1) We are given by a mixture of $|\uparrow\rangle$ and $|+\rangle$ with equal probabilities.

1.1 For the POVM measurement discussed in class, calculate the minimal probability for an undecidable answer.

1.2 Calculate the mutual information between a message encoded using this set of states and the measurement results for the POVM in 1.1.

2)

2.1 Calculate the expectation value of the position and momentum of a particle in a coherent state $|\alpha\rangle$.

2.2 Show that a coherent state can be used to obtain a resolution of the identity:

$$\frac{1}{\pi} \int d^2 \alpha |\alpha\rangle\langle \alpha| = \hat{1}$$

This means that $|\alpha\rangle$ form an overcomplete set: they are not orthogonal, but they span the space (prove that).

2.3 Use that to define a POVM $F_\alpha$. Given by an arbitrary state $\psi(x)$ of a particle, what is the expression for $\text{Prob}(\alpha)$.

3. Now let’s construct a possible realization of $F_\alpha$.

3.1 First, given by two particles with coordinates $x, p$ and $Q, P$, find the common eigenstates $|x_{rel} = a, p_{tot} = b\rangle$ of the operators $x_{rel} = x - Q$ and $p_{tot} = p + P$.

3.2 Consider next a combined measurement (that is performed on both particles) of a projector to $|x_{rel} = a, p_{tot} = b\rangle$. Given that the initial state of the particles is a product, $\psi(x)\phi(Q)$, show that

$$|\langle x_{rel} = a, p_{tot} = b | \psi \rangle\phi\rangle|^2 = \frac{1}{2\pi} |\langle \phi_{a,b} | \psi \rangle|^2$$

where $|\phi_{a,b}\rangle \equiv e^{iaP+ibQ} |\phi\rangle$

3.3 Hence show that $\phi_{a,b}$ can be used in order to form an overcomplete set, such that $\int dadb |\phi_{a,b} |\psi\rangle\langle \phi_{a,b} |$ gives a resolution of the identity.

3.4 Under what choice of $\phi(Q)$, we get the result of question 2? Show that with this choice the probability to get $|a, b\rangle$ coincides with the probability $\text{Prob}(\alpha)$ in question 2.