Comment on "Quantum Quasicrystals of Spin-Orbit-Coupled Dipolar Bosons"

In a recent Letter [1], Gopalakrishnan *et al.* show that quasi-two-dimensional dipolar Bose gases may exhibit a variety of crystalline phases, including a pentagonal quasicrystal. Realizing quasicrystalline condensates would provide new ways to study the quantum dynamics of their collective phason modes, where according to the authors "there are typically additional phasons in quantum-mechanical quasicrystals, when compared with their classical equivalents." I explain here that, on the contrary, the number of phason modes does not depend on whether they are classical or quantum.

Let us describe the crystalline state by a function $\psi_\alpha(\mathbf{r})$, possibly multicomponent—a real-valued scalar density or tensor field in the classical case or a complex-valued wave function or spinor in the quantum mechanical case—whose Fourier expansion is given by

$$\psi_\alpha(\mathbf{r}) = \sum_{\mathbf{k}\in L} \psi_\alpha(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{1}$$

where the (reciprocal) lattice L can be expressed as the set of all integral linear combinations of a finite number D of d-dimensional wave vectors [2]. If the smallest possible D, called the "rank" of the crystal, is equal to the physical dimension d, the crystal is periodic. More generally, for quasiperiodic crystals, $D \geq d$, and one refers to quasiperiodic crystals that are aperiodic (with $D > d$) as "quasicrystals" [3].

A generic free energy, expressed in Fourier space as

$$\mathcal{F} = \sum_{j\leq n} \sum_{\alpha_1,\ldots,\alpha_n} \sum_{\mathbf{k}_1,\ldots,\mathbf{k}_n} A_j^{\alpha_1,\ldots,\alpha_n}(\mathbf{k}_1,\ldots,\mathbf{k}_n) \times \psi_{\alpha_1}(\mathbf{k}_1)\cdots\psi_{\alpha_j}(\mathbf{k}_j)\psi^*_{\alpha_{j+1}}(-\mathbf{k}_{j+1})\cdots\psi^*_{\alpha_n}(-\mathbf{k}_n), \tag{2}$$

is restricted *only* by the requirements imposed by the symmetry of the disordered liquid phase. Phonons and phasons are Goldstone modes that result from breaking the continuous symmetries of $\mathcal{F}$ that are imposed by *translation invariance*, which requires the coefficients $A_j^{\alpha_1,\ldots,\alpha_n}(\mathbf{k}_1,\ldots,\mathbf{k}_n)$ to vanish unless $\mathbf{k}_1+\cdots+\mathbf{k}_n = 0$ [4]. We count the number of such broken symmetries by characterizing the set of all symmetry-broken minimum free-energy states since any broken symmetry operation, by definition, takes a particular minimum free-energy state into a different one. Surprisingly [5], as explained below, the number of these Goldstone modes is not equal to the number of independent translations in d dimensions, but rather to the rank D of the crystal.

A generic set of coefficients A can vary freely with external parameters subject to symmetry restrictions alone, while the field ψ_α adjusts itself accordingly to minimize $\mathcal{F}$. Thus, for two different fields ψ_α and ψ'_α to be *indistinguishable* [6], in the sense that they both minimize the same generic free energy, they must agree independently on all the products in $\mathcal{F}$ with nonvanishing coefficients. In the absence of additional symmetry, one can then show [6,7], using the identity of products of order 2 and 3 alone, that the condition for indistinguishability reduces to the requirement that

$$\psi'_\alpha(\mathbf{k}) = e^{2\pi i\chi(\mathbf{k})}\psi_\alpha(\mathbf{k}), \tag{3}$$

where $\chi(\mathbf{k})$, called a "gauge function" [8], is linear to within an additive integer on the lattice L.

In quantum crystals with additional U(1) symmetry $\psi_\alpha \to e^{i\theta}\psi_\alpha$ or in classical crystals with an additional $\mathbb{Z}_2$ symmetry $\psi_\alpha \to -\psi_\alpha$, the free energy (2) is restricted to contain products of only even order, questioning the linearity of gauge functions. Nevertheless, one can still use the identity of products of order 2 to establish Eq. (3); in the case of complex-valued fields, use products of order 4 to show that $\chi(-\mathbf{k}) = -\chi(\mathbf{k})$, which is trivially so for real-valued fields; and finally, use products of order 6 to obtain $\chi(\mathbf{k}_1+\mathbf{k}_2) = \chi(\mathbf{k}_1)+\chi(\mathbf{k}_2)$, where these equalities hold to within an overall constant coming from the additional symmetry. Suppression of sixth- or higher-order products at low density—whether in classical or in quantum crystals—does not affect this outcome. Such terms might be irrelevant at the critical fixed point, but dangerously so [9] in the sense that despite being small they still distinguish between states in the ordered phase.

The linearity of gauge functions implies that $\chi(\mathbf{k})$ is uniquely determined by specifying its values on a chosen basis for L, consisting of D linearly independent vectors over the integers. Thus, there are exactly D independent symmetry operations in $\mathcal{F}$ that are broken and therefore exactly D Goldstone modes. One *may* choose the basis so that d of the Goldstone modes are the familiar phonon modes. The remaining $D-d$ modes are the phason modes, regardless of whether the crystal is quantum or classical.

I am grateful to Michael Cross and Gil Refael for valuable discussions. This research is supported by the Israel Science Foundation through Grant No. 556/10.

Ron Lifshitz*
Condensed Matter Physics 149-33
California Institute of Technology
Pasadena, California 91125, USA

Received 4 December 2013; published 13 August 2014
DOI: 10.1103/PhysRevLett.113.079602
PACS numbers: 67.85.−d, 03.75.Mn, 61.44.Br,

*Permanent address: School of Physics & Astronomy, Tel Aviv University, Tel Aviv 69978, Israel.
ronlif@tau.ac.il

[1] S. Gopalakrishnan, I. Martin, and E. A. Demler, Phys. Rev. Lett. **111**, 185304 (2013).

[2] In Ref. [1], the dimension d is 2, and the rank D of an M-fold symmetric star of two-dimensional wave vectors is given by the Euler totient function $\phi(M) = M\Pi_{P_i}(1 - 1/P_i)$, where P_i are the distinct prime factors of M.
[3] R. Lifshitz, Found. Phys. **33**, 1703 (2003); Z. Kristallogr. **222**, 313 (2007).
[4] The constraints imposed by rotational invariance are immaterial here and treated elsewhere, for example, in Ref. [5].
[5] R. Lifshitz, Isr. J. Chem. **51**, 1156 (2011).
[6] N. D. Mermin, Rev. Mod. Phys. **64**, 3 (1992).
[7] R. Lifshitz, Rev. Mod. Phys. **69**, 1181 (1997), see the Appendix.
[8] D. S. Rokhsar, D. C. Wright, and N. D. Mermin, Acta Crystallogr. Sect. A **44**, 197 (1988).
[9] D. J. Amit and L. Peliti, Ann. Phys. (N.Y.) **140**, 207 (1982).