# Theoretical Model for Faraday Waves with Multiple-Frequency Forcing

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A simple generalization of the Swift-Hohenberg equation is proposed as a model for the pattern-forming dynamics of a two-dimensional field with two unstable length scales. The equation is used to study the dynamics of surface waves in a fluid driven by a linear combination of two frequencies. The model exhibits steady-state solutions with 2-, 4-, 6-, and 12-fold symmetric patterns, similar to the periodic and quasiperiodic patterns observed in recent experiments. [S0031-9007(97)03894-5]

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Parametric excitations of surface waves have been extensively studied since their first discovery by Faraday [1] over a century and a half ago. In the basic experimental setup an open container of fluid is subjected to vertical sinusoidal oscillations, which periodically modulate the effective gravity. When the driving amplitude $a$ exceeds a critical threshold $a_c$ a standing-wave instability occurs with temporal frequency $\omega$ one-half that of the driving frequency. The characteristic spatial wavelength of the standing-wave pattern is selected through the dispersion relation $\omega(k)$ of the fluid. One typically observes patterns of stripes or squares in such experiments. It is only in recent years that a variety of additional patterns—some with quasiperiodic rather than periodic long range order—have been observed [2–6]. We focus here on a particular set of experiments, performed by Edwards and Fauve [3], in which a fluid was driven by a linear combination of *two* frequencies, forming periodic patterns with 2-, 4-, and 6-fold symmetry, and quasiperiodic patterns with 12-fold symmetry.

Previous theoretical work [6–11] has focused mainly on a description through amplitude equations with an angle-dependent interaction $\beta(\theta_{ij})$ between pairs of modes. Such an interaction, which is either postulated or derived from the underlying microscopic dynamics, can be chosen to stabilize $N$-fold symmetric patterns for arbitrary $N$. Müller [10] has also used a set of two coupled partial differential equations, where the pattern of a primary field is stabilized by coupling to a secondary field which provides an effective space-dependent forcing. Newell and Pomeau [11] have coupled multiple fields in a similar way. In both cases the coupling between the different fields is achieved through resonant triad interactions, similar to the interactions we shall introduce below.

We propose a simple rotationally invariant model equation, governing the dynamics of a real field $u(x, y, t)$, which describes the amplitude of the standing-wave pattern. Our approach is different in that it searches for the minimal requirements for reproducing the steady states, which are observed in the experiments of Edwards and Fauve [3]. We incorporate into our model only the two most essential aspects of the system. (i) The dynamics is damped at frequencies away from the two forcing frequencies, and therefore the wavelengths involved in the selected pattern lie in narrow bands about two critical wavelengths. (ii) The driving used in the experiments is such that the up-down symmetry, taking $u$ to $-u$, is broken allowing interactions among triplets of standing plane waves to exist. These triad interactions are the *only* stabilizing mechanism for nontrivial patterns in our rotationally invariant model equation.

We capture the essential dynamics with a single field and without *a priori* specifying any angle-dependent interactions among critical modes. This allows for a meaningful comparison of the stability of different $N$-fold symmetric states. We find patterns of 2-, 4-, 6-, and 12-fold symmetry that are globally stable, but none with 8- or 10-fold symmetry, which is in agreement with the experimental observations of Edwards and Fauve [3].

The supercritical instability of a homogeneous state to a striped state is often modeled by the Swift-Hohenberg equation [12]

$$\partial_t u = \varepsilon u - (\nabla^2 + 1)^2 u - u^3, \tag{1}$$

which is variational,

$$\partial_t u = -\delta \mathcal{F}/\delta u, \tag{2}$$

driving the field $u(x, y, t)$ towards a minimum of the Lyapunov functional (effective "free energy")—

$$\mathcal{F} = \int dx\, dy \{-\tfrac{1}{2}\varepsilon u^2 + \tfrac{1}{2}[(\nabla^2 + 1)u]^2 + \tfrac{1}{4}u^4\}. \tag{3}$$

The first term in the Lyapunov functional (3) favors the growth of the instability whereas the quartic term is responsible for its saturation by providing a lower bound for $\mathcal{F}$. The growth rate $\varepsilon$ of the instability is proportional to the reduced driving amplitude $(a - a_c)/a_c$. The positive-definite gradient term is small only near the critical wave number $k_c = 1$, and thus inhibits the growth of any instabilities with wave numbers away from this value.

If the parametric forcing is such that the $u \to -u$ symmetry is broken, then the Swift-Hohenberg free energy is modified by the addition of a cubic term, $-\alpha u^3/3$. Such a term allows triad interactions of standing plane waves to lower the value of $\mathcal{F}$ and form hexagonal patterns. The analysis of the Swift-Hohenberg equation in the presence

  

of this term is summarized, for example, in the review by Cross and Hohenberg [13]. With single-frequency forcing one cannot break the $u \to -u$ symmetry, but with certain combinations of two frequencies the up-down symmetry is broken and triad interactions become important.

We model the two-frequency parametric excitation of a fluid by replacing the wavelength-selecting term in the Swift-Hohenberg equation (1) by a similar term which damps out all modes except those near one of *two* critical wavelengths:

$$\partial_t u = \varepsilon u - c(\nabla^2 + 1)^2(\nabla^2 + q^2)^2 u + \alpha u^2 - u^3. \tag{4}$$

The parameter $c$ can be scaled out, but we include it here because it is used in the numerical simulations, shown later. Other model equations with similar wavelength-selection properties are possible. We choose this equation because it is the simplest one that incorporates the physics we are interested in—it allows two unstable length scales and contains triad interactions among the different modes. Since (4) can be applied to any pattern-forming system satisfying these requirements, it is not our intention to provide a detailed derivation of it from any specific underlying microscopic dynamics.

Let us turn now to an analytic investigation of the model equation (4). When both $\varepsilon$ and $\alpha$ are sufficiently small (or $c$ sufficiently large) the wavelength selection by the gradient term is nearly perfect and the Lyapunov functional may be written in Fourier space as

$$\begin{aligned}\mathcal{F} = &-\frac{1}{2}\varepsilon \sum_{|\mathbf{k}|=1,q} u_{\mathbf{k}} u_{-\mathbf{k}} \\ &-\frac{1}{3}\alpha \sum_{|\mathbf{k}_i|=1,q} u_{\mathbf{k}_1} u_{\mathbf{k}_2} u_{-\mathbf{k}_1-\mathbf{k}_2} \\ &+\frac{1}{4} \sum_{|\mathbf{k}_i|=1,q} u_{\mathbf{k}_1} u_{\mathbf{k}_2} u_{\mathbf{k}_3} u_{-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3}\,,\end{aligned} \tag{5}$$

where the summations are restricted to wave vectors whose magnitude is either 1 or $q$, lying on two rings in Fourier space. The set of Fourier coefficients $u_{\mathbf{k}}$, which give rise to the lowest value of $\mathcal{F}$ for a given choice of the parameters $\varepsilon$ and $\alpha$, determines the most favorable steady-state solution of the model equation (4). We are interested only in finding the global minimum of $\mathcal{F}$, thus establishing that our model indeed predicts the existence of the patterns observed in the two-frequency parametric forcing experiments. Of course, this approach may overlook metastable states or local minima of the free energy. Note that with the omission of the gradient term, one may perform a rescaling of the field $u \to \alpha u$. The rescaled free energy $\alpha^{-4}\mathcal{F}$ is then controlled by a single control parameter

$$\varepsilon^* = \varepsilon/\alpha^2. \tag{6}$$

To study the formation of dodecagonal patterns we choose $q = 2\cos(\pi/12)$, which is the magnitude of the vector sum of two unit vectors separated by an angle of 30°. We minimize the Lyapunov functional (5) with respect to the Fourier coefficients $u_{\mathbf{k}}$ describing four different pattern candidates: (a) a striped pattern with space group $P2mm$, whose Fourier spectrum contains two opposite wave vectors of equal length; (b) a pattern of perfect hexagons with space group $P6mm$, whose Fourier spectrum contains a single 6-fold star of wave vectors; (c) a pattern of compressed hexagons with space group $P2mm$, whose Fourier spectrum contains four vectors on one ring and two vectors on the other ring; and (d) a dodecagonal pattern with space group $P12mm$, whose Fourier spectrum contains two 12-fold stars of wave vectors, one on each ring.

We use standard methods [14] to calculate $\mathcal{F}$ for each of the cases. Because all the candidate patterns have symmorphic space groups [15] which are also centrosymmetric we may always take all the Fourier coefficients on a given ring to be equal and their phases may all be chosen such that they are either 0 or $\pi$. The minimization of the Lyapunov functional is therefore always with respect to no more than two real variables. We find the values of the Lyapunov functional for the different patterns to be

$$\mathcal{F}_2 = -\frac{1}{6}\varepsilon^{*2}, \tag{7a}$$

$$\mathcal{F}_6 = \mathcal{F}_{4-2} = -\frac{4}{15^3}(1 + \sqrt{1 + 15\varepsilon^*}) - \frac{2}{15^2}(3 + 2\sqrt{1 + 15\varepsilon^*})\varepsilon^* - \frac{1}{10}\varepsilon^{*2}, \tag{7b}$$

$$\mathcal{F}_{12} = -\left(\frac{10}{67}\right)^3(1 + \sqrt{1 + 67\varepsilon^*/75}) - \frac{20}{67^2}\left(1 + \frac{2}{3}\sqrt{1 + 67\varepsilon^*/75}\right)\varepsilon^* - \frac{9}{67}\varepsilon^{*2}. \tag{7c}$$

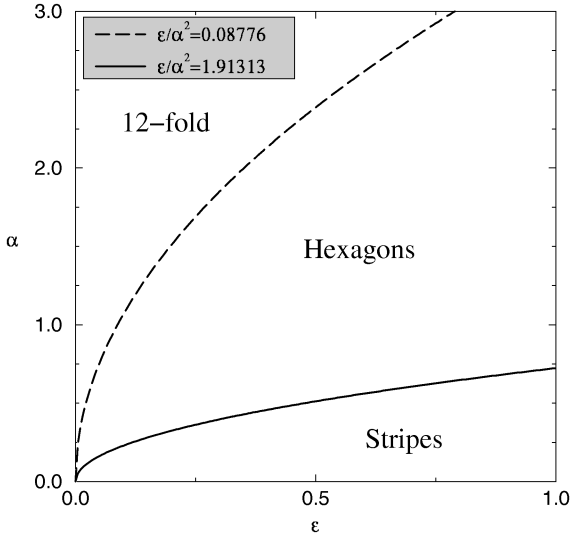


FIG. 1. Phase diagram of the lowest-energy steady-state solutions of the model equation (4) for $q = 2\cos(\pi/12)$. The phase boundaries are lines of constant $\varepsilon^* = \varepsilon/\alpha^2$.

For $\varepsilon^* > 1.913\,13$ the striped pattern has the lowest free energy. For $1.913\,13 > \varepsilon^* > 0.087\,76$ the 6-fold pattern of perfect hexagons and the 2-fold pattern of compressed hexagons (denoted by 4-2), which are degenerate, are most favorable. For $\varepsilon^* < 0.087\,76$ the dodecagonal pattern is the most stable. These analytical results are depicted in the phase diagram of Fig. 1. Note that the phase diagram depicts only the boundaries between global minima; in certain regions of the phase diagram additional states may be locally stable.

The model equation (4), supplemented with periodic boundary conditions, was solved numerically on a square domain using a pseudospectral method. The unit cell was typically chosen so that the simulation region held about 30 wavelengths. The simulation was performed on a $256 \times 256$ grid, with Adams-Bashforth second-order time stepping. The value of $c$ was taken to be between 10 and 100. Figures 2(a)–2(d) show the real-space and Fourier-space results of the simulations with $q = 2\cos(\pi/12)$ for varying values of the control parameter $\varepsilon^*$. The results are consistent with the Lyapunov-functional analysis and the phase diagram of Fig. 1.

Eight-fold and 10-fold symmetric patterns are not observed in our model for any choice of $q$. An analytic calculation of the Lyapunov functional (5) for these patterns shows that it is greater than the free energy $\mathcal{F}_6$ (7b) of the 6-fold state, for any value of the control parameter $\varepsilon^*$. This is in accord with the experiments of Edwards and

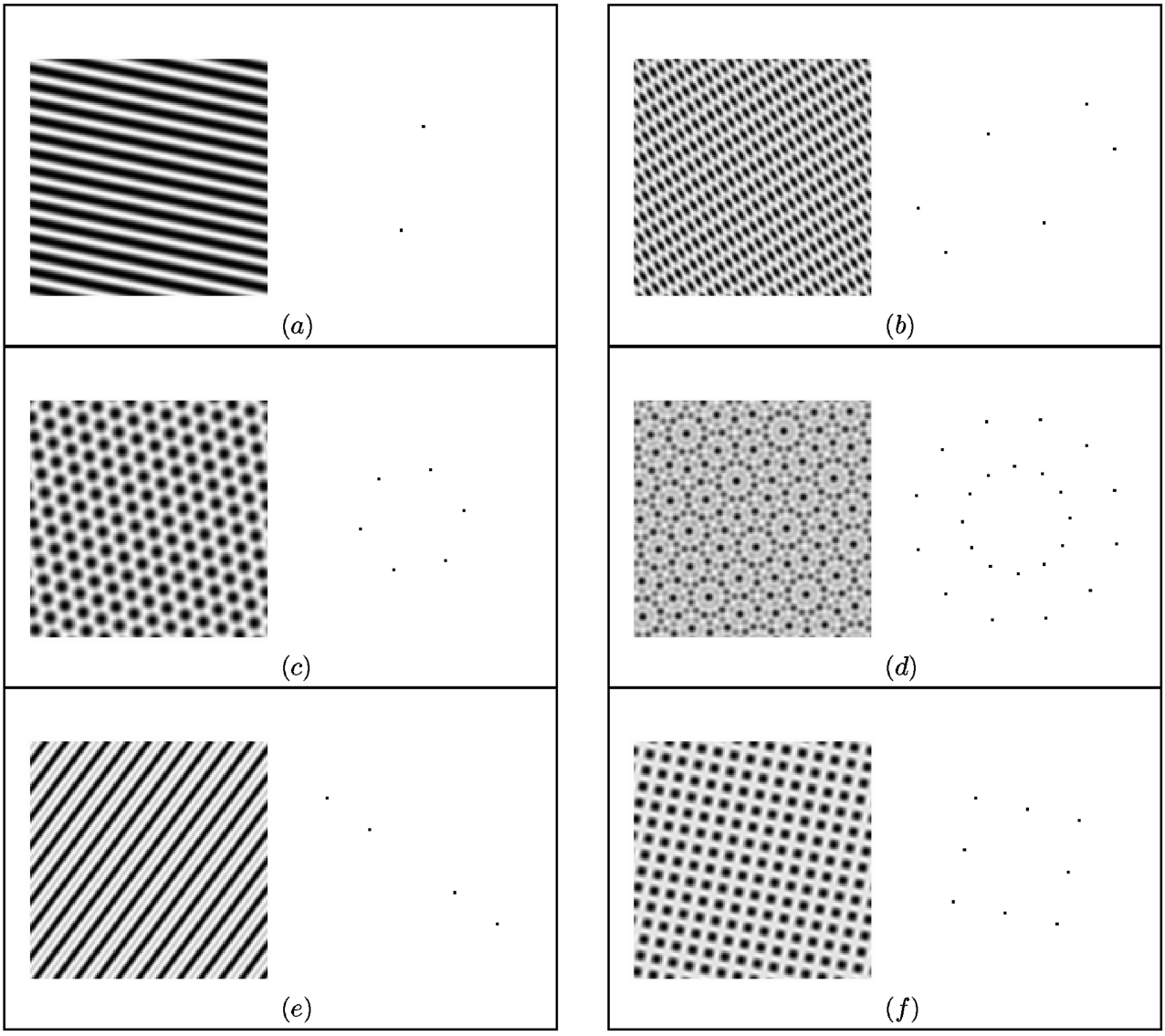


FIG. 2. Numerical solutions of the model equation (4) showing real-space patterns along with their Fourier spectra for different values of the control parameter $\varepsilon^* = \varepsilon/\alpha^2$. The real-space images of $u(x, y, t \to \infty)$ show one-quarter of the simulation cell with darker shades corresponding to larger values of the field. All figures are drawn to the same scale. In cases (a)–(d) $q = 2\cos(\pi/12) = (2 + \sqrt{3})^{1/2}$: (a) a 2-fold pattern of stripes for $\varepsilon^* = 2$, (b) a 2-fold pattern of compressed hexagons for $\varepsilon^* = 0.1$, (c) a 6-fold pattern of perfect hexagons for $\varepsilon^* = 1.8$, (d) a 12-fold pattern for $\varepsilon^* = 0.015$. In (e) $q = 2$, $\varepsilon^* = 0.04$, yielding a 2-fold superstructure of stripes. In (f) $q = \sqrt{2}$, $\varepsilon^* = 0.04$, giving rise to a 4-fold pattern of squares.

Fauve [3], where such patterns are not observed. This does not rule out the possibility that octagonal and decagonal patterns are locally stable but only that within the limits of our model they are not globally stable. Two additional patterns that are observed in our model are a superposition of stripes of periodicities $2\pi$ and $\pi$ [shown in Fig. 2(e)] and a square pattern for $q = \sqrt{2}$ [shown in Fig. 2(f)]. The latter has been reported by Edwards and Fauve. If one examines the Lyapunov functional (5) in its full generality by allowing the value of $q$ and all the amplitudes and phases to vary independently, other patterns might be discovered. We have examined only the symmetric patterns discussed here.

The simplicity of our model shows that for continuous media very little is required to stabilize structures with quasiperiodic long range order: two length scales and triad interactions. The reason that 12-fold patterns are stable and 8- and 10-fold patterns are not is purely geometrical. In view of the Lyapunov functional (5), the crucial issue is the competition between the number of modes, which tends to increase the value of $\mathcal{F}$, and the number of triad interactions, which tends to decrease the value of $\mathcal{F}$. The dodecagonal pattern of Fig. 2(d) contains 24 nonzero Fourier modes and *32* distinct triangles. The octagonal and decagonal patterns do not contain a sufficient number of triangles to compete with the 6-fold pattern of Fig. 2(c). Our model confirms the conclusion of Edwards and Fauve that "12-fold patterns are more common than previously supposed."

Our simplistic model is clearly not adequate for studying the structural stability quasicrystals in the solid state, yet it may offer a very simple system in which to study general questions regarding quasiperiodic order. These may include such questions as the formation and propagation of defects and phase boundaries as well as the dynamics of phason modes [16]. Moreover, we note that (4) may apply to situations other than Faraday waves. Any physical system that can be tuned such that two wavelengths undergo a simultaneous supercritical bifurcation can be described by an equation similar to (4).

An equation similar to (4) could be used to study multiple-frequency forcing of Faraday waves with more than just two frequencies, as suggested by the title of this Letter. We may speculate that with three or four forcing frequencies it might be possible to stabilize quasiperiodic patterns with even higher orders of symmetry, such as 18 or 24. We leave the stability of higher-order symmetric patterns as an open theoretical and experimental question.

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