News and Archival Information in Games*

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Abstract

I enrich the typology of players in the standard model of games with incomplete information, by allowing them to have incomplete "archival information" - namely, piecemeal knowledge of correlations among relevant variables. A player is characterized by the conventional Harsanyi type (a.k.a "news-information") as well as the novel "archive-information", formalized as a collection of subsets of variables. The player can only learn the marginal distributions over these subsets of variables. The player extrapolates a well-specified probabilistic belief according to the maximum-entropy criterion. This formalism expands our ability to capture strategic situations with "boundedly rational expectations." I demonstrate the expressive power and use of this formalism with some examples, and also characterize the environments in which the interim and ex-ante versions of equilibrium coincide.

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1 Introduction

When people engage in a static strategic interaction, they make use of various kinds of information. Part of that information is about the *current* realization of exogenous variables. I refer to this kind of information as "news-information", or N-information for brevity. Another kind is what I refer to as "archive-information", or R-information - namely, background data about joint realizations of exogenous and endogenous variables in past instances of the game.

To use a military-intelligence metaphor, N-information can be about the armies' current objectives, constraints and knowledge, whereas R-information is about past realizations of these variables and how they were correlated with the armies' behavior. To use a different, journalistic image, N-information is akin to a news flash about a corporate scandal, whereas R-information is what a reporter gets when he starts digging the newspaper's database for evidence about how various actors acted in past scandals. Thus, while N-information is knowledge of characteristics of the current strategic situation, R-information enables the player to make sense of this knowledge and draw conclusions about the possible consequences of his actions.

Standard game theory treats the two types of information very differently. Harsanyi's model of static games with incomplete information offers a rich general description of players' incomplete information regarding the current realization of exogenous variables, including high-order information: players' information about players' information about players' information etc. In other words, Harsanyi's type-space formalism is exclusively about news-information; it leaves the task of describing archive-information to the solution concept. One-shot solution concepts like rationalizability or level-k reasoning ignore archive-information altogether. At the other extreme, Nash equilibrium presumes that players have complete archive-information.

The last two decades saw various proposals for solution concepts for games that retain the steady-state approach of Nash equilibrium, while relaxing its assumption of complete archive-information and replacing it with some notion of limited learning feedback that players receive regarding the steady-state distribution, coupled with some model of how players form beliefs given their partial feedback. I provide a detailed literature review in Section 1.2. At this stage, it suffices to say that virtually all previous proposals assume that the feedback limitation that characterizes each player is fixed. And neither of them model players' imperfect information (of either kind) regarding their opponents' archive-information.

However, it is easy to describe in natural language real-life situations in which one player has incomplete news-information about another player's archive-information. Following up the military-intelligence imagery, suppose that army 1 receives news from a dubious source that army 2 has just gained access to records of army 1's behavior in other situations, some of which share the present situation's exogenous characteristics. Likewise, we can meaningfully talk about one player having incomplete archive-information about another player's archive-information. For instance, army 1 may receive access to a computer file that documents army 2's archival access in other situations. And one can easily extend these descriptions to high-order statements that involve both news-information and archive-information and make one's head swim, just as we do for news-information in the standard Harsanyi model.

In this paper I present a new type space for static games that combines both types of incomplete information. This formalism adds a bit of pedantry to the standard model, by requiring the game's description to include an explicit list of *variables* and classify them as exogenous variables, players' actions and consequence variables. It also makes heavy use of an explicit notational distinction between variables and their labels. Denote the set of variable labels by L.

A player's type has two components. The first component is the player's N-information, defined as the *realization* of some subcollection of the exogenous variables. This is essentially the standard, Harsanyi notion of a player's type: the player is partially informed about the current realization of the exogenous variables. The novelty lies in the second component of the player's type, namely his R-information. This is defined as a collection of subsets of L. It represents the player's "archival access", or "database privilege", to use a different metaphor. For any possible steady-state probability distribution p over all variables, the player gets to learn the marginal of p over each of the subsets of variables that correspond to the subsets of L constituting the player's R-information. In other words, rather than learning the entire joint distribution p, the player only has piecemeal knowledge of it, given by some marginals. Although the player may fail to receive archival data about certain variables, I assume that his payoff function is always measurable with respect to the variables about which he does get data.

The player forms a belief in two stages. First, he extrapolates a subjective probabilistic belief over the variables about which he has data, thus forming a distorted perception of the objective steady-state distribution p. There are infinitely many extrapolation rules one could follow. However, a recurring theme in the literature on equilibrium models with boundedly rational expectations is that players apply an "Occam's razor" parsimony criterion when extrapolating from limited feedback. In this spirit, I assume that players use the *maximum-entropy* criterion. That is, the player's belief is the distribution (over the variables about which he has archival data) that maximizes entropy subject to being consistent with the marginals he knows. In the second stage, he conditions this extrapolated belief on his N-information via Bayes' rule, to form a subjective belief over payoff-relevant outcomes as a function of his action.

Thus, each component of the player's type corresponds to a different stage in his belief-formation process. The first stage makes use of the player's R-information via the extrapolation rule of maximum entropy, while the second stage makes use of his N-information via the conditioning rule of Bayesian updating. To revisit the journalistic metaphor, players first use "archival research" to extrapolate an unconditional belief, and then condition it on the "news flash". Equilibrium is defined in a completely standard way: each player always plays a best-reply to his subjective belief.

1.1 An Example: Prisoner's Dilemma

The following is a basic illustration of the formalism. Two players, denoted 1 and 2, play the following version of the Prisoner's Dilemma:

$a_1 \backslash a_2$	C	D
C	$3,3$	$0,4$
D	$4,0$	$1,1$

There is no uncertainty regarding the game's payoff structure; the only uncertainty will be about players' archive-information.

Let R_i and a_i denote player i's R-information and action, respectively. The set of variable labels is $L = \{\delta_{a_1}, \delta_{a_2}, \delta_{R_1}, \delta_{R_2}\}$. With probability $1-\alpha$, both players have complete archive-information - that is, $R_1 = R_2 = \{L\}$. This means that the players have a full grasp any steady-state distribution over all four variables. With probability α, players have incomplete archive-information, where R_1 and R_2 are given as follows:

$$\begin{aligned} R_1 &= \{\{\delta_{R_1}, \delta_{a_1}\}, \{\delta_{a_1}, \delta_{a_2}\}\} \\ R_2 &= \{\{\delta_{R_2}, \delta_{a_2}\}, \{\delta_{a_1}, \delta_{a_2}\}\} \end{aligned}$$

That is, player i learns the joint distribution over his archive-information and action, as well as the joint distribution over the action profile.

The interpretation is as follows. The joint distribution over (a_1, a_2) represents a large, publicly available record of past game outcomes, each independently drawn from a steady-state joint distribution p over all four variables. Because the record is public, players always have access to it and they can learn the steady-state distribution over the action profile. In contrast, the record of past joint realizations of player i's archive-information and his action may be privately accessed by player i alone. Indeed, this is the case with probability α. With probability $1-\alpha$, however, the complete historical record of all variables becomes publicly available.

Finally, let us define the players' conventional news-information. Assume $t_i = R_i$ - i.e., each player knows what archival data he has access to, but he does not know his opponent's. This assumption is in fact immaterial for the equilibrium analysis of this example, but it simplifies the exposition because it means that player i's type is entirely pinned down by his R-information.

The exogenous component of p is $p(R_1, R_2)$ - i.e., the distribution over players' types. The endogenous components are the players' strategies, given by the conditional probability distributions $p(a_1 \mid R_1)$ and $p(a_2 \mid R_2)$. The assumption of simultaneous/independent moves means that p satisfies the conditional-independence property $a_i \perp R_j, a_j \mid R_i$ for each $i \neq j$. When player i's type is R_i, he forms his belief in two stages. First, he extrapolates an unconditional subjective belief p_{R_i} over the variables he has data on. Then, he conditions this belief on his news-information t_i and his action a_i to evaluate the payoff consequences of his actions.

Let us derive players' beliefs as a function of their types. Complete archive-information means rational expectations. That is, when $R_i = \{L\}$, $p_{R_i} = p$. Because D is a strictly dominant action, it follows that when player i's type is $R_i = \{L\}$, he will necessarily play D in any equilibrium. In contrast, suppose that $R_i = \{\{\delta_{R_i}, \delta_{a_i}\}, \{\delta_{a_1}, \delta_{a_2}\}\}$. Then, the player learns the marginal distributions $p(R_i, a_i)$ and $p(a_1, a_2)$. The maximum-entropy extension of these marginals is

$$p_{R_i}(R_i, a_1, a_2) = p(R_i, a_i)p(a_j \mid a_i)$$

Conditioning this belief on the player's news information $t_i = R_i$ and his action, we obtain

$$p_{R_i}(a_j \mid R_i, a_i) = p(a_j \mid a_i)$$

Thus, when player i's type is $R_i = \{\{\delta_{R_i}, \delta_{a_i}\}, \{\delta_{a_1}, \delta_{a_2}\}\}$, he forms a conditional subjective belief regarding a_j as if he thinks that his own action *causes* the opponent's action. In other words, he acts as if he mistakes the correlation between a_i and a_j (due to their respective dependence on players' correlated types) for a causal effect of the former on the latter.

Equilibrium in this environment is defined conventionally, in the spirit of termbling-hand perfection (Selten (1975)). A profile of completely mixed strategies constitutes an ε-equilibrium if whenever $p(a_i \mid t_i, R_i) > \varepsilon$, a_i maximizes player i's expected utility with respect to his conditional subjective belief $p_{R_i}(\cdot \mid t_i, a_i)$. An equilibrium is simply a limit of a sequence of ε-equilibria, where $\varepsilon \to 0$. We can now characterize the set of symmetric equilibria in this example. When equilibria are not sensitive to the perturbation's form, I will omit this part and go straight to the equilibria, without describing ε-equilibria first.

One equilibrium is for players to play D regardless of their type - this is the conventional game-theoretic prediction. Because C is never played, this equilibrium needs to be sustained by a perturbation. Suppose that players choose D with probability $1 - \varepsilon$, independently of their type, where ε is arbitrarily small. Then, $p(a_j \mid a_i) = 1 - \varepsilon$ for all a_i, and therefore $a_i = D$ is strictly dominant, hence the players' strategies constitute an ε-equilibrium. Taking the $\varepsilon \to 0$ limit gives us the equilibrium.

Another equilibrium is for each player i to play D if and only if $R_i = \{L\}$. To see why, we only need to establish that when $R_i = \{\{\delta_{R_i}, \delta_{a_i}\}, \{\delta_{a_1}, \delta_{a_2}\}\}$, $a_i = C$ is a best-reply (because D is strictly dominant under rational expectations). Under this candidate equilibrium, $p(a_j = C \mid a_i = C) = p(a_j = D \mid a_i = D) = 1$. We saw that when $R_i = \{\{\delta_{R_i}, \delta_{a_i}\}, \{\delta_{a_1}, \delta_{a_2}\}\}$, player i's belief can be described as if he interprets the perfect correlation between a_i and a_j causally, and therefore believes that player j will play C if and only if i plays C. Therefore, C is a subjective best-reply.

There is a third, "hybrid" equilibrium, in which each player i plays C with probability $\lambda \in (0,1)$ when $R_i = \{\{\delta_{R_i}, \delta_{a_i}\}, \{\delta_{a_1}, \delta_{a_2}\}\}$. (As before, he plays D with probability one when $R_i = \{L\}$.) Best-replying requires the player to be indifferent between the two actions, given his conditional subjective belief:

$$\begin{aligned} 3 \cdot p(a_j &= C \mid a_i = C) + 0 \cdot p(a_j = D \mid a_i = C) \\ &= 4 \cdot p(a_j = C \mid a_i = D) + 1 \cdot p(a_j = D \mid a_i = D) \end{aligned}$$

Plug

$$\begin{aligned} p(a_j &= C \mid a_i = C) = \frac{\alpha\lambda^2}{\alpha\lambda} = \lambda \\ p(a_j &= C \mid a_i = D) = \frac{\alpha\lambda(1-\lambda)}{1-\alpha\lambda} \end{aligned}$$

and obtain the solution

$$\lambda = \frac{1}{3-2\alpha}$$

Thus, the formalism enables us to capture the idea that some types of players perceive any long-run correlation between players' actions in causal terms. The common variation in players' archive-information creates the correlation between their actions, and one realization of their archive-information gives rise to the mistaken causal interpretation of this correlation.

1.2 Related Literature

As mentioned earlier in the Introduction, the literature contains a number of important game-theoretic solution concepts in which players receive partial feedback regarding equilibrium behavior. It is helpful to define each of these proposals by two ingredients: the way it formalizes partial feedback, and the belief-formation rule it assumes.

The approaches in the literature that are closest to the present paper are those in which players' partial feedback takes a concrete form, and they extrapolate a belief from their feedback according to an explicit rule that intuitively follows the Occam's razor principle. Osborne and Rubinstein (1998) assume that a player's feedback takes the form of a collection of finite samples taken from the conditional distributions over outcomes that is induced by each action. Players ignore sampling error and believe that the sample associated with each action is perfectly representative of its true conditional distribution over outcomes. Osborne and Rubinstein (2003) study a variant of this concept, in which each player's feedback consists of one sample drawn from the *unconditional* distribution over outcomes. In Esponda (2008), the feedback sample is not finite but it is *selective* - i.e., it is drawn from the distribution over outcomes conditional on players' equilibrium behavior. Players' rule for extrapolating from the sample reflects unawareness of its selectiveness. Jehiel (2005) and Jehiel and Koessler (2008) present a formalism that is the closest in spirit to the present paper, in the sense that a player's feedback limitation is a personal characteristic, rather than part of the definition of the solution concept (as in, say, Osborne and Rubinistein (1998)). Under this approach, each player partitions the set of possible contingencies (histories in extensive games, states of the world in Bayesian games) into "analogy classes", such that the feedback that he receives is the average distribution over contingencies within each analogy class. The player best-replies to this coarsening of the true equilibrium distribution.

In other approaches, players do not extrapolate a belief from limited feedback. Instead, they arrived at the game with a subjective, possibly misspecified prior model, and they fit this model to the feedback. For example, Esponda and Pouzo (2016) formalize feedback abstractly (in the tradition of self-confirming equilibrium (Battigalli (1987), Fudenberg and Levine (1993)) as a general consequence variable (in applications, it typically coincides with the player's payoff, or with the observed history in an extensive game). Each player has a prior belief over a set of possible distributions over consequences conditional on the game's primitives and the players' actions. This set represents the player's model, and it is misspecified if it rules out the true conditional distribution. In equilibrium, the player's belief is a conditional distribution in this set that is closest (according to a modified Kullback-Leibler Divergence) to the true equilibrium distribution. Battigalli et al. (2015) assume a similar notion of feedback and adopt a non-probabilistic model of beliefs in the decision-theoretic "ambiguity" tradition.

These two general approaches to belief formation - extrapolation from feedback vs. fitting a subjective model to feedback - are not mutually exclusive, and sometimes there are dualities between them. In particular, in Section 3 I will form a link between the approach in this paper and Spiegler (2016), where decision makers fit a prior subjective (causal) model to feedback.

Eyster and Rabin (2005) take a different interpretation to modeling distorted equilibrium beliefs in games. In "fully cursed" equilibrium, a player wrongly believes that the distribution over his opponents' actions is a measurable function of his own signal. In "partially cursed" equilibrium, a player's belief is a convex combination between the rational-expectations and fully cursed beliefs. Eyster and Rabin regard this belief distortion as a behavioral bias and do not attempt to derive it from explicit partial feedback or from an explicit subjective model. However, one can show (Jehiel and Koessler (2008), Spiegler (2016)) that fully cursed beliefs can easily be reinterpreted along these lines. Spiegler (2017a) shows that in some examples, partially cursed beliefs can be partially reconciled with some model of naive extrapolation from partial feedback.

The crucial difference between all the approaches described above and the present paper is that in all these papers, limited feedback is either part of the solution concept or a fixed characteristic of each player. The modeler allows for no uncertainty about this feature of the game. The only exception is the appendix to Eyster and Rabin (2005), where the convexification parameter that defines the degree of a player's cursedness is drawn from some distribution. Neither of the existing approaches involves an explicit model of players' uncertainty regarding other players' feedback.

2 The Formalism

For expositional simplicity, I restrict attention to interactions that involve two players, 1 and 2. The extension to more than two players is entirely straightforward. Let $x = (x_1, ..., x_n)$ be a collection of variables, where $x_k \in X_k$ and X_k is some finite set. The *label* of variable x_i is i. When the notation for x_i has a particular meaning (as in the Introduction) I use the alternative notation l_{x_i}. Denote the set of variable labels by L. We can decompose x as follows, $x = (x_E, a_1, a_2, x_C)$, where E and C are two disjoint subsets of L, such that x_E is the collection of *exogenous* variables, a_1 and a_2 denote the players' actions, and x_C is the collection of *consequence* variables (in many applications, C will be empty). Player i's *type* is defined by two (possibly overlapping) projections of x_E: R_i denotes the player's archive-information, and t_i denotes his news-information. Each player i is a subjective expected utility maximizer, with vNM function $u_i : X \to \mathbb{R}$.

Let $p \in \Delta(X)$ represent a steady-state distribution over all variables. The most convenient interpretation of p is that it represents a historical record of many similar interactions. The individual game is a one-shot interaction between players who may lack direct, personal experience with similar situations - p records the collective experience of many other agents who assumed the roles of the two players.

The exogenous components of p are $(p(x_E))$ and $(p(x_C \mid x_E, a_1, a_2))$. Assume that these components have full support. (In applications, we can sometimes afford to relax this restriction.) The endogenous components of p are the players' strategies, $\sigma_i = (p(a_i \mid R_i, t_i))$ for every $i = 1, 2$. Strategies have full support, unless indicated otherwise. To reflect the assumption that players' moves are independent, assume that conditionally on t_i, R_i, a_i is independent of all other exogenous variables as well as of a_j.

Let us now impose explicit structure on R_i, in order to substantiate its interpretation as archive-information. Define R_i as a collection of subsets of L - i.e., the set of variable labels. For any steady-state distribution p and every $S \in R_i$, player i learns the marginal of p over x_S (denoted p^S). That is, R_i represents the player's limited access to archival data - his "database privileges", so to speak. Let $N(R_i)$ denote the union of the members of R_i. I restrict R_i to satisfy the property that u_i is measurable with respect to $(x_{N(R_i)}, a_i)$. That is, the player always has data on variables that unambiguously determine his payoffs.

Note that I do not require $l_{a_i} \in N(R_i)$ - i.e., player i's archival information need not include data about a_i. This is not an obvious modeling choice. On one hand, such data may be easier to retrieve (as explained in the Prisoner's Dilemma example). On the other hand, player i is mostly interested in predicting *player j's* behavior; learning about the past actions of agents who assumed the role of player i need not be relevant for this purpose.

To make a decision, player i forms a probabilistic belief over payoff-relevant variables, as a function of his type. In the standard model of games of incomplete information, the player's type is defined solely in terms of his news-information, and he forms his conditional probabilistic assessment in a single step, using Bayes' rule. In the present model, the player's type consists of two kinds of information, and so he forms his conditional probabilistic assessment in two stages, where each stage makes use of a different kind of information.

The first stage involves *extrapolation from archive-information*: the player forms the unconditional belief $p_{R_i} \in \Delta(X_{N(R_i)})$ according to the *maximum-*

entropy criterion. That is:

$$p_{R_i} = \arg \max_{q \in \Delta(X_{N(R_i)})} - \sum_{z \in X_{N(R_i)}} q(z) \ln(q(z))$$

subject to the constraint that $q \equiv p^S$ for every $S \in R_i$.[1] That is, the player's unconditional belief (over the variables about which he has data) maximizes entropy subject to being consistent with the marginals he learns. The second stage involves *conditioning on news-information*. Here the conditioning rule is conventional Bayesian updating. The player's conditional belief is $p_{R_i}(x_{N(R_i)} \mid t_i)$. This is the belief that the player uses to evaluate actions.

Thus, each component of the player's type is associated with a particular *operation* that he performs on the objective joint distribution. The first stage involves *extrapolation*; the player's archive-information tells us what he extrapolates from. The second stage involves *conditioning*; the player's news-information tells us what he should condition on. This stage utilizes the "canonical" rule of Bayesian updating. By comparison, there is no "canonical" extrapolation rule. Nevertheless, there is a common intuition that extrapolating a belief from partial data should follow an "Occam's razor" parsimony criterion. This principle is implicit in many of the solution concepts cited in the Introduction. The maximum-entropy criterion (which has a rich tradition in statistical physics and data analysis (see Jaynes (1957)) is one way to implement the parsimony criterion. It basically views minimal assumptions on correlations as parsimonious, and thus looks for the distribution that exhibits maximal independence, subject to being consistent with the correlation patterns that *are* observed in the data.

For illustration, let $L = \{1, 2\}$ and suppose that $R = \{\{1\}, \{2\}\}$. Suppose that x_i gets n_i values, such that any pair (x_1, x_2) can be identified with a distinct pair (i, j), $i = 1, ..., n_1$ and $j = 1, ..., n_2$. The known marginals are, for each i, j, $r_i = \sum_{j=1}^{n_2} p_{ij}$ and $s_j = \sum_{i=1}^{n_1} p_{ij}$. The maximum-entropy

[1] The solution to the constrained maximization problem will always be unique.

extension of these marginals is the joint probability distribution $(q_{ij})_{i,j=1,\dots,n}$ that minimizes $\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} q_{ij}\ln q_{ij}$ subject to the constraints $r_i = \sum_{j=1}^{n_2} q_{ij}$ and $s_j = \sum_{i=1}^{n_1} q_{ij}$. We can now write down the first-order conditions of the Lagrangian and obtain the solution $q_{ij} = r_i s_j$. That is, $p_R(x_1, x_2) = p(x_1)p(x_2)$

Having defined players' beliefs as a function of their types, we are ready to introduce the notion of equilibrium, which is a standard trembling-hand perfection concept.

Definition 1 *Fix $\varepsilon > 0$ and the exogenous components of the steady-state distribution p. A profile of full-support strategies (σ_1, σ_2) is an ε-equilibrium if for every $i = 1, 2$ and every a_i, t_i, R_i for which $p(a_i \mid t_i, R_i) > \varepsilon$,*

$$a_i \in \arg\max_{a_i'} \sum_{z \in X_{N(R_i)}} p_{R_i}(z \mid t_i, a_i')u_i(z)$$

A strategy profile (σ_1^*, σ_2^*) (which need not satisfy full support) is an equilibrium if it is the limit of a sequence of ε-equilibrium with $\varepsilon \to 0$.[2]

Establishing existence of equilibrium is straightforward. Because p_{R_i} is a continuous function of p, the proof is essentially the same as in the case of standard trembling-hand perfect equilibrium.

I conclude this section with two comments about the formalism. First, the language I use to describe the strategic interaction is slightly different from the conventional description of simultaneous-move games. Instead of a game form, I represent the simultaneity assumption by a conditional independence assumption that I impose on the joint distribution over all variables. In addition, the primitives of the model contain an explicit list of variables. In particular, the state of Nature or players' news-information or archive-information can be described as collections of variables. This level of detail

[2] The conditioning of p_{R_i} on a_i' is irrelevant if $l_{a_i} \notin N(R_i)$.

is important. For example, when the state of Nature is described by multiple variables, the player need not have perfect archive-information about the correlation among these variables.

Second, the definition of the player's subjective belief $p_{R_i}(\cdot \mid t_i, a_i)$ does not make any explicit assumption about whether the player actively conditions on his own archive-information. One could assume that he is perfectly aware of his current archival-data limitations, and therefore conditions his belief on it. Alternatively, one might want to capture situations in which the player is not perfectly introspective, and therefore fails to realize that his current data limitations may have informational content. Any assumption we wish to make in this regard is fully encoded in the joint distribution of t_i and R_i. In particular, full introspection will be captured by a specification in which R_i is a component of t_i (as in the Prisoner's Dilemma example of the Introduction).

3 A Graphical Representation

Calculating the maximum-entropy extension of a collection of marginal distributions can be non-trivial. However, there are cases in which it takes a simple, transparent form. To take an extreme case, if all subsets that constitute R are disjoint, then

$$p_R(x_{N(R)}) = \prod\nolimits_{S \in R} p(x_S)$$

More generally, when R satisfies a certain combinatorial property, p_R can be described as if it is the result of fitting a causal model to the objective distribution p. In this section I briefly explain this observation. The material here is completely unoriginal, and borrowed from the literature on Bayesian networks (e.g., Cowell et al. (1999), Pearl (2009), Koller and Friedman (2009)). The exposition is essentially taken from Spiegler (2017a), which examines a more behaviorally motivated extrapolation procedure that coincides with maximum entropy under the combinatorial property referred to above.

The collection R of subsets of L satisfies the *running intersection property* if the subsets that constitute R can be ordered $S^1, ..., S^m$ such that for every $k = 2, ..., m$, $S^k \cap (\cup_{j<k} S^j) \subseteq S^i$ for some $i = 1, ..., k-1$. This property obviously always holds for $m = 2$. The collection $\{\{1,2\},\{2,3\},\{3,4\}\}$ satisfies the property, whereas the collection $\{1,2\},\{2,3\},\{1,3,4\}\}$ violates it. It can be shown (see Hajek et al. (1992)) that when R satisfies the running intersection property, the maximum-entropy extension of $(p^S)_{S\in R}$ is given by

$$p_R(x_{N(R)}) = \prod\nolimits_{S^1,...,S^m} p(x_{S^k - (\cup_{j<k} S^j)} \mid x_{S^k \cap (\cup_{j<k} S^j)}) \tag{1}$$

where the enumeration $1, ..., m$ validates the running intersection property. For instance, when $R = \{\{1,2\},\{2,3\},\{3,4\}\}$, we get

$$\begin{aligned} p_R(x_1,x_2,x_3,x_4) &= p(x_1,x_2)p(x_3 \mid x_2)p(x_4 \mid x_3) \\ &= p(x_1)p(x_2 \mid x_1)p(x_3 \mid x_2)p(x_4 \mid x_3) \end{aligned} \tag{2}$$

Thus, the running intersection property implies that p_R can be presented as a factorization of $p(x_{N(R)})$ into a product of marginal and conditional distributions. Moreover, the factorization has a *causal* interpretation. For instance, (2) looks as if it is consistent with a causal chain that can be described diagrammatically by the directed graph $1 \to 2 \to 3 \to 4$.

This is a general property. Suppose that R satisfies the running intersection property. Define a *directed acyclic graph* $G = (N, E)$, where $N = N(R)$ is the set of nodes, and $E \subset N \times N$ is the set of directed links (that is, $(j,i) \in E$ means that there is a link $j \to i$). Slightly abusing notation, define $G(i) = \{j \in N \mid (j,i) \in E\}$. As usual, a subset of nodes $C \subset N$ is a *clique* in G if there is a link between any pair of nodes in C (that is, for every $i,j \in C$, $i \neq j$, $(i,j) \in E$ or $(j,i) \in E$). A clique is *maximal* if it is not contained by any other clique. For any objective distribution p, define p_G as the factorization of p according to G, given by the formula

$$p_G(x_N) = \prod_{i \in N} p(x_i \mid x_{G(i)})$$

This is known in the literature as a Bayesian-network factorization formula.

The following is a restatement of a result by Hajek et al. (1992) - see Spiegler (2017a) for more details.

Proposition 2 *Suppose that R satisfies the running intersection property. Then, there exists a directed acyclic graph G such that $p_G \equiv p_R$. Moreover, G satisfies two additional properties. First, it is perfect - i.e., $G(i)$ is a clique for every $i \in N$. Second, the set of maximal cliques in G is R.*

This result establishes that when R satisfies the running intersection property, the maximum-entropy extension of the agent's archival data can be equivalently described as the outcome of fitting a subjective causal model (formalized as a directed acyclic graph) to the objective distribution. Moreover, the relation between the graph and R is simply that R is the set of maximal cliques in the graph. Finally, the graph is perfect, which implies a number of important and useful properties (see Spiegler (2017a,b). In particular, it induces correct marginal distributions over all individual variables.

The graphical interpretation of p_R has two important roles. First, it offers a convenient visualization of the player's type. This in turn makes the conditional independence properties of p_R visible and facilitates calculations. Second, it highlights the feature that the belief that the player extrapolates from partial data looks as if he imposes a false causal interpretation on observed correlations. Recall that in the Prisoner's Dilemma example of the Introduction, a player's subjective belief could sometimes be described as if he falsely believes that his action causes the opponent's action and explained the observed long-run correlation between the players' actions in terms of this wrong causal perception.

It is important to note that the perfect-DAG representation of p_R is not unique. If two perfect DAGs R and R' have the same set of maximal cliques, then $p_R = p_{R'}$. For example, (2) could be equivalently written as

$$p_R(x_1, x_2, x_3, x_4) = p(x_4)p(x_3 \mid x_4)p(x_2 \mid x_3)p(x_1 \mid x_2)$$

which factorizes p according to the DAG $4 \to 3 \to 2 \to 1$. See Spiegler (2017a,b) for extended discussions of this property.

The running intersection property is not an assumption that has a strong a priori motivation. However, in game-theoretic contexts, it is a natural property to focus on. To see why, suppose that the player thinks that he plays a two-player Bayesian game, in which the two players receive conditionally independent signals t_1 and t_2 of the state of Nature θ, and that each player's action is purely a function of his own signal. If this perception is correct, then the only joint distributions that player i needs to learn are $p(\theta, t_1)$, $p(\theta, t_2)$ and $p(t_j, a_j)$. The collection $R_i = \{\{\delta_\theta, \delta_{t_1}\}, \{\delta_\theta, \delta_{t_2}\}, \{\delta_{t_2}, \delta_{a_2}\}$ satisfies the running intersection property. Equivalently, his perception of the situation means that he believes that p is consistent with a causal model given by the following perfect graph:

$$a_1 \leftarrow t_1 \leftarrow \theta \rightarrow t_2 \rightarrow a_2$$

Thus, a natural perception of the situation lends itself to archival data that induces a collection R_i that satisfies the running intersection property - or, equivalently, to a subjective causal model that is given by a perfect graph. However, this is not always the case, and when the running intersection property is violated, calculating the maximum-entropy extension can be cumbersome. At any rate, all the examples in this paper will satisfy the property.

4 Examples

In this section I illustrate the formalism with three examples. For expositional clarity, I fix the conventional aspects and vary the novel component. The game that serves as a template is familiar from the "global games" literature since Rubinstein (1989), Carlsson and van Demme (1993) and Morris and Shin (1998). Its payoff structure makes high-order strategic reasoning crucial for players' behavior, and therefore enables us to illustrate the novel types of high-order reasoning that the present formalism can capture.

Thus, throughout the section, we will examine a 2×2 game in which a_1 and a_2 take values in $\{0, 1\}$ and the payoff matrix is

$a_1 \backslash a_2$	1	0
1	$\theta - \varepsilon, \theta - \varepsilon$	$-1, 0$
0	$0, -1$	$0, 0$

where $\varepsilon > 0$ is arbitrarily small, $\theta \in \{0, 1\}$ is the state of Nature, and $p(\theta = 1) = \frac{1}{2}$. In all the examples, players have complete news-information of θ. Therefore, under rational expectations, they would both find $a = 0$ a strictly dominant action when $\theta = 0$, whereas under $\theta = 1$ they know they are playing a coordination game with two Nash equilibria: $(0, 0)$ and $(1, 1)$.

In most of this section, I will assume that player i does not obtain archival data about his own past actions. That is, $l_{a_i} \notin N(R_i)$ for every realization of R_i. This means that the player's extrapolated belief treats a_j as independent of a_i conditional on player i's news-information. As a result, he will play $a_i = 0$ when $\theta = 0$, regardless of his R-information.

4.1 N-Information about R-Information

Suppose that player 1 has complete R-information with probability one. Therefore, we can omit R_1 as a variable. The only relevant variables are θ, R_2, a_1, a_2, where only θ, a_1, a_2 are directly payoff-relevant. Player 2's R-

information is random and independent of θ. With probability $\alpha \in (0, \frac{1}{2})$, $R_2 = \{\{l_\theta, l_{a_1}\}\}$ - a realization also denoted $R_2 = 1$. With the remaining probability $1 - \alpha$, $R_2 = \{\{l_\theta\}, \{l_{a_1}\}\}$ - a realization also denoted $R_2 = 0$. Thus, R_2 records whether player 2 learns the correlation between player 1's action and the state of Nature.

As to players' N-information, assume for now that $t_1 = \theta$ and $t_2 = (\theta, R_2)$. That is, player 1 knows the current state of Nature but receives no information about player 2's current R-information - we will later relax this assumption. Player 2 knows the current state of Nature as well as his own current R-information.

Let us construct player 2's beliefs as a function of his type:

$$\begin{aligned} p_{R_2=1}(\theta, a_1) &= p(\theta, a_1) \\ p_{R_2=0}(\theta, a_1, R_2, a_1) &= p(\theta)p(a_1) \end{aligned}$$

The derivation of $p_{R_2=1}$ is trivial, because $R_2 = 1$ enables player 2 to learn precisely the joint distribution of θ and a_1. The derivation of $p_{R_2=0}$ is essentially the same as in the example we used in Section 2 to illustrate the maximum-entropy rule.

Therefore,

$$\begin{aligned} p_{R_2=1}(a_1 &\mid \theta, R_2, a_2) = p(a_1 \mid \theta) \\ p_{R_2=0}(a_1 &\mid \theta, R_2, a_2) = p(a_1) \end{aligned}$$

Thus, the realization $R_2 = 0$ captures a "cursed" player (as in Eyster and Rabin (2005)), who does not perceive the correlation between player 1's action and the state of Nature. Following Ettinger and Jehiel (2010), this case can be interpreted as a situation in which player 2 commits the Fundamental Attribution Fallacy - i.e., he does not realize that player 1's behavior can be influenced by the state of Nature. More concretely, imagine that the interpretation of players' dilemma in this example is that they need to decide whether to act considerately in some social situation. In state $\theta = 0$, players cannot afford to be considerate, regardless of what the opponent does. In state $\theta = 1$, there are gains from mutually considerate behavior, and players are willing to act considerately if they expect similar behavior from the opponent. Player 1's action is potentially responsive to the social situation. However, when $R_2 = 0$, player 2 lacks access to the record of player 1's past behavior and does not get to learn this correlation; he extrapolates a belief that treats player 1's behavior as a non-situational statistical pattern.

Proposition 3 *There is a unique equilibrium in this example, in which both players always play $a = 0$.*

Proof. When $\theta = 0$, both players choose $a = 0$. To see why, note first that player 1 and player 2 under $R_2 = 1$ have rational expectations, and therefore correctly recognize that $a = 0$ is a dominant action under $\theta = 0$. When $R_2 = 0$, we saw that player 2 believes that player 1 mixes over actions independently of θ, R_2, a_2. Therefore, he finds $a_2 = 0$ to be the unique best-reply. The same argument holds when $R_2 = 0$ and $\theta = 1$.

Let us try to sustain an equilibrium in which $p(a_1 = 1 \mid \theta = 1) > 0$. As we saw in the previous paragraph, player 2 plays $a_2 = 0$ when $R_2 = 0$, independently of θ. Thus, player 2 plays $a_2 = 1$ with positive probability only if $\theta R_2 = 1$, such that $p(a_2 = 1 \mid \theta = 1) \leq \alpha$. Player 1's unique best-reply is $a_1 = 0$, a contradiction. It follows that player 1 always plays $a_1 = 0$ in any equilibrium. Completing the proof is straightforward. ■

The plot thickens: A second-order attribution error

What would happen if player 1 had partial N-information regarding player 2's R-information - i.e., he received a signal about whether player 2 exhibits the Fundamental Attribution Error? To incorporate this possibility, let us add the explicit variable $s_1 \in \{0, 1\}$, which represents player 1's signal regarding the value of R_2. Assume that the signal is binary with accuracy q - i.e.,

$p(s_1 = R_2) = q$ for every R_2, where $q \in (\frac{1}{2}, 1)$. Player 1' N-information is now $t_1 = (\theta, s_1)$. Player 2's N-information is as before: $t_2 = (\theta, R_2)$.

Continue to assume that player 1 has complete R-information. As to player 2, we now need to assume whether he has archive-information about the new variable s_1. Suppose that R_2 is distributed exactly as in the original specification of this example. In particular, player 2 never receives any data about s_1. Therefore, p_{R_2} is exactly as in the original specification. The difference is that now $R_2 = 1$ does *not* induce rational expectations. Rather, it captures a *"second-order" attribution error*: the player does not understand that player 1's behavior is responsive to his news-information about whether player 2 exhibits an attribution error.

Proposition 4 *There is a unique equilibrium in this example, in which both players always play $a = 0$.*

Proof. Let us try again to sustain an equilibrium in which $p(a_1 = 1 \mid \theta = 1) > 0$. First, derive player 1's posterior belief regarding R_2 as a function of his signal s:

$$\begin{aligned} p(R_2 &= 1 \mid s_1 = 1) = \frac{\alpha q}{\alpha q + (1-\alpha)(1-q)} \\ p(R_2 &= 1 \mid s_1 = 0) = \frac{\alpha(1-q)}{\alpha(1-q) + (1-\alpha)q} \end{aligned}$$

Because p_{R_2} is as in the original specification, the result that player 2 plays

$a_2 = 1$ with positive probability only if $\theta R_2 = 1$ continues to hold. By our assumptions on α and q, $p(R_2 = 1 \mid s_1 = 0) < \frac{1}{2}$. Therefore, when player 1 observes $s_1 = 0$, his unique best-reply is $a_1 = 0$. It follows that

$$p(a_1 = 1 \mid \theta = 1) \leq p(s_1 = 1) = \alpha q + (1-\alpha)(1-q) < \frac{1}{2}$$

Therefore, player 2's best-reply is $a_2 = 0$, regardless of R_2. Player 1's best-reply is necessarily $a_1 = 0$ regardless of s_1, a contradiction. It follows that player 1 always plays $a_1 = 0$ in any equilibrium. Completing the proof is straightforward. ■

Note that player 2's second-order attribution error is key to this negative result. If $R_2 = 1$ represented complete R-information, player 2 would be able to infer from $R_2 = 1$ that player 1 is likely to observe $s_1 = 1$ and (if q is high enough) play $a_1 = 1$, such that player 2's best-reply would be $a_2 = 1$. In contrast, our definition of $R_2 = 1$ means that player 2 does not condition his forecast of player 1's action on R_2 and ends up underestimating the conditional probability that player 1 will choose $a_1 = 1$.

4.2 R-Information about R-Information

Suppose that for each player i, $R_i = 0$ signifies $\{\{l_{R_1}, l_{R_2}\}, \{l_{R_j}, l_{a_j}\}, \{l_\theta\}\}$ and $R_i = 1$ signifies $\{\{l_{R_1}, l_{R_2}\}, \{l_{R_j}, l_{a_j}\}, \{l_\theta, l_{a_j}\}\}$. That is, player i does not always learn the joint distribution over (θ, a_j), but he always learns the joint distributions over the pairs of variables (R_1, R_2) and (R_j, a_j). Crucially, the player never learns how player j's action is correlated with the *joint* realization of player j's news-information and archive-information.

Assume that the R-information profile (R_1, R_2) is distributed independently of θ. The probability of (R_1, R_2) is denoted $\alpha_{R_1 R_2}$. The distribution is symmetric in the sense that $\alpha_{01} = \alpha_{10}$. Denote $\alpha = \alpha_{11} + \alpha_{10}$ (this is the probability that $R_i = 1$ for each i). Denote $q = \alpha_{11}/\alpha$ and assume $q < \frac{1}{2}$. This is the probability that $R_j = 1$ conditional on $R_i = 1$. Player i's N-information is $t_i = (\theta, R_i)$.

To derive player i's belief as a function of his type, we can employ the graphical representation presented in Section 3. The collection of subsets given by $R_i = 1$ satisfies the running intersection property. Therefore, we can view the three sets as maximal cliques in a perfect DAG, e.g.

$$R_i \to R_j \to a_j \to \theta \tag{3}$$

such that $p_{R_i=1}$ factorizes p according to this DAG:

$$p_{R_i=1}(\theta, R_i, R_j, a_j) = p(R_i, R_j)p(a_j \mid R_j)p(\theta \mid a_j)$$

Likewise, the three subsets that constitutes $R_i = 0$ satisfies the running intersection property, and can be viewed as maximal cliques in a perfect DAG that simply omits the link $a_j \to \theta$ from the DAG given by (3). Therefore,

$$p_{R_i=0}(\theta, R_i, R_j, a_j) = p(R_i, R_j)p(a_j \mid R_j)p(\theta)$$

It follows that

$$\begin{aligned}
p_{R_i=0}(a_j \quad | \quad \theta, R_i, a_i) &= p(R_j \mid R_i)p(a_j \mid R_j) \\
p_{R_i=1}(a_j \quad | \quad \theta, R_i, a_i) &= \frac{\sum_{R_j} p(R_j \mid R_i)p(a_j \mid R_j)p(\theta \mid a_j)}{\sum_{R_j} p(R_j \mid R_i) \sum_{a'_j} p(a'_j \mid R_j)p(\theta \mid a'_j)}
\end{aligned}$$

As in the previous example, and using essentially the same arguments, each player i necessarily plays $a_i = 0$ in equilibrium whenever $\theta R_i = 0$.

Proposition 5 *There is an equilibrium in which each player i plays $a_i = 1$ if and only if $\theta R_i = 1$, as long as*

$$q > \frac{1-\alpha}{3-2\alpha}$$

Proof. Let us try to sustain this equilibrium. Best-replying for player i requires that

$$p_{R_i=1}(a_j = 1 \mid \theta = R_i = 1) > \frac{1}{2}$$

Given the players' assumed strategies,

$$
\begin{aligned}
p_{R_i=1}(a_j &= 1 \mid \theta = R_i = 1) = \frac{\sum_{R_j} p(R_j \mid R_i = 1) p(a_j = 1 \mid R_j) p(\theta = 1 \mid a_j = 1)}{\sum_{R_j} p(R_j \mid R_i = 1) \sum_{a'_j} p(a'_j \mid R_j) p(\theta = 1 \mid a'_j)} \\
&= \frac{q \cdot \frac{1}{2} \cdot 1}{q \cdot \frac{1}{2} \cdot 1 + (1-q) \cdot p(\theta = 1 \mid a_j = 0)} \\
&= \frac{q \cdot \frac{1}{2} \cdot 1}{q \cdot \frac{1}{2} \cdot 1 + (1-q) \cdot \frac{\frac{1}{2}(1-\alpha)}{\frac{1}{2}(1-\alpha)+\frac{1}{2}}} = \frac{q}{q + (1-q)\frac{1-\alpha}{2-\alpha}}
\end{aligned}
$$

The requirement that this expression exceeds $\frac{1}{2}$ boils down to the inequality in the statement of the proposition. The condition is not vacuous because there are values of $q < \frac{1}{2}$ that satisfy it, for any α. ∎

Compare this result with an alternative specification in which $R_i = 1$ represents complete R-information. In this case, we would be unable to sustain the above equilibrium, because

$$
p(a_j = 1 \mid \theta = R_i = 1) = p(R_j = 1 \mid R_i = 1) = q < \frac{1}{2}
$$

Thus, the fact that $R_i = 1$ represents imperfect R-information enables an exaggerated belief that $a_j = 1$ conditional on $\theta = R_i = 1$, which facilitates good coordination. The reason is that player i's failure to account for the joint effects of θ and R_j on a_j leads him to regard θ and R_i as two conditionally independent signals of a_j, thus exaggerating the informative content of $\theta R_i = 1$. By itself, this is a correlation-neglect effect that appears in earlier work (e.g., in the context of voting behavior in Levy and Razin (2015)). The novelty here is that this effect is embedded in high-order reasoning: a particular realization of player i's archive-information leads him to neglect the correlation between player j's news-information.

4.3 A Hierarchical Type Space

In both the Prisoner's Dilemma example of the Introduction and the example of the previous sub-section, one player's R-information was one of the variables about which the other player had R-information. The formalism's capacity for such cross-references is one of its prime virtues - in analogy to the Harsanyi formalism's ability to define one player's N-information regarding another player's N-information. And as in the Harsanyi model, it is natural to think of hierarchical constructions of this inter-dependence.

The starting point of a hierarchical definition of players' R-information is a collection of *basic* variables. Let B be the set of labels of the basic variables. These would include variables that define the state of Nature, players' conventional signals and actions, and consequence variables. For each player i, define an m-tuple $(R_i^1, ..., R_i^m)$, $m \geq 2$, where R_i^1 is a collection of subsets of B; and for every $k = 2, ..., m$, R_i^k is a collection of subsets of $B \cup \{l_{R_i^1}, ..., l_{R_i^{k-1}}\}$, each of which includes $l_{R^{k-1}}$. Define $R_i = \cup_{k=1,...,m} R_i^k$.

The interpretation of this hierarchical construction is natural: R_i^1 is the player's first-order archive-information, describing his knowledge of correlations among basic variables; R_i^2 is the player's second-order archive-information, describing his knowledge of how players' first-order archive-information is correlated with the basic variables; and so forth. The formalism makes the hierarchical specification particularly natural, because R-information is conceived as a *collection* (of subsets of variable labels) to begin with; therefore, we can define the player's R-information as a *union* of such lower-order collections.

The following is a simple example of hierarchically defined R-information in the context of our "global game". For simplicity, assume that players have common R-information, which is distributed independently of θ. Assume in addition that players have complete N-information - i.e., $t_i = (\theta, R)$ for both $i = 1, 2$. The basic variables are θ, a_1, a_2. Suppose that for every $k = 1, ..., m$,

R^k takes two values, 0 and 1, which are defined as follows:

k	$R^k = 0$	$R^k = 1$
1	$\{\{l_\theta\}, \{l_{a_1}\}, \{l_{a_2}\}\}$	$\{\{l_\theta, l_{a_1}, l_{a_2}\}\}$
2	$\varnothing$	$\{\{l_\theta, l_{a_1}, l_{a_2}, l_{R^1}\}\}$
3	$\varnothing$	$\{\{l_\theta, l_{a_1}, l_{a_2}, l_{R^1}, l_{R^2}\}\}$
$\vdots$	$\vdots$	$\vdots$
m	$\varnothing$	$\{\{l_\theta, l_{a_1}, l_{a_2}, l_{R^1}, ..., l_{R^{m-1}}\}\}$

The only values of R that are realized with positive probability are those for which $R^k = 1$ implies $R^{k-1} = 1$, for every $k = 2, ..., m$. Therefore, it is convenient to represent R by the largest number $k = 0, ..., m$ for which $R^k = 1$. Specifically, let $p(R = k) = \delta(1 - \delta)^k$ for every $k < m$, and $p(R = m) = (1 - \delta)^m$, where $\delta > \frac{1}{2}$. Note that $R = k$ means that players perceive actions as a function of $\theta, R^1, ..., R^{k-1}$.

Proposition 6 *There is a unique equilibrium, in which players always play $a = 0$.*

Proof. The proof is by induction on R. As a first step, observe that by the same argument of the previous sub-sections, $a_i = 0$ whenever $\theta R^1 = 0$. Suppose that we have shown that $a_i = 0$ when $\theta = 1$ and $R < k$, and consider the case of player 1, say, when $\theta = 1$ and $R = k$. The player will find it optimal to play $a_1 = 1$ only if $p_{R=k}(a_2 = 1 \mid \theta = 1, R = k, a_1 = 1) > \frac{1}{2}$, but this does not hold since

$$\begin{aligned} p_{R=k}(a_2 &= 1 \mid \theta = 1, R = k, a_1) = p(a_2 = 1 \mid \theta = R^1 = \cdots = R^{k-1} = 1) \\ &\leq \frac{p(R \geq k)}{p(R \geq k-1)} = 1 - \delta < \frac{1}{2} \end{aligned}$$

■

The intuition for this result is simple. When players have $R = k$, they only perceive correlations between actions and R-information of order $k - 1$

and below. By the assumption that $\delta > \frac{1}{2}$, players are more likely to lack R-information of order k conditional on having R-information of order $k-1$, and by the inductive step, they play $a = 0$ in that case. It follows that when a player has $R = k$, he believes it is more likely that the opponent will play $a = 0$, hence the best-reply is to play $a = 0$, too.

I conclude this sub-section with two comments regarding hierarchical constructions. First, while they are natural in some contexts, I do not believe they are more fundamental or "legitimate" than the non-hierarchical type spaces of previous examples. A value of R can be identified with a tangible information source - e.g. a written report - that can be identified by a distinct title. It is natural for one player's report to contain data about the correlation between various variables and the report (identified by its title) that is available to the opponent. Second, unlike the Harsanyi formalism, hierarchical type spaces in the present context are necessarily *finite* because there are finitely many variables. In order to incorporate infinite hierarchies, one would need to extend the model by allowing L to be infinite.

5 A Market with Diversely Sophisticated Traders

In this section I present a simple model of a market for a financial asset, where two large traders interact in the presence of exogenous external demand. As in conventional models of financial trade, traders have diverse news-information regarding the value of the traded asset. However, unlike standard models, the traders will also vary in their archive-information. I will refer to traders with better archive-information as sophisticated, and those with poorer archive-information as naive. The difference between the two types is that sophisticated traders' data encompasses the sophistication variable, whereas naive traders' data is limited to the variables that would suffice if all traders had rational expectations. We will see that this extension has subtle effects on market trade.

Let us present the formal model. A single financial asset is traded in a market that is populated by two large traders, denoted 1 and 2. Each trader i simultaneously choose a position $a_i \in \mathbb{R}$ - i.e., short-selling is frictionless. Exogenous external demand is given by the function $a_0(\pi) = 1 - \pi$, where π is the asset's price. There is a fixed supply of the asset, normalized to 1. The asset's price π is given by the market clearing condition

$$a_0(\pi) + a_1 + a_2 = 1$$

such that the market clearing price is $\pi = a_1 + a_2$ (i.e., the model treats negative and positive prices symmetrically). Trader i's payoff is $a_i(\theta - \pi)$, where $\theta \sim N(0, \sigma_\theta^2)$ is the asset's intrinsic value. Plugging the external demand and market clearing condition, the trader i's payoff is reduced to

$$u_i(a_i, a_j) = a_i(\theta - a_i - a_j)$$

Let $\psi_i \in \{0, 1\}$ denote the *type* of trader i, where $\psi_i = 1$ with independent probability ρ. Types differ in both N-information and R-information. Let us begin with traders' N-information. A trader of type ψ receives a signal $s_\psi = \theta + \varepsilon_\psi$, where ε_ψ is independently distributed noise, $\varepsilon_\psi \sim N(d_\psi, \sigma_\psi^2)$. Thus, d_ψ and σ_ψ^2 indicate the signal's bias and precision, respectively. The trader's N-information is $t_i = (\psi_i, s_i)$. That is, each trader is N-informed of his type. The traders' types uniquely pin down their R-information. When $\psi_i = 0$, trader i's R-information is

$$R_0 = \{\{\delta_\theta, \delta_{s_i}\}, \{\delta_\theta, \delta_{s_j}\}, \{\delta_{s_j}, \delta_{a_j}\}\}$$

whereas when $\psi_i = 1$, trader i's R-information is

$$R_1 = \{\{\delta_\theta, \delta_{s_i}, \delta_{\psi_i}\}, \{\delta_\theta, \delta_{s_j}, \delta_{\psi_j}\}, \{\delta_{s_j}, \delta_{a_j}, \delta_{\psi_j}\}\}$$

This completes the specification of this example.

The collections R_0 and R_1 satisfy the running intersection property. Therefore, p_{R_0} factorizes p according to the perfect DAG

$$s_i \leftarrow \theta \rightarrow s_j \rightarrow a_j$$

and p_{R_1} factorizes p according to the perfect DAG

$$\begin{array}{ccccc} \psi_i & \leftarrow & \theta & \rightarrow & \psi_j \\ \downarrow & \swarrow & \downarrow & \swarrow & \downarrow \\ s_i & & s_j & \rightarrow & a_j \end{array}$$

This facilitates the derivation of trader i's subjective belief as a function of his news-information and archive-information:

$$\begin{aligned} p_{R_0}(\theta, a_j \quad &| \quad \psi_i = 0, s_i) = p(\theta \mid \psi_i = 0, s_i) p(s_j \mid \theta) p(a_j \mid s_j) \\ &= \quad p(\theta \mid \psi_i = 0, s_i) \sum_{\psi_j} p(\psi_j) p(s_j \mid \theta, \psi_j) \sum_{\psi_j} p(\psi_j) p(a_j \mid s_j, \psi_j) \\ p_{R_1}(\theta, a_j \quad &| \quad \psi_i = 1, s_i) = p(\theta \mid \psi_i = 1, s_i) \sum_{\psi_j} p(\psi_j) p(s_j \mid \theta, \psi_j) p(a_j \mid s_j, \psi_j) \end{aligned}$$

In a standard model, p_{R_0} would coincide with rational expectations, because the conditional-independence properties $s_i \perp s_j \mid \theta$ and $a_i \perp \theta \mid s_i$ would be satisfied by assumption. However, in the present model, ψ_i is a confounding variable that affects both the distribution of s_i conditional on θ and (potentially) the distribution of a_i conditional on s_i. Therefore, the above conditional-independence properties need not hold, which means that p_{R_0} can depart from rational expectations. However, p_{R_1} coincides with rational expectations.

Throughout this section, I focus on symmetric equilibria in strategies that are linear in the trader's signal s. That is, a trader of type ψ who receives a signal s plays the position $a_\psi(s) = \alpha_\psi + \beta_\psi s$. I refer to such equilibria as *symmetric linear equilibria*.

As a benchmark, consider the case in which traders have complete archive-information, regardless of their type. Then, there is a unique symmetric linear equilibrium:

$$\alpha_\psi = -\frac{\lambda_\psi d_\psi}{2+\bar{\lambda}}$$
$$\beta_\psi = \frac{\lambda_\psi}{2+\bar{\lambda}}$$

where

$$\lambda_\psi = \frac{\sigma_\theta^2}{\sigma_\theta^2+\sigma_\psi^2}$$
$$\bar{\lambda} = \rho\lambda_1 + (1-\rho)\lambda_0$$

The trader's strategy can be rewritten as follows:

$$a_\psi(s) = \frac{\lambda_\psi}{2+\bar{\lambda}}(s-d_\psi)$$

Thus, in this rational-expectations benchmark, each trader's position is a constant fraction of his fully debiased signal, where this constant is a function of the precision of his signal (measured by λ_ψ) and the average precision of the opponent's signal.

I will not present a proof of this conventional result here, because it will follow transparently from the analysis of the example with incomplete archive-information. At this stage, it is important to note two features of this benchmark equilibrium. First, the trader's response to his (debiased) signal increases unambiguously with the precision of his signal. Second, the trader's position gets arbitrarily small as his (debiased) signal tends to zero (the expectation of θ).

We are now ready to present the characterization of symmetric linear equilibria in the present example.

Proposition 7 *There is a unique symmetric linear equilibrium. Trader of type $\psi = 0, 1$ plays*

$$a_\psi(s) = \frac{\lambda_\psi}{2 + \bar{\lambda}}(s - d_0) + C_\psi$$

where

$$\begin{aligned} C_0 &= \frac{\rho(1-\rho)(\lambda_1 - \lambda_0)(d_1 - d_0)}{2(2 + \bar{\lambda})} \\ C_1 &= -\frac{1-\rho}{2+\rho} C_0 \end{aligned}$$

Proof. Trader i of type ψ_i who receives a signal s_i chooses a_i to maximize

$$E_{R_{\psi_i}}[a_i(\theta - a_i - a_j) \mid s_i, \psi_i] = -a_i^2 + a_i[E_{R_{\psi_i}}(\theta \mid s_i, \psi_i) - E_{R_{\psi_i}}(a_j \mid s_i, \psi_i)]$$

Because both types learn the joint distribution of θ, s_i, ψ_i,

$$E_{R_\psi}(\theta \mid s, \psi) = E(\theta \mid s, \psi) = \lambda_\psi(s - d_\psi)$$

for every s, ψ.

Let us calculate $E_{R_1}(a_j \mid s_i, \psi_i = 1) = E(a_j \mid s_i, \psi_i = 1)$ - i.e., the sophisticated trader i's (rational expectations) subjective expectation of a_j conditional on his type. First,

$$\begin{aligned} E(a_j \quad | \quad \theta) &= \rho[\alpha_1 + \beta_1(\theta + d_1)] + (1-\rho)[\alpha_0 + \beta_0(\theta + d_0)] \\ &= \bar{\alpha} + \bar{\beta}\theta \end{aligned}$$

where

$$\begin{aligned} \bar{\alpha} &= \rho(\alpha_1 + \beta_1 d_1) + (1-\rho)(\alpha_0 + \beta_0 d_0) \\ \bar{\beta} &= \rho\beta_1 + (1-\rho)\beta_0 \end{aligned}$$

Then,

$$\begin{aligned} E(a_j \quad &| \quad s_i = s, \psi_i = 1) = \bar{\alpha} + \bar{\beta} E(\theta \mid s_i = s, \psi_i = 1) \\ &= \quad \bar{\alpha} + \bar{\beta}\lambda_1(s - d_1) \end{aligned}$$

Therefore, trader i of type $\psi_i = 1$ chooses a_i to maximize

$$-a_i^2 + a_i[\lambda_1(s - d_1) - \bar{\alpha} - \bar{\beta}\lambda_1(s - d_1)]$$

The first-order condition gives us

$$a_i = \frac{1}{2}\lambda_1(1 - \bar{\beta})s - \frac{1}{2}[\lambda_1 d_1(1 - \bar{\beta}) + \bar{\alpha}]$$

such that

$$\begin{aligned} \beta_1 &= \frac{1}{2}\lambda_1(1 - \bar{\beta}) \\ \alpha_1 &= -\frac{1}{2}[\lambda_1 d_1(1 - \bar{\beta}) + \bar{\alpha}] \end{aligned}$$

Let us now calculate $E_{R_0}(a_j \mid s_i, \psi_i = 0)$ - i.e., the naive trader i's subjective expectation of a_j conditional on his type. First,

$$\begin{aligned} E(a_j \quad &| \quad s_j = s) = \rho(\alpha_1 + \beta_1 s) + (1 - \rho)(\alpha_0 + \beta_0 s) \\ &= \quad \rho\alpha_1 + (1 - \rho)\alpha_0 + (\rho\beta_1 + (1 - \rho)\beta_0)s \end{aligned}$$

Therefore,

$$\begin{aligned} E_{R_0}(a_j \quad &| \quad \theta) = E[\rho\alpha_1 + (1 - \rho)\alpha_0 + (\rho\beta_1 + (1 - \rho)\beta_0)s_j \mid \theta] \\ &= \quad \rho\alpha_1 + (1 - \rho)\alpha_0 + (\rho\beta_1 + (1 - \rho)\beta_0)E(s_j \mid \theta) \\ &= \quad \rho\alpha_1 + (1 - \rho)\alpha_0 + (\rho\beta_1 + (1 - \rho)\beta_0)[\rho(\theta + d_1) + (1 - \rho)(\theta + d_0)] \\ &= \quad \alpha^* + \beta^*\theta \end{aligned}$$

where

$$
\begin{aligned}
\alpha^* &= \rho\alpha_1 + (1-\rho)\alpha_0 + (\rho\beta_1 + (1-\rho)\beta_0)(\rho d_1 + (1-\rho)d_0) \\
\beta^* &= \rho\beta_1 + (1-\rho)\beta_0 = \bar{\beta}
\end{aligned}
$$

Then,

$$
\begin{aligned}
E(a_j &\mid s_i = s, \psi_i = 0) = \alpha^* + \bar{\beta}E(\theta \mid s_i = s, \psi_i = 0) \\
&= \alpha^* + \bar{\beta}\lambda_0(s - d_0)
\end{aligned}
$$

Therefore, trader i of type $\psi_i = 0$ chooses a_i to maximize

$$
-a_i^2 + a_i[\lambda_0(s-d_0) - \alpha^* - \bar{\beta}\lambda_0(s-d_0)]
$$

The first-order condition gives us

$$
a_i = \frac{1}{2}\lambda_0(1-\bar{\beta})s - \frac{1}{2}[\lambda_0 d_0(1-\bar{\beta}) + \alpha^*]
$$

such that

$$
\begin{aligned}
\beta_0 &= \frac{1}{2}\lambda_0(1-\bar{\beta}) \\
\alpha_0 &= -\frac{1}{2}[\lambda_0 d_0(1-\bar{\beta}) + \alpha^*]
\end{aligned}
$$

(As an aside, note that if type $\psi_i = 0$ had rational expectations, we would have the same expressions for α_0, β_0, except that α^* would be replaced with $\bar{\alpha}$.) We can now solve for $\alpha_0, \alpha_1, \beta_0, \beta_1$ and obtain the solution. ■

Thus, the only departure from the rational-expectations benchmark is in the fixed term C_ψ. That is, traders' responsiveness to their debiased signal is exactly as in the benchmark, but their position has an additional fixed term, which depends on several factors. This fixed term represents an element of speculative trade between the two types of traders. This is not due to a misperception of the asset's value θ. Indeed, all traders - regardless of their type - reach the rational-expectations estimate of the asset's value conditional on their signal. However, a naive trader i misperceives the mapping from θ to the opponent's position, because he only learns $p(\theta, s_j)$ and $p(s_j, a_j)$ and extrapolates a belief in a way that ignores the confounding effect of ψ_i. The misperception exists only if both d and σ^2 vary across types. This co-variation - as well as the relative proportions of the two trader types - determines the sign and magnitude of the misperception, and it leads trader i to lean against the misperceived constant, state-independent component in trader j's position. This leads to the fixed term in trader i's strategy. A sophisticated trader has rational expectations, and correctly predicts the average constant term in his opponent's strategy, and leans against it. This leads to the fixed term in the sophisticated trader's strategy.

We can now revisit the two key features of the symmetric linear equilibrium in the rational-expectations benchmark. First, it is no longer true that a trader with a more precise signal takes an unambiguously larger position, because the fixed term is larger in absolute terms for naive traders. If the naive trader also has a less precise signal, then for small debiased signals, the naive trader's position is larger (in absolute terms) than the sophisticated trader's. Second, traders' positions are bounded away from zero even when they receive a debiased signal that is close to zero.

From a methodological point of view, this example illustrates two features of the formalism: the possibility of correlation between a player's news-information and archive-information, and the implications of whether one player's archive-information encompasses another player's archive-information.

6 Concluding Remarks

The general idea of equilibrium modeling without rational expectations has received considerable attention in the last two decades. Relaxing rational expectations often takes the form of limited learning feedback, where players apply some notion of parsimony to extrapolate a subjective belief from their limited feedback. Alternatively (yet sometimes equivalently), players arrive at the scene with a subjective model and they fit it to their partial feedback. Although the literature has explored a variety of equilibrium concepts along these lines, virtually all attempts capture the departure from rational expectations as an aspect of the solution concept, or as a permanent fixture of individual players.

The formalism presented in this paper enriches the scope of equilibrium modeling with non-rational expectations, by including individual players' limited feedback (called their R-information) in their description of their *type*. Importantly, it describes R-information in terms of the collections of *variables* about which the player receives feedback. This enables us to capture new and realistic kinds of "high-order" reasoning, such as N-information or R-information about another player's R-information, as well as hierarchical elaborations. The novel aspect of the formalism is the new type space and the model of how players form beliefs given their type. The notion of equilibrium that I applied is conventional - a trembling-hand implementation of the conventional assumption that players maximize expected utility with respect to their subjective beliefs.

Interim vs. ex-ante maximization

The definition of equilibrium in this paper takes an interim approach: it tests the optimality of players' actions *given* their (extended) type. In the standard model of games with incomplete information, Nash equilibrium has an equivalent ex-ante definition, where players commit to their strategy before their type is realized. Is such a reformulation possible in the present model?

When R_i is random, it is not even clear how one could define ex-ante optimization in a way that will bear a reasonable connection to the possible realizations of R_i. However, even when R_i is fixed, there is an ambiguity in how to define ex-ante optimization, and generally the interim and ex-ante approaches will diverge. This is already an issue in single-player decision problems. Consider the following example, which is abstracted from the Dieter's Dilemma example in Spiegler (2016). Suppose that the player receives no N-information, and that his R-information is fixed. Specifically, let a denote the player's action; let z denote some payoff-relevant consequence variable; and let w denote a third, payoff-irrelevant variable. The player's payoff is $u(z)$. Suppose that $R = \{\{l_a, l_w\}, \{l_w, l_z\}\}$. Then, $p_R(a, w, z) = p(a, w)p(z \mid w)$, such that $p_R(z \mid a) = \sum_w p(w \mid a)p(z \mid w)$. In other words, p_R satisfies the property that z and a are independent conditional on w. Suppose that the objective distribution p violates this property. Then, the term $p(z \mid w)$ in the expression for $p_R(z \mid a)$ is not necessarily invariant to changes in $(p(a))$. In other words, changes in the player's strategy will affect the subjective conditional distribution over payoff consequences, hence his evaluation of individual actions.

This means that when a player contemplates a mixed strategy σ, the expression we would like to write down for his ex-ante expected utility,

$$\sum_a \sigma(a) \sum_z p_R(z \mid a)u(z) \tag{4}$$

is ambiguous, because $p_R(z \mid a)$ is only well-defined with respect to some prescribed marginal over a. One way to fix this ambiguity is to assume that this marginal is σ itself. An alternative definition (which is more plausible to my taste) is to define ex-ante optimization as an equilibrium object: $p_R(z \mid a)$ is defined with respect to some mixed strategy σ^*, and σ^* in turn maximizes (4). The interim version of (perturbed) subjective maximization is also defined as an equilibrium object: $p_R(z \mid a)$ is defined with respect to some mixed strategy σ^* with full support, such that every action a for which $\sigma^*(a) > \varepsilon$ maximizes $\sum_z p_R(z \mid a)u(z)$. Spiegler (2016) defines this notion of interim subjective maximization and refers to as a "personal equilibrium".

It can shown (via an example) that personal equilibrium is not equivalent to either of the two notions of ex-ante subjective maximization suggested above.

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