ is given by $\epsilon_\lambda = \sum_n |\langle \psi_n | a_\lambda | \Phi \rangle|^2 (E_0 - E_n)$ with $E_0 = \langle \Phi | H | \Phi \rangle$ the energy of the Hartree–Fock ground state.
   (b) Show that $\epsilon_\lambda = \langle \Phi | a_\lambda^\dagger [a_\lambda, H] | \Phi \rangle$.
   (c) Prove the relation $E_0 = \frac{1}{2}\langle \Phi | H_0 | \Phi \rangle + \frac{1}{2}\sum_\lambda \epsilon_\lambda$.
3. (a) Calculate to first order in the inter-particle interaction $v(\mathbf{r} - \mathbf{r}')$ the energy of an $(N+1)$-particle system of spin $\frac{1}{2}$ fermions with one particle of momentum $\mathbf{p}$ outside an $N$-particle Fermi sea (a quasi-particle state). Repeat the computation for the state of $N-1$ particles with a particle of momentum $\mathbf{p}$ removed from an $N$-particle Fermi sea (a hole state). Measure the energies from the $N$-particle ground state energy.
   (b) Evaluate the quasi-particle and hole energies for a Coulomb interaction in the jellium model (uniform positive background).