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Exogenous Growth Under International Capital and Labor Mobility

The world economy is characterized by diverse levels of income and patterns of growth across countries and across time. To understand the evolution of such diversity and whether there are tendencies toward cross-country equality, we have to study the mechanics of the growth process in each country individually and in the world equilibrium at large. In order to provide a systematic analysis of such process, we devote this chapter to a simplified world where the engine of growth is exogenously determined. As a natural extension, this growth engine will be endogenized in the next chapter.

Evidently, factor mobility could potentially influence the convergence/ divergence tendencies across countries within a growing world economy. We therefore focus our analysis on the roles that capital mobility and labor mobility can play in such a process. In addition, we examine the potential welfare gains that capital and labor mobility can generate. Also studied are the effects of factor mobility on the speed of adjustment to long run equilibria and on the rate of convergence (if any) across countries.

12.1 The Closed Economy

In this section, we provide an exposition of the standard growth model that emphasizes the role of saving and physical capital accumulation in driving short run output growth and that of (exogenous) human capital accumulation in driving long run growth.[1]

Consider a closed economy with a Cobb-Douglas technology that transforms physical capital (K_t) and human capital (H_t) into a single composite good (Y_t):

$$Y_t = A_t K_t^{1-\alpha} H_t^{\alpha}. \tag{12.1}$$

The productivity level A_t is, in general, time-varying and can be a source of growth and fluctuations. To make things simple at this stage, we assume that $A_t = A$, a deterministic constant. In Section 12.4 below, we will assume it to follow a stochastic process in order to analyze growth under uncertainty.[2] Human capital H_t is a product of the size of the labor force N_t (treated synonymously with the size of population, assuming constant labor force participation and inelastic work hours) and their skill level (h_t). An important feature of this model is constant returns to scale technology.[3] While increasing returns to scale will give rise to unbounded growth (which is thus inconsistent with the existence of a stable long run equilibrium), decreasing returns to scale cannot generate sustainable long run growth. [See Appendix A.]

Physical and human capital are accumulated according to the following laws of motion respectively:

$$K_{t+1} = I_t + (1-\delta_k)K_t. \tag{12.2}$$

where I_t is gross investment in, and δ_k the rate of depreciation of, physical capital.

$$H_{t+1} = (1+g_H)H_t. \tag{12.3}$$

where g_H is the exogenous rate of growth of human capital. Since $H_t = N_t h_t$ and both N_t and h_t are assumed to grow at exogenous constant rates g_N and g_h, we can decompose (13.3) into $N_{t+1} = (1+g_N)N_t$ and $h_{t+1} = (1+g_h)h_t$, with $(1+g_N)(1+g_h) = 1+g_H$.

The representative consumer is the owner of both physical and human capital. He/she accumulates these two forms of capital over time, and supplies them as factor inputs to the representative firm for production at the competitive wage (w_t) and rental (r_{kt}) rates, period by period. To derive its demand for physical capital (K^d_t) and human capital (H^d_t), the firm in turn solves a static profit maximization problem[4]:

$$Max_{\{K^d_t, H^d_t\}} \ A(K^d_t)^{1-\alpha}(H^d_t)^{\alpha} - w_t H^d_t - r_{kt} K^d_t$$

This yields the standard marginal productivity-factor price relations:

$$w_t = \alpha A x_t^{1-\alpha} \quad and \quad r_{kt} = (1-\alpha) A x_t^{-\alpha}, \tag{12.4}$$

where x_t is the ratio of physical capital to human capital (K_t/H_t).

Preferences of the dynastic head of the family are given by an isoelastic utility function:

$$U = \sum_{t=0}^{\infty} \beta^t N_t \left(\frac{c_t^{1-\sigma} - 1}{1-\sigma} \right), \tag{12.5}$$

where $0 < \beta < 1$ is the subjective discount factor, and $\sigma > 0$ the reciprocal of the intertemporal elasticity of substitution in consumption (c_t). In the limiting case where $\sigma = 1$, the utility function is logarithmic. The household derives wage income from the supply of human capital ($w_t H_t$) and rental income from the supply of physical capital ($r_{kt} K_t$). He/she can either consume ($N_t c_t$) or save these incomes in the form of physical capital ($K_{t+1} - (1-\delta_k)K_t$). He/she can also borrow B_{t+1} at the rate of interest r_{t+1} in any period t, and repay $(1+r_t)B_t$ for his/her borrowing B_t from the previous period t−1.[5] The household's net saving is therefore given by $[K_{t+1} - (1-\delta_k)K_t] - [B_{t+1} - (1+r_t)B_t]$. His/her period-t budget constraint can thus be written as

$$[K_{t+1} - (1-*_k)K_t] = w_t H_t + r_{kt}K_t + [B_{t+1} - (1+r_t \tag{12.6}$$

The consumer chooses $\{c_t, K_{t+1}, B_{t+1}\}$ to maximize (12.5) subject to (12.2), (12.3), and (12.6). The first order conditions for this problem imply:

$$R_{B_{t+1}} = \frac{1}{\$}\left(\frac{c_{t+1}}{c_t}\right)^{F} = R_{k_{t+1}}, \tag{12.7}$$

where $R_{Bt+1} = 1 + r_{t+1}$ and $R_{kt+1} = 1 + r_{kt+1} - \delta_k$. Recall that, as in Chapter 5, in making his borrowing decision, the consumer equates the gross rate of interest (R_{Bt+1}) to his/her intertemporal marginal rate of substitution (IMRS). In addition, as an owner of physical capital, he/she also equates the return on physical capital (R_{kt+1}) to the same IMRS. Arbitrage through capital market forces will drive equality between the interest rate and rental rate on capital (net of depreciation, i.e., $r_{t+1} = r_{kt+1} - \delta_k$.

The equilibrium values of the wage rate, rental rate, and interest rate are determined by the following market clearing conditions: $H_t = H_t^d$ in the labor market, $K_t = K_t^d$ in the physical capital market, and $B_{t+1} = 0$ in the financial capital market.[6] By the Walras law, the consumer budget constraint (12.6), the profit-maximizing conditions (12.4), and these market clearing conditions together imply the economy-wide resource constraint:

$$N_t c_t + K_{t+1} - (1-*_k)K_t = F(K_t, H_t). \tag{12.8}$$

That is, output is split between consumption and saving/investment.

The Mechanics of Economic Growth

The dynamics of our example economy are driven by the two fundamental laws of motion (12.2) and (12.3), which can be combined to yield a single difference equation in the physical capital-human capital ratio (or the capitals ratio, X_t):

$$X_{t+1} = \left(\frac{A}{1+g_H} \right) s_t X_t^{1-\alpha} + \left(\frac{1-\delta_k}{1+g_H} \right) X_t , \tag{12.9}$$

where s_t is the saving rate at time t, defined as the ratio between saving S_t (gross and net) and output Y_t (= I_t/Y_t since $S_t = I_t$ in the closed economy equilibrium). In general, the equilibrium saving rate is determined by solving the consumer's intertemporal optimization problem and imposing the market clearing conditions. We spell out the dynamics of the system in Appendix B, and illustrate how the saving rate is derived in an example economy in Appendix C.

We can express per capita output as y_t (= Y_t/N_t) = $A(X_t)^{1-\alpha}h_t$, where $h_t = h_0(1+g_h)^t$. Evidently, the dynamic path of y_t depends on that of X_t and h_t. In particular, the growth rate of y_t (g_{yt}) can be approximated (using $\ln(1+g) = g$) by

$$\begin{aligned} \ln\left(\frac{y_{t+1}}{y_t} \right) &= (1-\alpha)\ln\left(\frac{X_{t+1}}{X_t} \right) + g_h \\ &= (1-\alpha)\ln\left[\left(\frac{A}{1+g_H} \right) s_t X_t^{-\alpha} + \frac{1-\delta_k}{1+g_H}\right] + g_h \\ &= (1-\alpha)\{\ln[A s_t X_t^{-\alpha} + (1-\delta_k)] - g_N\} + \alpha g_h . \end{aligned} \tag{12.10}$$

Observe that g_{yt} depends positively on the saving rate (s_t) and the skill growth rate (g_h), and negatively on the capitals ratio (X_t) and the population growth rate (g_N). If, however, the economy converges to a long run time-invariant equilibrium where $X_{t+1} = X_t$ so that $\ln(X_{t+1}/X_t) = 0$, then g_y will converge to g_h.

Steady State Growth and Transitional Dynamics: Policy Implications

While equation (12.9) describes the entire growth path of the capitals ratio X_t, it can conveniently be divided into two components: the transition (to steady state) path and the steady state growth path. *Steady state* growth is defined as the particular pattern of growth where growth rates of all variables are constant over time, but possibly different from one another. (It is sometimes called `*balanced* growth' as well.) The constant returns to scale assumption, combined with isoelastic preferences (which implies equality between the average and marginal propensities to consume), are necessary for the existence of such long run equilibrium. With $X_t = X_{t+1} = \hat{X}$ and $s_t = s_{t+1} = \hat{s}$ in the steady state, the capitals ratio and the level of per capita output can be expressed as (positive) functions of the saving rate as follows:

$$\hat{X} = \left(\frac{sA}{g_H+\delta_k}\right)^{1/\alpha} \quad and \quad \hat{y}_t = \left(\frac{s}{g_H+\delta_k}\right)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} h_0(1+g_h)^t .$$

The capital-(raw) labor ratio $\hat{k}_t$ ($= K_t/N_t$) is equal to $\hat{X}h_t = \hat{X}h_0(1+g_h)^t$. Observe that $\hat{y}_t$ and $\hat{k}_t$ both grow at the same rate g_h along the steady state growth path, implying that the capital-output ratio must be constant. Since the rate of return on capital ($\hat{r}_{kt}$) is given by $(1-\alpha)\hat{y}_t/\hat{k}_t$, it must also be constant. Indeed, these steady state properties of $\hat{y}_t$, $\hat{k}_t$, and $\hat{r}_{kt}$ are all consistent with a set of empirical regularities about patterns of growth established by Kaldor (1963).

Another component of the growth path exhibits transitional dynamics described by equation (12.9) for $X_t \neq X_{t+1} \neq \hat{X}$ and portrayed by Figure 12.1. For any initial stocks of physical and human capital (hence X_0), the capitals ratio in the next period (X_1) can be read off the concave curve for the corresponding equilibrium value of the saving rate (s_0, say s'). Given X_1, the capitals ratio in the period after (X_2) can be obtained from a point on the concave curve correspending to the equilibrium saving rate in period 1 (s_1, say, s"). Eventually, the capitals ratio will converge to the steady state at the intersection between the 45^0 line and the concave curve with constant saving rate $\hat{s}$ (point A). As shown in the example in Appendix C, the Solow (1956) model is a special case of our model (when $\sigma = \delta_k = 1$) where the saving rate is time-invariant (say, $\hat{s}$). In this case, for $X_t < \hat{X}$, $X_{t+1} > X_t$ (as indicated by the arrows pointing to the right); and for $X_t > \hat{X}$, $X_{t+1} < X_t$ (as indicated by the leftward pointing arrows).

[insert Figure 12.1 here]

In Appendix B, we derive two measures of the speed of convergence to the long run equilibrium. The first one measures the half-life of the dynamical system, i.e., the time it takes to reach half of the distance between the initial position and the steady state. The second one, analogous to Barro and Sala-i-Martin (1992), measures the ratio between the deviations of output from its steady state value in two consecutive periods.

[Explain dependence of $(1+g_h)$ on underlying parameters.]

From the arguments just presented, it should now be evident that, in the exogenous growth paradigm, policies which target saving rates (such as taxation of capital income) can influence growth only along the transition path, but not in the long run steady state equilibrium. They can nonetheless affect the long run level of output per capita. In other words, these policies only have *level* effects, but no *growth* effects.

12.2 The Open Economy: Capital Mobility

From a global perspective, there is an issue as to whether countries with different levels of initial income will converge (with income gaps narrowing) over time. We have just shown that income convergence can occur even among isolated economies if the long run (capital account autarky) steady state positions they converge to are similar. This entails similar technologies (A, α, and δ_k) and preferences (β and σ), and most importantly similar exogenous growth rates (g_H, or g_N and g_h).

When the economies are open, it is more likely that such converegence will take place because openness can potentially enhance technology transmission. Moreover, when capital markets are integrated into the world capital market, the process of convergence (its speed as well as the particular transition path) will likely be altered. International borrowing and lending can facilitate consumption smoothing and create additional investment opportunities, which will improve the efficiency in the allocation of consumption over time and the allocation of capital across countries. For these reasons, the capital market integration may generate welfare gains as well. In this section, therefore, we introduce capital mobility in this section to study its role for the growth process and for international convergence of income levels.

Perfect Capital Mobility

Consider integration of our example economy into the world capital market. Assume free trade in commodities. Since we have a single composite good here, although a country either exports or imports (but will not do both) within any given period, it can definitely engage in intertemporal trade in this single commodity (exporting in some periods and importing in others). This implies mobility in financial capital (in the form of various financial securities). Naturally, trade in goods can occur in the form of physical capital as well. In other words, physical capital also becomes internationally mobile.

Accordingly, we modify the consumer budget constraint (12.6) as follows:

$$\begin{aligned} & N_t c_t + (K^H_{t+1} + K^{H*}_{t+1}) - (1-*_k)(K^H_t + K^{H*}_t) \\ {}^{H} & + r^*_{kt}K^{H*}_t + (B^H_{t+1} + B^{H*}_{t+1} - B^{*H}_{t+1}) - (1+r_t)(B^H_t + B^{*H}_t) \end{aligned} \tag{12.11}$$

K^H and B^H denote the domestic consumer's claims on physical capital residing in the home country and domestic debt respectively. Likewise, K^{H*} and B^{H*} stand for the domestic consumer's claims on physical capital residing in the foreign country and foreign debt. The domestic rate of interest and rental rate are denoted by r and r_k, and their foreign counterparts by r^* and r^*_k respectively.

Similarly, the resource constraint of the domestic economy (12.8) will have to be modified.

$$\begin{aligned} & N_t c_t + (K^H_{t+1} + K^{H*}_{t+1}) - (1-*_k)(K^H_t + K^{H*}_t) \\ & + r^*_{kt}K^{H*}_t - r_{kt}K^{*H}_t + (B^{H*}_{t+1} - B^{*H}_{t+1}) - [(1+r^*_t)B^{H*}_t - (\end{aligned} \tag{12.12}$$

where $K = K^H + K^{*H}$, the latter being the stock of foreign direct investment in the domestic economy.

We can rearrange terms in (12.12) in order to derive the country's balance of payments accounts $CA_t + KA_t = 0$ as follows:

$$CA_t = GNP_t - N_t C_t - I^H_t$$
$$[F(K_t,H_t) + (r_t B^{*H} - r^*_t B^{H*}_t) + (r^*_{kt}K^{H*}_t - r_{kt}K^{*H}_t)] - N_t C_t - [K^H_{t+1} - (1-*_k)K^H_t], \text{ a}$$
$$KA_t = -(FDI_t + FPI_t)$$
$$= -[K^{H*}_{t+1} - (1-*_k)K^{H*}_t] + [(B^{*H}_{t+1} - B^{*H}_t) - (B^{H*}_{t+1} - B^{H*}_t)].$$

Here, CA stands for the current account balance, KA the capital account balance, FDI foreign direct investment, and FPI foreign portfolio investment.

In this deterministic setting, capital flows will be unidirectional, the direction being determined by the relative magnitudes of the domestic and foreign rates of interest. Furthermore, since $r = r_k - \delta_k$ and $r^* = r_k^* - \delta_k$ in equilibrium, physical capital and financial capital are perfect substitutes. Assume without loss of generality that $r > r^*$, so $r_k > r_k^*$. Then, $B^{*H} = 0$ and $K^{H*} = 0$.

To highlight the role of capital mobility for international convergence of per capita output levels, we assume that the rest of the world is initially in a long run steady state growth equilibrium of the sort studied in the previous section, whereas the home economy starts initially with a lower physical capital-human capital ratio ($X_0 < X_0^* = \hat{X}^*$).

Assume further that the rest of the world is large so that the home country is a price-taker in the world market. Capital market integration then ensures that $r_t = r^*$ ($= (1-\alpha)A(\hat{X}^*)^{-\alpha} - \delta_k$ from (13.4)). As a result, the capital inflow will generate an immediate convergence of the capitals ratio ($X_1 = \hat{X}^*$) and hence convergence of per capita GDPs ($y_t = \hat{y}_t^*$). From (13.7), one can verify that the consumption growth rate will converge at the same time to its long run value, g_h. Per capita GNPs and consumption will not converge, however, because the capital inflow effectively equalizes labor income between the home country and the rest of the world, while the former starts with a lower K_0. Evidently, the home economy reaps all the benefits from the intertemporal trade--since prices in the rest of the world remain unchanged while, for the domestic economy, the initial marginal productivity of capital net of depreciation exceeds the world rate of interest. Therefore, the consumption ratio c_t/c_t^* will initially rise, and then stay constant thereafter (given equality between their consumption growth rates).[7]

Free capital flows must generate welfare gains from intertemporal trade, akin to the standard gains from trade argument. Obviously, the magnitude of the gains is directly related to the difference in the initial capitals ratios between the home country and the rest of the world. (Recall that this is assumed to be the only source of heterogeneity in our analysis.) In Appendix D, we compute such gains and relate them to the initial cross-country difference. These gains are measured in utility terms. An alternative welfare concept , measured in terms of compensating variations in consumption, is defined implicitly by

$$\sum_{t=0}^{\infty} \beta^t U(c_t^{autarky}(1+\omega)) = \sum_{t=0}^{\infty} \beta^t U(c_t^{kmobility}) .$$

[calculate numerical values of as a function of $Y_0/\hat{Y}_0$]

Constrained Capital Mobility

Free capital mobility is an extreme situation based on ideal market structure with full information and absence of risk, default, and time inconsistent behavior (both for private agents and governments). When such elements are taken into account, borrowing and lending may be subject to significant constraints and regulations. To account for these possibilities in our aggregative model, we now introduce an upper bound on financial capital flows. No such bound is imposed, however, on foreign direct investment.

We assume that households (in their capacity as owners of the firms) can only borrow up to a certain limit. The limit is the collateral based on the ownership by the domestic households of the domestic firm's net worth, which is equal to K_t^H.[8] Recall, however, that an increase in domestic consumption can come from three sources: international borrowing, domestic output net of returns to FDI, and reduction in investment by domestic residents in domestic capital.

Therefore, if borrowing from abroad is constrained, domestic consumers can still offset the effect of this constraint by reducing their investment in domestic capital. Only when a lower bound on that form of investment is imposed will the consumption boom following the open-up of the international capital market become constrained. We therefore make the realistic assumption that gross investment of this kind be non-negative. (Effectively, this constraint will eliminate the possibility of using the domestic firm's capital stock, which is partly owned by foreigners, as an additional collateral for borrowing from abroad to finance domestic consumption.) If binding, these two constraints together with the resource constraint (12.12) will imply

$$ _t = F(K_t, H_t) - r_{kt} K_t^{*H} - r_{kt}^{*} K_t^{H} = w_t H_t + (r_{kt} - r_{kt}^{*}) \tag{12.12} $$

At the same time, the domestic rate of interest will not be set equal to the world rate of interest because of the borrowing constraint. From the intertemporal condition (12.7), we have $c_{t+1}/c_t > [\beta(1+r^*)]^{1/\sigma}$. In comparison to the free capital mobility case, therefore, while the jump in consumption immediately following the open-up of international capital markets is smaller, the consumption growth rate is higher as long as the constraints remain binding. Thus, while the borrowing constraint lengethens the transitional dynamics, it generates a more accelerated rate of growth in consumption and income during that phase compared to the long run growth rate. Eventually, the constraints will become slack, and the growth rate will converge to the same long run value, g_h.

This is a stylized example of real world constraints that tend to slow down the adjustment process. The welfare gains from the opening up of credit markets will evidently be lower than those under free capital mobility.

12.3 Open Economy: Labor Mobility

Since both labor and physical capital are primary inputs in production, labor mobility is expected to play a similar role in the growth process as capital mobility. The skill levels are typically embodied in labor. Consequently, we have international transfer of skills and human capital as an integral part of labor mobility.

We now open up the home country to free labor flows. For simplicity, we assume that capital (financial as well as physical) is immobile internationally. As before, the rest of the world is assumed to be in the long run steady state initially, whereas the home economy starts with a lower physical capital-human capital ratio ($X_0 < X_0^* = \hat{X}^*$). This implies that $w_0 < w_0^* = \hat{w}^*$. We retain the assumption that the rest of the world is large so that the home country is a price-taker in the world market. Labor market integration then ensures that $w_t = \hat{w}^*$ ($= \alpha A(\hat{X}^*)^{1-\alpha}$ from (12.4)). As a result of the labor outflow, we obtain an immediate convergence of the capitals ratio ($X_1 = \hat{X}^*$) and hence convergence of per capita GDPs ($y_1 = \hat{y}^*$). Again, the consumption growth rate will converge at the same time to its long run value, g_h because wage rate equalization implies interest rate equalization. Per capita GNPs and consumption will not converge, however, because the home country starts out with a lower K_0, hence a lower initial wealth. Evidently, the home economy reap all the benefits from the intertemporal trade since prices in the rest of the world remain unchanged while, for the domestic economy, the initial marginal productivity of labor falls short of the world wage rate. Therefore, the consumption ratio c_t/c_t^* will initially rise, and then stay constant thereafter (given equality between their consumption growth rates). What we have just shown is that labor mobility is a perfect substitute for capital mobility, with the same welfare implications.

The assumption of free labor flows abstracts, however, from the significant costs of adjustment associated with such mobility (e.g., cultural, ethnic, family, and other differences). When such costs are accounted for, labor mobility becomes a less efficient convergence mechanism in comparison with capital mobility (even after incorporating the above-mentioned borrowing constraints).

12.4 Stochastic Growth

So far, we have been assuming that growth is deterministic. In order to highlight the effects of random shocks on the growth dynamics, we introduce in this section stochastic elements. To simplify the analysis, we revert to the closed economy setting developed in Section 12.1. The complexity in characterizing the stochastic dynamics excludes the possibility of an analytic inspection of the growth mechanism except for relatively low dimensional cases, one of which is elegantly analyzed by Campbell (1994) and is presented in this section. At the same time, however, we develop here tools of analysis that will be useful also for analyzing higher dimensional problems such as those in an open economy. These techniques will be further developed in Chapter 13.

To incorporate random disturbances into the model of Section 12.1, we assume that the productivity level a_t ($= \ln(A_t)$) follows a first order autoregressive process:

$$a_{t+1} = \mathrm{D}a_t + \,,_{t+1}, \tag{12.13}$$

where $_{t+1}$ is an i.i.d. shock with zero mean and constant variance, and $0 \leq \quad \leq 1$ measures the persistence of the shock. In the presence of such shocks, the expectation operator will have to be inserted into equations (12.5) and (12.7).

Defining X = K/H and Z = c/h, we can rewrite the two dynamic equations (12.7) and (12.8) in K and c as (B.1) and (B.2) in Appendix B. Together with equation (12.13), these two laws of motion form a system of nonlinear expectational difference equations in (Z_t, X_t, A_t). To get a handle on the solution, we take a loglinear approximation of this nonlinear system around its deterministic steady state (with $_t = 0$) to obtain a linear system of expectation difference equations in x_t (= $\ln(X_t)$) and z_t = ($\ln(Z_t)$). Details of the solution procedure are laid out in Appendix E, from which we derive the approximate dynamic behavior of the economy as a function of the shock as follows:

$$a_t = \frac{1}{(1-\mathrm{D}L)} ,_t \, . \tag{12.14}$$

$$x_{t+1} = \frac{0_{xa}}{(1-0_{xx}L)(1-\mathrm{D}L)} ,_t \, . \tag{12.15}$$

$$z_t = \frac{0_{za} + (0_{zx}0_{xa} - 0_{za}0_{xx})L}{(1-0_{xx}L)(1-\mathrm{D}L)} ,_t \, . \tag{12.16}$$

$$\ln\left(\frac{y_t}{h_t}\right) = (1-{}'')x_t + a_t = \frac{1 + [(1-{}'')0_{xa} - 0_{xx}]L}{(1-0_{xx}L)(1-\mathrm{D}L)} ,_t \, . \tag{12.17}$$

Above, `L' denotes the lag operator, and $_{zx}$, $_{za}$, $_{xx}$, and $_{xa}$ represent the partial elasticities of

z_t with respect to x_t, z_t with respect to a_t, x_{t+1} with respect to x_t, and x_{t+1} with respect to a_t respectively. The explicit expressions for these elasticities in terms of the fundamental parameters are spelled out in Appendix E. Observe from equations (12.15)−(12.17) that the logarithm of the ratio between physical capital and human capital (x_t) follows an AR(2) process, while the logarithm of the ratio between concumption and human capital (z_t) as well as the logaritm of the ratio between output and human capital ($\ln(y_t/h_t)$) follow an ARMA(2,1) process. Note also that they have identical autogressive roots $_{xx}$ and .

We now flesh out some properties of the partial elasticities:

(1) $_{zx}$, which measures the effect on current consumption of an increase in the capital stock for a fixed level of productivity, naturally does not depend on the persistence of the productivity shock ;

(2) $_{za}$, which measures the effect on current consumption of a productivity increase with a fixed stock of physical capital, is increasing in for low values of σ but decreasing for high values of σ. The inituition behind this is as follows. When substitution effects are weak (small σ), the consumer responds mostly to income effects, which are stronger the more persistent the shock. When substitution effects are strong (high σ), a persistent shock which raises the furture rate of interest will stimulate saving at the expense of current consumption, making the response in consumption small;

(3) $_{xx}$, which measures the effect on the future level of capital stock of an increase in the current level of capital stock with a fixed level of productivity, does not depend on but declines with σ. In fact, in our model of Section (12.1) where technology is time-invariant, 1− $_{xx}$ reflects the speed of convergence to the steady state, similar to our second measure of convergence on p.7.

We now turn to the general equilibrium implications of the persistence parameter . Observe that if productivity follows a random walk (= 1), we have $_{zx}$ + $_{za}$ = 1 and $_{xx}$ + $_{xa}$ = 1. Consequently, capital, consumption, and output will follow cointegrated random walk processes, so that the difference between any two of them is stationary.

Appendix A: Returns to Scale and Long Run Growth

Consider a Cobb-Douglas production function that allows for decreasing, increasing, and constant returns to scale.

$$Y_t = AK_t^{\beta}H_t^{\alpha}. \tag{A.1}$$

where $\beta + \alpha$ can be < 1, > 1, or $= 1$.

Following Solow (1956), assume for simplicity a constant saving rate s, so that investment (I_t) is equal to sY_t. The physical capital-human capital ratio (X_t) can be shown to evolve according to:

$$X_{t+1} = \left(\frac{s}{1+g_H}\right) AX_t^{\beta}H_t^{\alpha+\beta-1} + \left(\frac{1-\delta_K}{1+g_H}\right) X_t.$$

If $\alpha + \beta > 1$ (increasing returns), we will have unbounded growth in the long run. If $\alpha + \beta < 1$ (decreasing returns), the economy will vanish over time.

Appendix B: Local Dynamics of Our Example Economy

The dynamics of our example economy is governed by the system of difference equations (12.2), (12.3), and (12.7) in $\tilde{c}_t$, K_t, H_t, and A_t. For the sake of the analysis in Section 12.4, the productivity level A is allowed to be time-varying. Defining X_t as K_t/H_t as before and Z_t as c_t/h_t, we can combine (12.2) and (12.3) as one single difference equation in X_t.

$$(1+g_H)X_{t+1} = A_t X_t^{1-"} + (1-*_k)X_t - Z_t . \tag{B.1}$$

Dividing (12.7) throughout by h_t yields a difference equation in Z_t.

$$Z_t^{-F} = \$E_t[(1+g_h)Z_{t+1}]^{-F}[(1-")A_{t+1}X_{t+1}^{-"} + (1-*_k)] \tag{B.2}$$

In the steady state, $A_t = A_{t+1} = A$, $X_t = X_{t+1} = \hat{X}$ and $Z_t = Z_{t+1} = \hat{Z}$. These steady state values are given by

$$\hat{X} = \left[\frac{(1-")A}{(1+g_h)^F/\$ - (1-*_k)}\right]^{1/"} \quad and \quad \hat{Z} = A\hat{X}^{1-"} - (g_H+*_k)\hat{X} .$$

Log-linearizing (B.1) and (B.2) around the steady state $(\hat{Z},\hat{X})$ yields

$$s_x(1+g_H)E_t\left(\frac{dX_{t+1}}{X_{t+1}}\right) - s_x(1-*_k)\left(\frac{dX_t}{X_t}\right) = (1-")\left(\frac{dX}{X}\right.$$
$$= \frac{\hat{X}^"}{A} = \frac{1-"}{r+*_k} \quad and \quad s_z = \frac{\hat{Z}}{\hat{Y}} = 1-(g_H+*_k)s_x = 1- \tag{B.3}$$

$$\left[E_t\left(\frac{dZ_{t+1}}{Z_{t+1}}\right)-\frac{dZ_t}{Z_t}\right]=\left(\frac{r+*_k}{1+r}\right)E_t\left[\left(\frac{dA_{t+1}}{A_{t+1}}\right)-"\left(\frac{dX_{t+1}}{X_{t+1}}\right)\right] \tag{B.4}$$

We can group (B.3) and (B.4) into a matrix difference equation as follows:

$$AE_t\begin{pmatrix}\frac{dX_{t+1}}{X_{t+1}}\\ \frac{dZ_{t+1}}{Z_{t+1}}\end{pmatrix}=B\begin{pmatrix}\frac{dX_t}{X_t}\\ \frac{dZ_t}{Z_t}\end{pmatrix}+CE_t\left(\frac{dA_{t+1}}{A_{t+1}}\right)+D\left(\frac{dA_t}{A_t}\right), \tag{B.5}$$

where the matrices A, B, C, and D are given by

$$\begin{pmatrix} & 0\\ & F\end{pmatrix},\ B=\begin{pmatrix}(1-")+s_x(1-*_k) & -s_z\\ 0 & F\end{pmatrix},\ C=\begin{pmatrix}0\\ \frac{r+*_k}{1+r}\end{pmatrix},$$

Equation (B.5) can be rewritten as:

$$E_t\begin{pmatrix}\frac{dX_{t+1}}{X_{t+1}}\\ \frac{dZ_{t+1}}{Z_{t+1}}\end{pmatrix}=W\begin{pmatrix}\frac{dX_t}{X_t}\\ \frac{dZ_t}{Z_t}\end{pmatrix}+RE_t\left(\frac{dA_{t+1}}{A_{t+1}}\right)+Q\left(\frac{dA_t}{A_t}\right), \tag{B.6}$$

where $W = A^{-1}B$, $R = A^{-1}C$, and $Q = A^{-1}D$.

To study the deterministic dynamics of this system, we revert to the case where A is time-invariant (hence the expectation operators and the terms dA_t/A_t and dA_{t+1}/A_{t+1} can be dropped from (B.6)). We can then compute the eigenvalues (call them $_1$ and $_2$) by solving the equation det(W− I) = 0, which is a quadratic equation in . The values of the two roots $_1$ and $_2$ are complicated functions of the parameters $\alpha,\sigma,\beta,A,\delta_k,r$, and g_H. One can show that $_1$ and $_2$ satisfy the following conditions.

$$8_1 8_2 = \frac{1+r}{1+g_H}, \; and \; 8_1+8_2 = 1+\left(\frac{1}{1+g_H}\right)\left\{(1+r)+\left(\frac{''}{F}\right)\left[\frac{r+*_k}{1-''}-(g_H+*_k)\right]\left(\frac{r+*_k}{1+r}\right)\right\}.$$

Of the two eigenvalues of the fundamental matrix W, one exceeds unity (unstable root) and the other is less than unity (stable root, call it). Imposing the transversality condition: $\beta(1+g_H)^{1-\sigma} = (1+g_H)/(1+r) < 1$, we can eliminate the unstable root from the solution to obtain the following.

$$x_t - \hat{x} = (x_0 - \hat{x})8^t, \quad and \tag{B.7}$$

$$z_t - \hat{z} = (z_0 - \hat{z})8^t. \tag{B.8}$$

where $x = \ln(X)$ and $z = \ln(Z)$.

The dynamics in the vicinity of the steady state can be portrayed graphically by a `phase diagram'.[9]

[insert `phase diagram']

The stationary schedules corresponding to $x_t = x_{t+1}$ and $z_t = z_{t+1}$ are given by the vertical and the inverted-U curves respectively. The steady state $(\hat{z},\hat{x})$ is attained at the intersection of these schedules. The unstable trajectories diverge from the vicinity of this steady state in the northwest and southeast directions, while the stable trajectories approach the steady from the northeast and southwest directions. The equilibrium transitional dynamics correspond to movements along the stable paths, described by equations (B.7) and (B.8).

As a measure of the speed of adjustment to the steady state, we use the concept of half life (t^*), defined as

$$\frac{x_{t^*} - \hat{x}}{x_0 - \hat{x}} = \frac{1}{2} = \frac{z_{t^*} - \hat{z}}{z_0 - \hat{z}} .$$

Substituting from (A2.3) and/or (A2.4) and solving for t^*, we get $t^* = -\ln(2)/\ln(\ )$.

Alternatively, we can measure the speed of convergence from the implied ratio of $y_{t+1} - \hat{y}_{t+1}$ to $y_t - \hat{y}_t$.

$$\frac{y_{t+1} - \hat{y}_{t+1}}{y_t - \hat{y}_t} = (1+g_h)\left(\frac{x_{t+1}^{1-\alpha} - \hat{x}^{1-\alpha}}{x_t^{1-\alpha} - \hat{x}^{1-\alpha}} \right)$$

$$\approx (1+g_h)\left(\frac{[1+(1-\alpha)x_{t+1}] - [1+(1-\alpha)\hat{x}]}{[1+(1-\alpha)x_t] - [1+(1-\alpha)\hat{x}]} \right) = (1+g_h)\lambda .$$

where we have used $y_t = A(x_t)^{1-\alpha} h_0 (1+g_h)^t$ in the first equality and the approximation $x^{1-\alpha} = 1+(1-\alpha)x$ in the second equality. This is the discrete time analogue of the convergence rate (β) in Barro and Sala-i-Martin (1992)'s continuous time setup, where

$$\ln\left(\frac{y_t}{\hat{y}_t} \right) = e^{-\beta t} \ln\left(\frac{y_0}{\hat{y}_0} \right) .$$

Appendix C: An Example Economy with Full-Fledged Solution

In order to highlight the mechanics of growth in this model, we specialize for simplicity by setting $\sigma = \delta_k = 1$. From equations (12.4) and (12.7), we get

$$\frac{c_{t+1}}{\beta c_t} = (1-\alpha)AX_{t+1}^{-\alpha} = (1-\alpha)\frac{Y_{t+1}}{K_{t+1}}. \tag{C.1}$$

We can write: $c_t = (1-s_t)Y_t$ and $K_{t+1} = s_t Y_t$. Substituting these into (12.7)' yields

$$\frac{1-s_{t+1}}{\beta(1-s_t)} = \frac{1-\alpha}{s_t}.$$

This can be restated as a first order difference equation in s_t, with an unstable root. Thus, the unique (economically plausible) solution is a constant saving rate $s = \beta(1-\alpha)$. Substituting this saving rate into (12.9) gives the following fundamental difference equation in X_t.

$$X_{t+1} = \left[\frac{\beta(1-\alpha)A}{1+g_H}\right]X_t^{1-\alpha}. \tag{C.2}$$

We display this dynamic equation graphically in Figure 12.1. The concave curve, depicting the right hand side of (B.2), intersects the 45^o line at two points to produce two steady states, at the origin (unstable) and E (stable). For any given initial value $X_0 > 0$, the economy must converge to the long run equilibrium point E along the trajectories as indicated by the arrows. At E, the steady state value of the ratio between the two capital stocks is given by $\hat{X} = \{[\beta(1-\alpha)A]/(1+g_H)\}^{1/\alpha}$.

Given this closed form solution, it is straightforward to compute the consumer's utility along the entire growth path. Recursive substitution of (B.2) implies

$$x_t = \left[\frac{1-(1-\alpha)^t}{\alpha}\right]\ln\left[\frac{[1-\beta(1-\alpha)]A}{1+g_H}\right] + (1-\alpha)^t x_0 .$$

Substituting this into $c_t = (1-s_t)Y_t = [1-\beta(1-\alpha)]AX_t^{1-\alpha}h_0(1+g_h)^t$ and then the resulting expression into the utility function, we get

$$U = \sum_{t=0}^{\infty} \ln(c_t) = \left(\frac{1-\alpha}{1-\beta(1-\alpha)}\right)x_0$$
$$+ \left(\frac{1}{1-\beta}\right)\left[\left(1+\frac{1-\alpha}{1-\beta(1-\alpha)}\right)\ln\{[1-\beta(1-\alpha)]A\} + \ln(h_0) + \beta g_h - \left(\frac{1-\alpha}{1-\beta(1-\alpha)}\right)g_H\right].$$

This welfare measure will be used in the capital mobility section to evaluate the gains from intertemporal trade.

Appendix D: Welfare Gains from Free Capital Flows

To evaluate the welfare gains from capital flows, we first compute the equilibrium consumption path under such regime. From the intertemporal condition (12.7), we have $c_t = c_0[\beta(1+r^*)]^t$, where c_0 can be solved from the consumer's present value budget constraint, obtainable by consolidating the flow budget constraint in (12.11).

$$\sum_{t=0}^{\infty} \frac{N_t c_t}{(1+r^*)^t} = \sum_{t=0}^{\infty} \frac{Y_t}{(1+r^*)^t} + K_0 .$$

As we make clear in the text, the capitals ratio will jump to its steady state in one period. Consequently, output per capita is given by

$$Y_t = AX_t^{1-"}h_t = \begin{cases} AX_0^{1-"}h_0 & t = 0 \\ A(\hat{X})^{1-"}h_0(1+g_h)^t & t > 0 \end{cases}$$

Substituting this and $c_t = c_0[\beta(1+r^*)]^t$ into the present value budget constraint, we have

$$c_0 = [1-\$(1+g_N)]\left[X_0h_0 + AX_0^{1-"}h_0 + A\hat{X}^{1-"}h_0\left(\frac{1+r^*}{r^*-g_h}\right)\right].$$

Substituting into the utility function yields

$$U = \sum_{t=0}^{\infty} \ln(c_t)$$

$$\left.\frac{1}{1-\$}\right)\left\{\ln(1-\$) + \ln\left[X_0h_0 + AX_0^{1-"}h_0 + A\hat{X}^{1-"}h_0\left(\frac{1+r^*}{r^*-g_h}\right)\right] + \left(\frac{\$}{1-\$}\right)\ln[\$(1+r^*)\right.$$

$$where \quad r^* = (1-")A\left[\frac{[1-\$(1-")]A}{1+g_H}\right]^{1/"} - 1 .$$

Appendix E: Stochastic Dynamics

We follow Campbell (1994) by solving the equilibrium dynamics of the system of equations (B.3), (B.4), and (12.13) by the method of undetermined coefficients.

We guess a solution for z_t (= $\ln(Z_t)$) of the form:

$$z_t = 0_{zx} x_t + 0_{za} a_t , \tag{E.1}$$

where $_{zx}$ and $_{za}$ are unknown fixed coefficients, and $a_t = \ln(A_t)$. To verify this solution, substitute (E.1) into (B.3) and (B.4) and rearrange to get the following:

$$\begin{aligned} x_{t+1} &= 0_{xx} x_t + 0_{xa} a_t , \\ \text{where } 0_{xx} &= \frac{1+r}{1+g_H} + \left[(g_H + *_k) - \frac{r + *_k}{1 - "} \right] 0_{zx} \\ \text{and } \ 0_{xa} &= \frac{r + *_k}{(1-")(1+g_H)} + \left[(g_H + *_k) - \frac{r + *_k}{1 - "} \right] 0_{za} . \end{aligned} \tag{E.2}$$

$$\Xi_t (x_{t+1} - x_t) + 0_{za} E_t (a_{t+1} - a_t) = \mathrm{F} \left(\frac{r + *_k}{1+r} \right) E_t (a_{t+1} - x_{t+:} \tag{E.3}$$

Substituting (E.2) into (E.3) and using the fact that $E_t a_{t+1} = \ a_t$ from (12.13), we arrive at one equation in x_t and a_t:

$$0_{zx} x_t + [0_{zx} 0_{xa} + 0_{za} (\mathrm{D} - 1)] a_t = \mathrm{F} \left(\frac{r + *_k}{1+r} \right) [(\mathrm{D} - 0_{xa}) a \tag{E.4}$$

Recall from (E.2) that $_{xx}$ is a linear function of $_{zx}$ and $_{xa}$ is a linear function of $_{za}$. Comparing the coefficients on x_t on the two sides of equation (E.4), we can obtain a quadratic equation in $_{zx}$ as follows:

$$
\begin{gathered}
Q_2 0_{zx}^2 + Q_1 0_{zx} + Q_0 = 0, \\
where\ Q_2 = (g_H + *_k) - \frac{r + *_k}{1 - "}, \\
Q_1 = \frac{r - g_H}{1 + g_H} + \mathrm{F}\left(\frac{r + *_k}{1 + r}\right)\left[(g_H + *_k) - \frac{r + *_k}{1 - "}\right],\ and \\
Q_0 = \frac{\mathrm{F}(r + *_k)^2}{(1 - ")(1 + r)}.
\end{gathered}
\tag{E.5}
$$

Imposing the transversality condition (which implies that $r > g_H$), we know that $_{zx}$ must be positive. Taking only the positive root, we have

$$
0_{zx} = -\frac{1}{2Q_2}\left(Q_1 + \sqrt{Q_1^2 - 4Q_0 Q_2}\right). \tag{E.6}
$$

Given this solution for $_{zx}$, we can compare the coefficients on a_t on the two sides of equation (E.4) to solve for $_{za}$:

$$
a = \frac{-\frac{0_{zx}(r + *_k)}{(1 - ")(1 + g_H)} + \mathrm{F}\left(\frac{r + *_k}{1 + r}\right)\left(\mathrm{D} - \frac{r + *_k}{(1 - ")(1 + g_H)}\right)}{(\mathrm{D} - 1) + \left[(g_H + *_k) - \left(\frac{r + *_k}{1 - "}\right)\right]\left[0_{zx} + \mathrm{F}\left(\frac{r + *_k}{1 + r}\right)\right]} \tag{E.7}
$$

The last two coefficients $_{xx}$ and $_{xa}$ can be backed out from (E.2), givem (E.6) and (E.7).

Problems

1. Consider the closed economy example in Appendix C (with $\sigma = \delta_k = 1$). Suppose there is a time-invariant tax ($_k$) levied on capital income ($r_{kt}K_t$) and the tax proceeds are rebated as lump sum transfers to the households. Show how this tax rate will affect the saving rate (s_t), the speed of adjustment to the long run equilibrium, and the levels and rates of growth of per capita output in that equilibrium.

2. In the previous problem, how will your answers be affected if the tax proceeds ($_k r_{kt}K_t$) are used to finance government spending (g_t) in each period?

3. Consider the model with constrained capital mobility in Section 12.2. Calculate the jump in the initial level of consumption resulting from the opening up of the world capital market, in the presence of the two constraints described in the section. Compare this with the corresponding jump in initial consumption in the free capital mobility case.

4. Consider the case of labor mobility described in Section 12.3.

(a) Specify the consumer budget constraint and resource constraint, and derive the law of motion for the capitals ratio (x_t, redined as the ratio between physical capital and domestically employed labor).

(b) Consider a labor migration quota set by the receiving country (the rest of the world. What is the quota level (Q_t in each period t) which makes the constrained labor mobility regime equivalent to the constrained capital mobility regime?

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Endnotes

1. Solow (1956) interprets this factor as labor-augmenting technological progress.

2. On the other hand, we can also allow A_t to follow a deterministic growth trend, say, $A_{t+1} = (1+g_A)A_t$ (Hicks-neutral technological progress). We choose not to specify it this way because such process has already been subsumed under the accumulation process of human capital. In other words, one can interpret $1+g_H = (1+g_A)^{1/\alpha}$, i.e., technological progress is labor-augmenting.

3. Indeed, in most of the discussions below, we need not specialize the production function to the Cobb-Douglas form. We choose this specific form in order to simplify the exposition.

4. An equivalent specification, as in Chapter 7, assumes that the firm is the owner of physical capital and makes its investment and production decision by solving an intertemporal profit maximization problem.

5. Note that the time convention in our notations in this chapter is slightly different from that used in Chapter 5. In this chapter, we denote current period (t) borrowing by B_{t+1} rather than B_t and the corresponding rate of interest by r_{t+1} instead of r_t. This change is done in order to be consistent with the time convention for factor accumulation.

6. In the closed economy, equilibrium in the financial capital market implies that gross saving (saving in the form of physical capital) amounts to consumer's net saving.

7. If the rest of the world starts from an off-steady-state position initially, the economies will immmediately converge and then grow together at the same rates towards a common long run steady state in terms of per capita GDP.

8. Barro, Mankiw, and Sala-i-Martin (1995) consider similar international borrowing constraints. Their model focuses on the impossibility of financing investment in human capital through borrowing.

9. Strictly speaking, the phase diagram aparatus applies only to continuous time dynmaics described in terms of a system of differential equations. The discrete time dynamics that we examine here are just approximations to their continuous time counterparts.