Equilibrium Selection in Global Games with Strategic Complementarities*

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Abstract

We study games with strategic complementarities, arbitrary numbers of players and actions, and slightly noisy payoff signals. We prove *limit uniqueness*: as the signal noise vanishes, the game has a unique strategy profile that survives iterative dominance. This generalizes a result of Carlsson and van Damme (1993) for two player, two action games. The surviving profile, however, may depend on fine details of the structure of the noise. We provide sufficient conditions on payoffs for there to be noise-independent selection.

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1 Introduction

In two player, two action games with common knowledge of payoffs, there often exist two strict Nash equilibria. Carlsson and van Damme [5] showed a remarkable result: if each player instead observes a noisy signal of the true payoffs, and if the *ex ante* feasible payoffs include payoffs that make each action strictly dominant, then as the noise becomes small, iterative strict dominance eliminates all equilibria but one. In particular, if there are two Nash equilibria in the underlying complete information game, then the risk dominant equilibrium (Harsanyi and Selten [14]) must be played in the game with small noise. Carlsson and van Damme called the noisy game with dominance regions a "global game."

Carlsson and van Damme's result can be reconstructed in two logically separate parts. First, there is a *limit uniqueness* result: as the noise in the global game becomes arbitrarily small, for almost any payoffs there is a unique action that survives iterative elimination of dominated strategies. The second is a *noise independent selection* result: as the noise goes to zero, the equilibrium played (for a given realization of the payoffs) is *independent of the distribution of the noise.*

In this paper, we extend Carlsson and van Damme's model to many player, many action games.[1] Our main assumption is that the actions can be ranked such that there are strategic complementarities: an increase in one player's action raises the incentive for other players to raise their own actions. We show that the limit uniqueness result generalizes. In contrast, the noise-independent selection result does not hold in general. We present a counterexample (a two player, four action symmetric game) in which the equilibrium selected in the limit as the noise goes to zero does depend on the structure of the noise. We proceed to identify sufficient conditions for noise independent selection to hold in games with a finite number of players.

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[1]Our results also generalize the extensions of Carlsson and van Damme [6] and Kim [16], who show limit uniqueness in games with finitely many identical players and two actions, under a uniform prior assumption.

We consider the following setting. An unknown state of the world $\theta \in \mathbb{R}$ is drawn according to some prior. Each player i observes the signal $\theta + \nu \eta_i$, where $\nu > 0$ is a scale factor and η_i is a random variable with density f_i. Our main assumptions are (1) *strategic complementarities*: for any state θ, each player's best response is weakly increasing in the actions of her opponents; (2) for any given opposing action profile, a player's best response is increasing in the state θ; and (3) *dominance regions*: at sufficiently low (high) states θ, each player's lowest (highest) action is strictly dominant. Call this global game $G(\nu)$. Under these and some technical continuity assumptions, we show that limit uniqueness holds: as the noise scale factor ν goes to zero, there is an essentially unique[2] strategy profile surviving iterated deletion of dominated strategies in $G(\nu)$. In this unique surviving strategy profile, each player's action is a nondecreasing function of her signal. Moreover, for almost all states θ, players play a Nash equilibrium of the complete information game with payoff parameter θ.

We also show that there may not be noise-independent selection: the particular Nash equilibrium played at a state θ may depend on the noise densities f_i. This implies that different equilibria of a given complete-information game g may be selected, depending on the global game in which g is embedded. We proceed to identify conditions on the payoffs of complete-information games g that guarantee noise-independent selection. In particular, if g is a *local potential game* in which each player's payoffs are *quasiconcave* in her own action, then there is noise-independent selection at g: a unique Nash equilibrium of g must be played in the limit as the signal errors shrink, regardless of the global game in which g is embedded.

Local potential games include both the potential games of Monderer and Shapley [21] and games with low p-dominant equilibria of Morris, Rob and Shin [22] and Kajii and Morris [15]. In particular, local potential games include (1) all two player, two action games; (2) all many player, two action games with symmetric payoffs; and (3)

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[2]I.e., for almost all signals.

all two player, three action games with symmetric payoffs. In each of these cases, we characterize the selected equilibrium.[3]

Strategic complementarities are present in many settings, including macroeconomic coordination failures, technology adoption, oligopoly, R&D competition, coordination in teams, arms races, and pretrial bargaining.[4] In a number of recent applied papers, the global games approach has been used to select a unique equilibrium. Examples include models of currency crises (Morris and Shin [23]), bank runs (Goldstein and Pauzner [11]) and debt pricing (Morris and Shin [24]).[5] However, the applications have generally been limited to situations with homogenous agents and two actions. Our results make it possible to apply the global games approach to a wider class of games. In particular, our model allows for arbitrary mixtures of large and small (infinitesimal) players, who can choose their actions from arbitrary compact sets.[6] (For ease of exposition, we assume a finite number of players first and later generalize the model to include continua of players.)

This paper also contributes to a large literature on games with strategy complementarities, also known as supermodular games. These games were first studied as a class by Topkis [28] and are further analyzed by Vives [30] and Milgrom and Roberts [19]. The connection between our findings and existing results on supermodular games is discussed in the conclusion.

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[3]In cases (1) and (2), there is own-action quasiconcavity since there are only two actions. In case (3), noise-independent selection holds even without own-action quasiconcavity.

[4]See Milgrom and Roberts [19] for a survey of applications.

[5]Morris and Shin [25] survey such applications and describe sufficient conditions for limit uniqueness in games with a continuum of identical players and two actions, under slightly weaker technical assumptions than those assumed in this paper. Their results incorporate other applications of global games such as Morris and Shin [23, 24].

[6]Note that any finite set of actions is compact. For recent applications of our more general results, see, e.g., Goldstein [10] (four actions, continuum of players) and Goldstein and Pauzner [12] (two actions, two continua of players).

The remainder of this paper is organized as follows. The base model, with a finite number of players, is presented in section 2. We prove limit uniqueness in section 3 and partially characterize the unique equilibrium in section 4. In section 5, we show that CvD's noise independence result does not generalize. Section 6 introduces conditions on the payoffs of the game (own action quasiconcavity and the existence of a local potential maximizer) that suffice for noise independent selection and discusses classes of games that satisfy the conditions. In section 7, we show that the limit uniqueness and partial characterization results generalize to models with large and small players. Section 8 concludes. All proofs are relegated to the appendix.

2   The Game

The global game $G(\nu)$ is defined as follows. The set of players is $\{1, ..., I\}$.[7] A state $\theta \in \mathbb{R}$ is drawn from the real line according to a continuous density ϕ with connected support. Each player i observes a signal $x_i = \theta + \nu\eta_i$, where $\nu > 0$ is a scale factor and each η_i is distributed according to an atomless density f_i with support contained in the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$. The signals are conditionally independent: η_i is independent of η_j for all $i \neq j$.

The action set of player i, $A_i \subseteq [0, 1]$, can be any closed, countable union of closed intervals and points that contains 0 and 1.[8] If player i chooses action $a_i \in A_i$, her payoff is $u_i(a_i, a_{-i}, \theta)$; $a_{-i} = (a_j)_{j \neq i}$ denotes the action profile of i's opponents.

Let $\Delta u_i(a_i, a'_i, a_{-i}, \theta)$ be the difference in the utility of player i from playing a_i

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[7] As noted above, the case of a continuum of players is treated in section 7.

[8] That is, A_i is a closed union of disjoint closed intervals $\cup_{m=1}^{M} [b_m, c_m]$ where $M \geq 1$ can be infinity. (Isolated points are represented by setting $b_m = c_m$.) Requiring A_i to include 0 and 1 is not restrictive since a player's highest and lowest actions can always be normalized to 0 and 1, respectively, by rescaling the payoff functions.

versus a'_i against the action profile a_{-i} when the payoff parameter is θ.[9] Let us write $a_{-i} \geq a'_{-i}$ if actions are weakly higher under a_{-i} than under a'_{-i}: if $a_j \geq a'_j$ for each $j \neq i$. We make the following assumptions on payoff functions.

A1 (Strategic Complementarities) A player's incentive to raise her action is weakly increasing in her opponents' actions: if $a_i \geq a'_i$ and $a_{-i} \geq a'_{-i}$ then for all θ, $\Delta u_i \left(a_i, a'_i, a_{-i}, \theta \right) \geq \Delta u_i \left(a_i, a'_i, a'_{-i}, \theta \right)$.

A2 (Dominance Regions) For extreme values of the payoff parameter θ, the extreme actions are strictly dominant: there exist thresholds $\underline{\theta} < \overline{\theta}$, where $\left[\underline{\theta} - \nu, \overline{\theta} + \nu \right]$ is contained in the interior of the support of ϕ, such that, for all i and for all opposing action profiles a_{-i}, $\Delta u_i \left(0, a_i, a_{-i}, \theta \right) > 0$ if $a_i \neq 0$ and $\theta \leq \underline{\theta}$, and $\Delta u_i \left(1, a_i, a_{-i}, \theta \right) > 0$ if $a_i \neq 1$ and $\theta \geq \overline{\theta}$.

If each player's action space is finite, we can replace **A2** by the weaker assumption:

A2' (Unique Equilibrium Regions) For extreme values of the payoff parameter, the equilibrium is unique: there exist thresholds $\underline{\theta} < \overline{\theta}$ in the interior of the support of θ such that for all $\theta < \underline{\theta}$, the complete information game in which the payoff function of each player i is $u_i \left(\cdot, \theta \right)$ has a unique equilibrium $(\underline{a}_1, ..., \underline{a}_I)$; for all $\theta > \overline{\theta}$, the complete information game with payoffs $u_i \left(\cdot, \theta \right)$ has a unique equilibrium $(\overline{a}_1, ..., \overline{a}_I)$. By Theorem 6 in Milgrom and Roberts [19], $\underline{a}_i \leq \overline{a}_i$ for all i. We assume that $\underline{a}_i < \overline{a}_i$ for at least one player i.

Note that under assumption **A1**, **A2'** is equivalent to requiring that there is a unique action profile surviving iterated deletion of strictly dominated strategies if $\theta \notin [\underline{\theta}, \overline{\theta}]$ (Milgrom and Roberts [19]).

A3 (State Monotonicity) Higher states make higher actions more appealing. There is a $K_0 > 0$ such that for all $a_i \geq a'_i$ and $\theta, \theta' \in [\underline{\theta}, \overline{\theta}]$, $\theta \geq \theta'$, $\Delta u_i \left(a_i, a'_i, a_{-i}, \theta \right) - \Delta u_i \left(a_i, a'_i, a_{-i}, \theta' \right) \geq K_0 (a_i - a'_i)(\theta - \theta')$.

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[9]I.e., $\Delta u_i(a_i, a'_i, a_{-i}, \theta) = u_i(a_i, a_{-i}, \theta) - u_i(a'_i, a_{-i}, \theta)$.

A4 (Payoff Continuity) Each $u_i(a_i, a_{-i}, \theta)$ is continuous in all arguments.

If player i's action space is finite, continuity with respect to a_i is vacuous. The following assumption is also vacuous if action sets are finite.

A5 (Bounded Derivatives) A player's utility is Lipschitz continuous in her own action and her marginal utility of raising her action is Lipschitz continuous in other players' actions. More precisely, (a) for each θ and a_{-i}, there exists a constant K_1 such that for all a_i, a_i',

$$|\Delta u_i(a_i, a_i', a_{-i}, \theta)| \leq K_1 |a_i - a_i'|$$

(b) for each θ there exists a constant K_2 such that for all a_i, a_i', a_{-i}, and a_{-i}',

$$\left|\Delta u_i(a_i, a_i', a_{-i}, \theta) - \Delta u_i\left(a_i, a_i', a_{-i}', \theta\right)\right| \leq K_2 |a_i - a_i'| \sum_{j \neq i} |a_j - a_j'| \qquad (1)$$

A *pure strategy* for player i is a function $s_i : \mathbb{R} \to A_i$. A *pure strategy profile* is a vector of pure strategies, $s = (s_i)_{i=1}^I$. The profile s is *increasing* if $s_i(x_i)$ is weakly increasing in x_i for all i; it is *left (right) continuous* if each s_i is left (right) continuous. Profile s' is *higher than* profile s $(s' \geq s)$ if $s_i'(x_i) \geq s_i(x_i)$ for all i and $x_i \in \mathbb{R}$. A mixed strategy is a probability distribution over pure strategies, and a mixed strategy profile is an assignment of mixed strategies to players.[10] (Players are not restricted to pure strategies.)

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[10]More precisely, we follow Aumann [2] in modelling a mixed strategy for player i as a function σ_i from $[0, 1]$ (representing the possible outcomes of player i's randomizing device) to the space $A_i^{\mathbb{R}}$ (representing the space of pure strategies, which are functions from $\mathbb{R}$ to A_i). Thus, $\sigma_i(y)$ is the pure strategy played by i if her randomizing device yields the value y. Following Aumann, we restrict players to mixed strategies σ_i for which the corresponding function $h : [0, 1] \times \mathbb{R} \to A_i$ defined by $h(y, x_i) = \sigma_i(y)(x_i)$ is measurable. This property ensures that a mixed strategy and a distribution on the type space $\mathbb{R}$ induces a distribution of actions. With this assumption, players are able to calculate their expected payoffs against any vector of opposing mixed strategies.

3 Limit Uniqueness

Our solution concept is iterative strict dominance. First we eliminate pure strategies that are strictly dominated, as rational players will never pick (i.e., put positive weight on) such strategies. Then we eliminate a player's pure strategies that are strictly dominated if her opponents are known to mix only over the pure strategies that survived the prior round of elimination; and so on. Theorem 1 shows that as the signal errors shrink to zero, this process selects an essentially unique Bayesian equilibrium of the game.

Theorem 1 $G(\nu)$ *has an essentially unique strategy profile surviving iterative strict dominance in the limit as $\nu \to 0$. It is an increasing pure strategy profile. More precisely, there exists an increasing pure strategy profile s^* such that if, for each $\nu > 0$, s^ν is a pure strategy profile that survives iterative strict dominance in $G(\nu)$, then $\lim_{\nu \to 0} s_i^\nu(x_i) = s_i^*(x_i)$ for almost all $x_i \in \mathbb{R}$.*

Intuition for Limit Uniqueness

To see the intuition, consider the case of a symmetric, two player game with a continuum of actions ($A_i = [0, 1]$). Assume the two players have the same distribution of signal errors and that their prior over θ is uniform over a large interval that includes $\left[\underline{\theta} - \nu, \overline{\theta} + \nu\right]$. (At the end of this section we explain how the argument works with a nonuniform prior.) Further assume that a player's payoff function is strictly concave in her own action, so that her best response does not jump in response to small changes in her posterior distribution over the state θ or her opponent's action. Recall that a pure strategy is a function from a player's signal x_i to an action $a_i \in [0, 1]$, and players can choose mixtures over these pure strategies.

By the assumption of dominance regions (**A2**), we know that a player who observes a signal above some threshold must choose $a_i = 1$. This means that no player will ever choose (i.e., put positive weight on) a pure strategy that lies below the curve in Figure 1.

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Figure 1:

Knowing this, a player will never choose a strategy below the *best response* to this curve. This relies on strategic complementarities (**A1**): any pure strategy that lies above the curve would have a best response that lies weakly above the best response to the curve. So the best response to the curve is a new lower bound on the pure strategies that can ever be played. We iterate this process ad infinitum, and denote the limit by $\underline{S}$ (Figure 2). Note that $\underline{S}$ is a symmetric equilibrium of the game (and thus survives iterative strict dominance) and that any strategy that survives iterative strict dominance lies weakly above $\underline{S}$, and so $\underline{S}$ is the smallest strategy surviving iterative strict dominance.
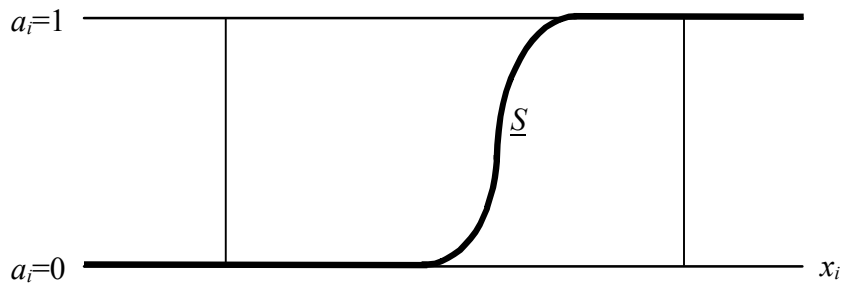

Figure 2:

By a symmetric argument, there must exist a largest strategy surviving iterated deletion of strictly dominated strategies, which we denote by $\overline{S}$, which must lie above $\underline{S}$ (see Figure 3). Now if we show that $\overline{S}$ must equal $\underline{S}$, we will have established the existence of a unique strategy profile surviving iterative strict dominance. This strategy for proving the dominance solvability of a game was discussed in Milgrom and Roberts [19].
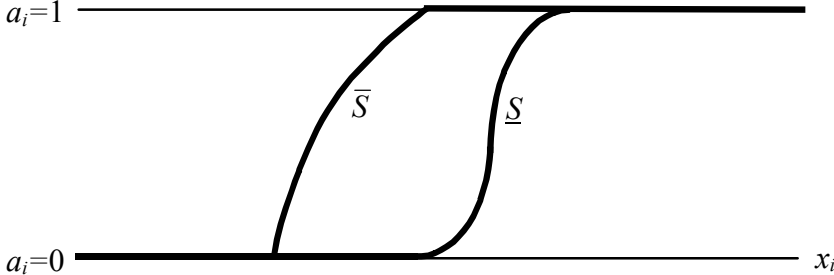
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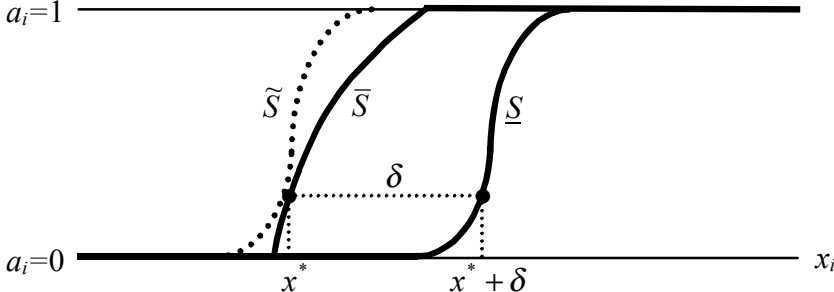
Figure 4:

Our argument establishing that $\underline{S}$ and $\overline{S}$ must coincide exploits the monotonicity properties **A1** and **A3**. We will note where each property is used in the following argument. Because of the dominance regions, the strategy $\overline{S}$ must prescribe playing 0 for low enough signals. Thus, there is a translation of $\underline{S}$ that lies entirely to the left of $\overline{S}$ and touches $\overline{S}$ at at least one point (see Figure 4). Label this translated curve

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$\widetilde{S}$. Let the amount of the translation be δ. Let x^* be the signal corresponding to the point at which $\overline{S}$ and $\widetilde{S}$ touch. Finally, write a^* for the best response of a player who has observed signal x^* and believes that his opponent is following strategy $\widetilde{S}$.

Since both $\overline{S}$ and $\underline{S}$ are equilibria, we know that

$$\overline{S}(x^*) = \underline{S}(x^* + \delta). \tag{2}$$

But since $\widetilde{S}$ is everywhere above $\overline{S}$, we know by strategic complementarities (**A1**) that

$$a^* \geq \overline{S}(x^*). \tag{3}$$

We will now show that with a uniform prior over θ, a^* must be strictly less than $\underline{S}(x^* + \delta)$ unless $\widetilde{S}$ and $\underline{S}$ *coincide*. Since this inequality would contradict (2) and (3), the two curves must coincide. As $\overline{S}$ lies entirely between them, it must also coincide with $\underline{S}$; this will show that a unique equilibrium survives iterative strict dominance.

With a uniform prior over θ, the posterior of a player with signal x^* over the error in her signal, $x^* - \theta$, is exactly the same as the posterior of a player with signal $x^* + \delta$ over the error in her signal, $x^* + \delta - \theta$. Since the signal error of the player's opponent is independent of θ, the player's posterior over the difference between her signal error and that of her opponent is also the same if her signal is x^* as if her signal is $x^* + \delta$. But the difference between the two players' signal errors is just the difference between their signals: $(x_i - \theta) - (x_j - \theta) = x_i - x_j$. Thus, a player's posterior over the difference between her signal and that of her opponent is the same at x^* as at $x^* + \delta$. Hence, since $\widetilde{S}$ is an exact translation of $\underline{S}$, a player who observes x^* and thinks that her opponent will play according to $\widetilde{S}$ expects the same *action distribution* as a player who observes $x^* + \delta$ and thinks that the opponent will play according to $\underline{S}$. But assumption **A3** implies that a player's optimal action is strictly increasing in her estimate of θ, controlling for her opponent's action distribution.

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Hence, if $\delta > 0$, then a^* must be less than $\underline{S}(x^* + \delta)$. Since in fact a^* is at least $\underline{S}(x^* + \delta)$, δ must equal zero, and thus $\underline{S}$ and $\overline{S}$ coincide if the prior over θ is uniform.

The same property still holds with a general prior, in the limit as the signal errors shrink to zero. When the signal errors are small, a player can be sure that the true payoff parameter θ is very close to her signal. Consequently, her prior over θ is *approximately* uniform for the small interval of values of θ that are still possible given her signal. (Recall that the model assumes a continuous prior over θ and a finite, very small support of the signal errors.) Thus, the above argument still holds in the limit: δ must shrink to zero (and thus $\underline{S}$ and $\overline{S}$ must coincide) as the signal errors become small.

4 A Partial Characterization

Theorem 2 partially characterizes the surviving equilibria of the global game when the noise is small. It states that in the limit, for all but a vanishing set of payoff parameters θ, players play arbitrarily close to some pure strategy Nash equilibrium of the complete information game with payoffs $u_i(\cdot, \theta')$ for some θ' that is close to θ. The intuition is that for small signal errors, players can precisely estimate both the payoff parameter θ and, for most signals, what other players will do.[11] Since players are best-responding to beliefs that are arbitrarily precise, the result must be very close to a Nash equilibrium of the underlying complete information game.

For any $\varepsilon > 0$ and $\nu > 0$, let $Q(\varepsilon, \nu)$ be the set of parameters θ for which the surviving strategy profiles in $G(\nu)$ do *not* all prescribe that players play ε-close to some common pure strategy Nash equilibrium of some complete information game whose

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[11] With small signal errors, there cannot be many signals at which players are very uncertain about opponents' possible actions. Otherwise, over a wide range of signals, opponents' strategies would have to rise considerably for small increases in their signals. (Recall that strategies must be weakly increasing when the signal errors are small.) This is impossible since the action space is bounded.

payoff parameter is ε-close to θ. More precisely, $Q(\varepsilon, \nu)$ is the set of parameters θ for which there is *no* Nash equilibrium action profile $a \in \times_{i=1}^{I} A_i$, of the complete information game with payoffs $(u_i(\cdot, \theta'))_{i=1}^{I}$ for some $\theta' \in [\theta - \varepsilon, \theta + \varepsilon]$, such that for every strategy s^{ν} surviving iterative strict dominance in $G(\nu)$, $|s_i^{\nu}(\theta) - a_i| \leq \varepsilon$ for all i. Theorem 2 shows that for any $\varepsilon > 0$, this set becomes arbitrarily small as ν shrinks to zero.

Theorem 2 *In $G(\nu)$ in the limit as $\nu \to 0$, for almost all payoff parameters θ, players play arbitrarily close to some pure strategy Nash equilibrium of the complete information game with payoffs $u_i(\cdot, \theta')$ for some θ' that is arbitrarily close to θ. More precisely, for any $\varepsilon > 0$ there is a $\overline{\nu} > 0$ such that for any $\nu < \overline{\nu}$, $Q(\varepsilon, \nu)$ is contained in a finite union of closed intervals of $\mathbb{R}$ whose measure is less than ε.[12]*

Theorem 2 has an interesting implication for symmetric games with a continuum of actions. Suppose that the action played in a locally stable (unstable) equilibrium of the underlying complete information game with payoff parameter θ monotonically rises (falls) if θ rises.[13] Then by Theorems 1 and 2, any strategy profile that survives iterative dominance must almost always prescribe that players play a *stable* Nash equilibria of the underlying complete information game. The strategy cannot coincide with an unstable Nash equilibrium of the underlying game as θ rises since surviving strategies are nondecreasing in players' signals.

This implication is illustrated in Figure 5. The dashed line shows the set of Nash equilibria of the underlying complete information games with payoff parameters equal to x_i. The upwards (downwards) sloping segments correspond to the locally stable (unstable) Nash equilibria of these games. The bold curve illustrates how the

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[12]For any ε, the number of intervals in the union is independent of ν. The number is finite for any given ε but may grow without bound as ε shrinks to zero.

[13]"Locally stable" refers to the traditional notion in which the best response function intersects the 45 degree line at an slope of less than 1.

essentially unique surviving strategy profile in the global game $G(\nu)$ must look in the limit as ν shrinks. It must coincide with a stable Nash equilibrium except at points where it jumps from a lower stable equilibrium to a higher one.
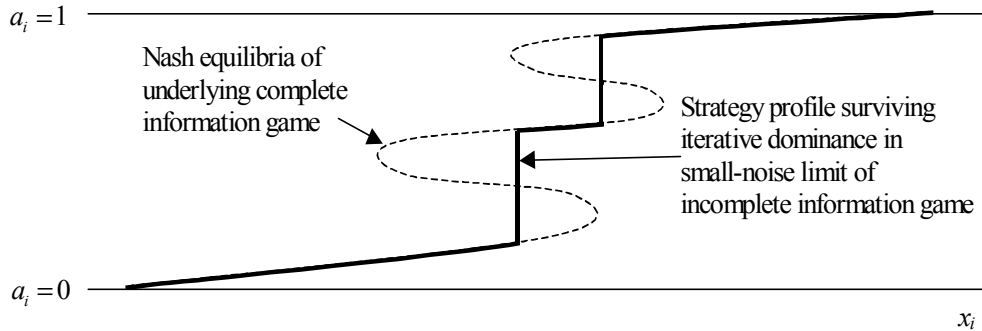
Figure 5:

5 Noise Independent Selection: A Counterexample

In showing limit uniqueness, we began with a given noise structure and scaled it down by taking the scale factor ν to zero. Our result does not imply that the selected equilibrium is independent of the structure of the noise (i.e., of the densities f_i).

Carlsson and van Damme's result implies that noise-independent selection holds in 2×2 games. Their proof method does not rely on properties of the game's payoffs. This method can be generalized to additional classes of games, but not to all. To obtain some intuition, suppose the game has two players, each with the same finite action set $\overline{A}$ and the same payoff function, and each player's noise term has the same symmetric distribution f. Assume that each player follows the same increasing strategy, $s : \mathbb{R} \to \overline{A}$. As $\nu \to 0$, what beliefs does each player have over the action of her opponent at the critical point where she switches from one action

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to another? Recall that for small ν, each player's posterior belief about the other's signal is computed approximately as if she had a uniform prior over θ.

Suppose first that there are two actions, 0 and 1. Consider a symmetric equilibrium given by a pure strategy profile s satisfying

$$s\left(x\right) = \begin{cases} 0, \text{ if } x < c \\ 1, \text{ if } x \geq c \end{cases}$$

A player observing signal c will assign probability $\frac{1}{2}$ to her opponent's choosing action 0 and probability $\frac{1}{2}$ to her opponent's choosing action 1. This is independent of the choice of c and the distribution f. Thus as the noise goes to zero, c must converge to the payoff parameter at which the player is indifferent between the two actions if she has a 50/50 conjecture over her opponent's action. This is simply the symmetric version of Carlsson and van Damme's [5] result.

Now suppose that there are three actions, 0, 1/2, and 1, so that

$$s\left(x\right) = \begin{cases} 0, \text{ if } x < c_1 \\ \frac{1}{2}, \text{ if } c_1 \leq x < c_2 \\ 1, \text{ if } c_2 \leq x \end{cases}$$

A player observing signal c_1 will assign probability $\frac{1}{2}$ to her opponent choosing action 0, some probability λ to her opponent choosing action $\frac{1}{2}$, and probability $\frac{1}{2} - \lambda$ to her opponent choosing action 1; a player observing signal c_2 will assign probability $\frac{1}{2} - \lambda$ to her opponent choosing action 0, probability λ to her opponent choosing action $\frac{1}{2}$ and probability $\frac{1}{2}$ to her opponent choosing action 1. For any distribution f, we can choose c_1 and c_2 so that λ takes any value in $[0, 1/2]$. In other words, the distribution of noise does not affect the limiting conjectures that each player may end up having over her opponent's actions.

This implies noise independent selection: any profile (c_1, c_2) that is an equilibrium as $\nu \to 0$ under a noise structure f must also be an equilibrium under any other noise structure f'. To see why, let us distinguish between two cases. In the first, $c_1 - c_2$ does

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not shrink to 0 as $\nu \to 0$. This means that λ converges to $1/2$: for sufficiently small ν, a player with signal c_1 is indifferent between 0 and $1/2$ and thinks that her opponent will play 0 or $1/2$ with equal probabilities; a player with signal c_2 is indifferent between $1/2$ and 1 and puts equal probabilities on her opponent's playing $1/2$ and 1. These signals clearly must converge to particular payoff parameters, independent of the structure of the signal errors (since the player's beliefs are independent of f).

In the second case, $\lim_{\nu \to 0} (c_1 - c_2) = 0$. Here, λ need not converge to $1/2$. But if we replace the signal error structure f with some other structure f', we can construct an equilibrium near the one given by f by simply adjusting the gap between c_1 and c_2 so as to keep λ the same under f' as under f. Since the gap between c_1 and c_2 asymptotically shrinks to zero, we can make this adjustment without changing the limit to which both cutoffs converge. Thus, under f' there is a sequence of equilibria of the global game that converges to the same limit as the sequence of equilibria under f. This explains why there is noise-independent selection. In the next section, we show that there is noise-independent selection for two player, three action, symmetric-payoff games, even with general asymmetric noise distributions f_i.

With four or more actions, the above property ceases to hold. The set of conjectures a player can have over her opponent's action can depend on the structure of the noise. With three actions, each profile (c_1, c_2) gave rise to one unknown, λ, which could be adjusted arbitrarily by changing the distance between c_1 and c_2. With four actions, each profile (c_1, c_2, c_3) gives rise to *three* unknowns. This is illustrated in Figure 6. The density centered at each threshold c_i represents the posterior distribution over the signal of the opponent of a player whose signal is c_i. The area under the segment marked b (respectively, c) of the posterior of a player with signal c_3 is the probability she assigns to her opponent's having seen a signal between c_1 and c_2 (respectively, between c_2 and c_3).[14]

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[14]Likewise, the area under the segment marked c (respectively, b) of the posterior of an agent with signal c_3 is the probability she assigns to her opponent's having seen a signal between c_2 and
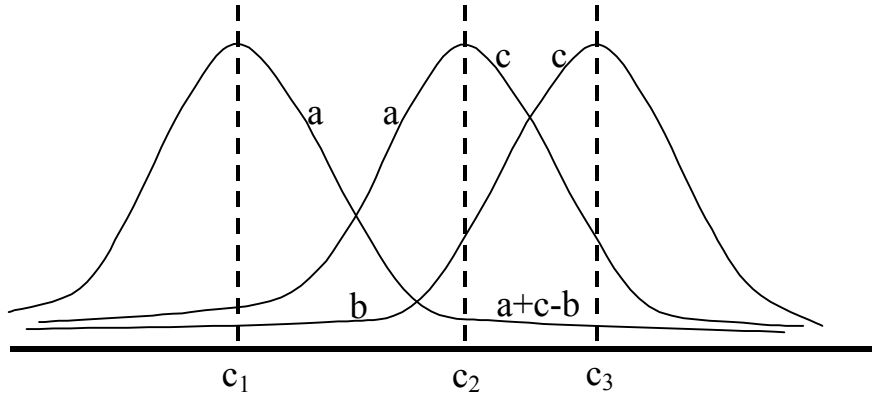
Figure 6:

The three probabilities a, b, and c are a minimal set of probabilities that suffice to determine the action distributions expected by players at each of the thresholds. However, by altering the profile we have only two degrees of freedom: we can change $c_3 - c_2$ and $c_2 - c_1$. This means that if we change the noise structure f, we will not necessarily be able to adjust the profile in order to preserve the action distribution seen by players at each of the thresholds c_1, c_2, and c_3. Thus, if f changes, we may not be able to keep the players indifferent between adjacent actions simply by adjusting the distance between the thresholds. It may also be necessary to shift the entire profile in order to change the payoff parameter that each player sees. In other words, changing the noise structure may alter the signals at which the unique surviving strategy profile jumps between any two actions. This is what it means for the equilibrium to depend on the structure of the noise.

Theorem 3 *There exists a two-person, four-action game satisfying A1-A5 in which for different noise structures, different equilibria are selected in the limit as the signal errors vanish.*

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c_3 (respectively, between c_1 and c_2). By symmetry of the signal errors, the two areas marked a are equal, as are the two areas marked c. Also by symmetry, the probability that an opponent of an agent with signal c_1 sees a signal between c_2 and c_3 must equal $a + c - b$, as indicated.

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Theorem 3 is proved by constructing such a game. It is clear from the proof that noise-independent selection would continue to fail if the payoffs of the game were perturbed by a small amount.

6   A Sufficient Condition for Noise-Independent Selection

The preceding counter example shows that for certain complete information games, different ways of embedding them into global games (with different noise structures) can lead to different predictions. That is, the payoffs of the complete information game may not tell us which equilibrium will be selected. We now focus on the complete information game corresponding to some parameter θ and identify conditions on payoffs of that game that guarantee that a particular equilibrium will be selected in the limit regardless of the noise structure. We will show that there is noise-independent selection in that game if (a) its payoffs are *own-action quasiconcave* and (b) it has a strategy profile that is a *local potential maximizer*.

A complete information game $\mathbf{g} = (g_1, ..., g_I)$ is a collection of payoff functions, with each $g_i : A \to \mathbb{R}$. For any player i, let A_{-i} be the set of all opposing action vectors $(a_j)_{j \neq i}$. The complete information game g is own-action quasiconcave if for all i and opposing action profiles $a_{-i} \in A_{-i}$ and for all constants c, the set $\{a_i : g_i(a_i, a_{-i}) \geq c\}$ is convex.[15] It has *local potential maximizer* a^* if there is a function $v(a)$ (where $a = (a_i)_{i=1}^I$ is an action profile), called a local potential function, which is strictly maximized by a^*, such that against any action profile $a_{-i} \in A_{-i}$, if moving i's action a bit closer to a_i^* raises v, then this also raises i's payoff. More formally:

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[15] Unlike concavity, this does not imply that the slope of i's payoff is decreasing in her action. It only guarantees that there are no local maxima other than the global maxima (which could be a single action or an interval).

Definition 1 *Action profile a^* is a local potential maximizer (LP-maximizer) of the complete information game g if there exists a local potential function $v : A \to \mathbb{R}$ such that $v(a^*) > v(a)$ for all $a \neq a^*$ and, for each i, a function $\mu_i : a_i \to \mathbb{R}_+$ such that for all $a_i \in A_i$ and $a_{-i} \in A_{-i}$,*

1. *if $a_i > a_i^*$ then there is an $a_i' \in A_i$ that is strictly less than a_i, such that for all $a_i'' \in A_i$ lying in $[a_i', a_i]$,*

$$v(a_i'', a_{-i}) - v(a_i, a_{-i}) \leq \mu_i(a_i) [g_i(a_i'', a_{-i}) - g_i(a_i, a_{-i})] \qquad (4)$$

2. *if $a_i < a_i^*$ then there is an $a_i' \in A_i$ that is strictly greater than a_i, such that for all $a_i'' \in A_i$ lying in $[a_i, a_i']$, (4) holds.*

The local potential function generalizes the notion of a potential function in Monderer and Shapley [21]. A potential function is a common payoff function v on action profiles such that the change in a player's payoff from switching from one action to another is always the same as the change in the potential function. For local potential function, we make the weaker requirement that the payoff change from switching away from a^* is always less (after multiplying by a constant) than the change in the potential function.

Importantly, the LP-maximizer property guarantees that a profile is a strict Nash equilibrium if payoffs are own-action quasiconcave:

Lemma 1 *If a^* is an LP-maximizer of the own-action quasiconcave complete information game $\mathbf{g}$, then a^* is a strict Nash equilibrium.*

The main result of this section (stated formally at the end) is that in the global game, if the payoffs at θ are own-action concave and have an LP-maximizing action profile a^*, then in the limit as the signal errors vanish, any strategy profile that survives iterative strict dominance must assign to each player i the action a_i^* at the signal θ. Thus, there is noise-independent selection at θ: the profile a^* depends only on the payoffs at θ and not on the shape of the signal errors.

An intuition is as follows. For concreteness, let us regard v as God's utility function. God is pleased when people take steps that are in their self-interest: "God helps those who help themselves" (Benjamin Franklin, *Poor Richard's Almanack*, 1732). More precisely, a change in a player's action pleases God (i.e., raises v) if and only if it raises the player's own payoff. Not all games have a local potential function; in those that don't, God does not have a preference ordering over strategy profiles. For example, in matching pennies, God must want player A to play B's action but must also want B to play the opposite of A's action, since both are best responses. There is no preference ordering with this property.

Suppose for simplicity that the game is symmetric and has a finite set of actions and that a player's utility depends directly on her signal, rather than on θ (which is approximately true anyway when the signal errors are small). Suppose also that for the complete information game corresponding to each signal x, there is a local potential function $v_x()$, which is maximized when all players take some action a_x^*. $v_x()$ represents God's preferences over action profiles when the payoff parameter equals x. Assume that a_x^* is nondecreasing in x. Figure 7 depicts a_x^* as a function of x.
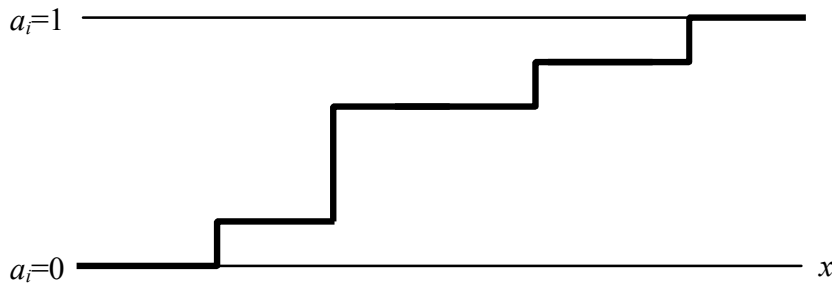

Figure 7:

We can interpret the function depicted in Figure 7 as a strategy profile: if player i's signal is x_i, the profile instructs her to play $a_{x_i}^*$. This profile is very pleasing to God, since each player plays according to the action profile that maximizes God's utility if

the payoff parameter equals her signal (which it approximately does for small signal errors). However, God may not be entirely pleased near the points of discontinuity of the strategy profile, since there might be miscoordination: some players will get signals above the threshold and others below, so a potential-maximizing strategy profile will not generally be played. But God is pleased as punch if the payoff parameter θ is at least ν away from any point of discontinuity since then players coordinate on the potential-maximizing action.

Starting with this profile, let us imagine what happens if we let players take turns in best-responding.[16] Critically, God likes it when people best-respond, since she wants them to do what is in their best interests. Thus, seeing a player best-respond can only increase God's pleasure. But since the original profile depicted in Figure 7 is *already* very pleasing to God, iterative best response cannot lead us to stray far from this profile. In particular, at a distance of more than ν from any discontinuity of the original strategy profile, the profile cannot change, since for signals received here players are already playing the profile that is most pleasing to God. Moreover, the limit of the iterations must be an equilibrium, since the best response to the limit is the limit itself.[17] However, any equilibrium must survive iterative strict dominance. By limit uniqueness, for small ν, all strategies surviving iterative strict dominance must be close to this limiting equilibrium, which itself must be close to the original profile depicted in Figure 7. This implies that for small ν, players *must* play close to the potential-maximizing action in any strategy profile that survives iterative strict dominance, regardless of the noise structure.

Theorem 4 *Let s^* be either the left- or the right-continuous version of the unique strategy profile surviving iterative strict dominance in $G(\nu)$ in the limit as $\nu \to 0$. If*

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[16]That is, player 1 switches to her best response, then 2 best responds to the resulting profile, and so on, repeatedly cycling through the players.

[17]In this intuition we assume such a limit exists.

the complete information game at some payoff parameter θ is own-action quasiconcave and has an LP-maximizer a^*, then $s^*(\theta) = a^*$, regardless of the noise structure.

The LP-maximizer conditions of Definition 1 are rather complex. In Sections 6.1 and 6.2, we describe simpler conditions that are sufficient for an action profile to be an LP-maximizer. In Sections 6.3 to 6.5, we apply those results to give a complete characterization of the LP-maximizer in certain special classes of games.

6.1 Weighted Potential Maximizers

One sufficient condition for a^* to be a local potential maximizer is that a^* is a weighted potential maximizer. This is a slight generalization of Monderer and Shapley [21]'s notion of an action profile that maximizes a potential function for a game.

Definition Action profile a^* is a *weighted potential maximizer* (*WP-maximizer*) of
$\mathbf{g}$ if there exists a vector $\mu \in \mathbb{R}_+^I$ and a weighted potential function $v : A \to \mathbb{R}$
with $v(a^*) > v(a)$ for all $a \neq a^*$, such that for all i, a_i, $a_i' \in A_i$ and $a_{-i} \in A_{-i}$,

$$v(a_i, a_{-i}) - v(a_i', a_{-i}) = \mu_i [g_i(a_i, a_{-i}) - g_i(a_i', a_{-i})].$$

6.2 p-Dominance Conditions

Let $\mathbf{p} = (p_i)_{i=1}^I$. The notion of $\mathbf{p}$-dominance is a many player, many action game generalization of risk dominance (see Kajii and Morris [15]). An action profile a^* is $\mathbf{p}$-dominant if it is a best response for each player i if she puts weight at least p_i on her opponents' playing according to a^*:

Definition 2 *Action profile a^* is $\mathbf{p}$-dominant in g if*

$$\sum_{a_{-i}} \lambda_i(a_{-i}) g_i(a_i^*, a_{-i}) \geq \sum_{a_{-i}} \lambda_i(a_{-i}) g_i(a_i, a_{-i}),$$

for all i, $a_i \in A_i$ and $\lambda_i \in \Delta(A_{-i})$ with $\lambda_i(a_{-i}^*) \geq p_i$.

For low enough $\mathbf{p}$, $\mathbf{p}$-dominance is a sufficient condition for an action profile to be an LP-maximizer.

Lemma 2 *(Morris and Ui [26]) If action profile a^* is $\mathbf{p}$-dominant for some p with $\sum_{i=1}^{I} p_i < 1$, then a^* is an LP-maximizer.*

6.3 Two Player, Two Action Games with Two Strict Nash Equilibria

Let $I = 2$ and $A_1 = A_2 = \{0, 1\}$. Let $g_1(0,0) > g_1(1,0)$, $g_1(1,1) > g_1(0,1)$, $g_2(0,0) > g_2(0,1)$ and $g_2(1,1) > g_2(1,0)$, so $(0,0)$ and $(1,1)$ are both strict Nash equilibria. Now let

$$q_1^* = \frac{g_1(0,0) - g_1(1,0)}{g_1(0,0) - g_1(1,0) + g_1(1,1) - g_1(0,1)}$$
$$q_2^* = \frac{g_2(0,0) - g_2(0,1)}{g_2(0,0) - g_2(0,1) + g_2(1,1) - g_2(1,0)}$$

A weighted potential function v is given by the following matrix:

	0	1
0	$q_1^* + q_2^*$	q_1^*
1	q_2^*	1

$(0,0)$ is a LP-maximizer if $q_1^* + q_2^* > 1$ and $(1,1)$ is a WP-maximizer if $q_1^* + q_2^* < 1$. Thus, generically, there is a WP-maximizer. The WP-maximizer is the risk dominant equilibrium in the sense of Harsanyi and Selten [14].

6.4 Many Player, Two Action Games with Symmetric Payoffs

Let $A_i = \{0, 1\}$ for each i and suppose $g_i(a_i, a_{-i}) = g(a_i, a_{-i})$ depends only on a_i and the number of players $j \neq i$ who play 1. Let $\xi(n)$ be the relative payoff to playing 1 versus 0 when n other players play 1. (I.e., $\xi(n) = g(1, a_{-i}) - g(0, a_{-i})$ for any

a_{-i} in which n players play 1.) Assume strategic complementarities: i.e., $\xi(n)$ is increasing in n. Let the potential function be

$$v(a) = \begin{cases} \sum_{k=0}^{m-1} \xi(k), \text{ if the number of players playing 1 in } a \text{ is } m > 0 \\ 0, \text{ if no players play 1 in } a \end{cases}$$

Also set $\mu_i = 1$ for all i. One can easily verify that $\mathbf{1} = (1,..,1)$ is the WP-maximizer if $\sum_{k=0}^{I-1} \xi(k) > 0$ and that $\mathbf{0}$ is the WP-maximizer if $\sum_{k=0}^{I-1} \xi(k) < 0$. Thus generically in this class of games, there exists a WP-maximizer.

An equivalent characterization of the WP-maximizer is the following. Suppose that a player believes that the number of her opponents playing action 1 is uniformly distributed (between 0 and $I-1$). If the action 1 is a best response to that conjecture, then the action profile $\mathbf{1}$ is the WP-maximizer; if 0 is a best response to that conjecture, then $\mathbf{0}$ is the WP-maximizer. These are equivalent since $\frac{1}{I}\sum_{k=0}^{I-1} \xi(k)$ is just the relative payoff to playing 1 if one has such beliefs. This case (2 actions, many players) was first studied by Carlsson and van Damme [6] and Kim [16], who obtained the same result using different techniques in more restrictive settings.

This characterization extends naturally to the case of a continuum of players (not formally treated here). In such games, the formula becomes $\int_0^1 \xi(n)\,dn$, where $\xi(n)$ is the relative payoff to playing 1 versus 0 when a proportion n of the other players play 1. Such 2-action, symmetric payoff games with continuum of players have been the focus of the applied literature using global games discussed in the introduction.

6.5 Two Player, Three Action Games with Symmetric Payoffs

Let $I = 2$, $A_1 = A_2 = \{0, 1, 2\}$; $g_1(a_1, a_2) = g_2(a_2, a_1) = w_{a_1}w_{a_2}$, where $w_{xx} > w_{yx}$ for all $y \neq x$ and $w_{xy} - w_{x'y} > w_{xy'} - w_{x'y'}$ if $x > x'$ and $y > y'$. Write $\Delta_{x'y'}^{xy}$ for the net expected gain of choosing action x rather than y against a 50/50 conjecture on

whether the opponent will choose action x' or y'. Thus

$$\Delta_{x'y'}^{xy} = w_{x'x} + w_{x'y} - w_{y'x} - w_{y'y}.$$

Note that $\Delta_{x'y'}^{xy} = \Delta_{x'y'}^{yx}$ and $\Delta_{x'y'}^{xy} = -\Delta_{y'x'}^{xy}$. Note that $\Delta_{xy}^{xy} > 0$ implies that action profile (x, x) pairwise risk dominates action profile (y, y). Now we have the following complete (for generic games) characterization of the LP-maximizers.

- $(0, 0)$ is the LP-maximizer if $\Delta_{01}^{01} > 0$ and either (1) $\Delta_{12}^{12} > 0$ or (2) $\Delta_{21}^{21} > 0$ and $\frac{\Delta_{10}^{02}}{\Delta_{01}^{01}} < \frac{\Delta_{12}^{02}}{\Delta_{21}^{21}}$.

- $(1, 1)$ is the LP-maximizer if $\Delta_{10}^{10} > 0$ and $\Delta_{12}^{12} > 0$.

- $(2, 2)$ is the LP-maximizer if $\Delta_{21}^{21} > 0$ and either (1) $\Delta_{10}^{10} > 0$ or (2) $\Delta_{01}^{01} > 0$ and $\frac{\Delta_{10}^{02}}{\Delta_{01}^{01}} > \frac{\Delta_{12}^{02}}{\Delta_{21}^{21}}$.

The following example illustrates these conditions:

(g_1, g_2)	0	1	2
0	$4, 4$	$0, 0$	$-6, -3$
1	$0, 0$	$1, 1$	$0, 0$
2	$-3, -6$	$0, 0$	$2, 2$

$(0, 0)$ is the LP-maximizer, since $\Delta_{01}^{01} = 3$, $\Delta_{21}^{21} = 1$, $\Delta_{10}^{02} = 2$ and $\Delta_{12}^{02} = 1$. Note that $(2, 2)$ pairwise risk dominates both $(1, 1)$ and $(0, 0)$, but nonetheless is not the LP-maximizer.

Proving the above claims (i.e., constructing the local potential functions) involves tedious algebra. Here, we will just note two cases to illustrate the issues.

Case 1: $\Delta_{10}^{10} > 0$ and $\Delta_{12}^{12} > 0$. Consider the following local potential function:

v	0	1	2
0	$-\Delta_{10}^{10}$	$w_{01} - w_{11}$	$-\varepsilon$
1	$w_{01} - w_{11}$	0	$w_{21} - w_{11}$
2	$-\varepsilon$	$w_{21} - w_{11}$	$-\Delta_{12}^{12}$

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for some small but strictly positive ε. Setting $a^* = (1,1)$ and $\mu_1(0) = \mu_1(2) = \mu_2(0) = \mu_2(2) = 1$, one can verify that the conditions of Definition 1 are satisfied.

Case 2: $\Delta_{01}^{01} > 0$, $\Delta_{21}^{21} > 0$, $\Delta_{10}^{02} > 0$, $\Delta_{12}^{02} > 0$ and $\frac{\Delta_{10}^{02}}{\Delta_{01}^{01}} < \frac{\Delta_{12}^{02}}{\Delta_{21}^{21}}$. Consider the following local potential function:

v	0	1	2
0	ε	$\varepsilon + \lambda_1 [w(1,0) - w(0,0)]$	$\lambda_1 [w_{02} - w_{12}] + \lambda_2 [w_{12} - w_{22}]$
1	$\varepsilon + \lambda_1 [w_{10} - w_{00}]$	$-\lambda_2 \Delta_{21}^{21}$	$\lambda_2 [w_{12} - w_{22}]$
2	$\lambda_1 [w_{02} - w_{12}] + \lambda_2 [w_{12} - w_{22}]$	$\lambda_2 [w_{12} - w_{22}]$	0

for some small but strictly positive ε and positive λ_1 and λ_2 such that

$$\frac{\Delta_{21}^{21}}{\Delta_{01}^{01}} < \frac{\lambda_1}{\lambda_2} < \frac{\Delta_{12}^{02}}{\Delta_{10}^{02}}.$$

Setting $a^* = (0,0)$, $\mu_1(1) = \mu_2(1) = \lambda_1$, $\mu_1(2) = \mu_2(2) = \lambda_2$, one can verify that the conditions of Definition 1 are satisfied.

7 A Continuum Player Generalization

The results stated above are for a finite set of players. However, many applications of global games assume a continuum (or several different continua) of players. We now show that our limit uniqueness results (Theorems 1 and 2) extend to this case.

We make the following changes to the global game $G(\nu)$. The set of players, denoted I, is partitioned into a finite set T of "types" (subsets) of players. Each type contains either a single player or a continuum of identical players of finite measure. This can capture, e.g., the presence of both large and small players. For each player $i \in I$ let $\tau(i) \in T$ be the type of i: the element of T to which i belongs. Each player i observes a signal $x_i = \theta + \nu \eta_i$, where each η_i is distributed according to an atomless density $f_{\tau(i)}$ with support contained in the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$. Signals are conditionally

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independent.[18]

Let A_t denote the action set of players of type t. Let $O(i)$ be the set of types of player i's opponents.[19] If a type-t player i chooses action $a_i \in A_t$, her payoff is $u_t(a_i, a_{-i}, \theta)$; $a_{-i} = (a^{\hat{t}})_{\hat{t} \in O(i)}$ denotes the action profile of i's opponents, where $a^{\hat{t}}$ is the cumulative distribution function of actions chosen by type-$\hat{t}$ players.[20] This implies that opponents of a given type are interchangeable: the c.d.f. of their actions is all that player i cares about and all that the action profile a_{-i} captures. We assume players always play measurable action profiles (those that can be expressed as a vector of c.d.f.'s). Note that all players of a given type have the same action set, signal error distribution, and payoff function.[21]

Let $\Delta u_t(a_i, \hat{a}_i, a_{-i}, \theta) = u_t(a_i, a_{-i}, \theta) - u_t(\hat{a}_i, a_{-i}, \theta)$. Let us write $a_{-i} \geq \hat{a}_{-i}$ if actions are weakly higher under a_{-i} than under $\hat{a}_{-i}$: if $a^{\hat{t}}(c) \leq \hat{a}^{\hat{t}}(c)$ for each opposing type $\hat{t} \in O(i)$ and for all c. For any $t \in T$, we define the distance $|a^t - \hat{a}^t|$ to be the largest difference in actions between players with the same rank in the two distributions.[22] We make the same assumptions **A1** through **A5** on payoff functions as in the finite player, where a_{-i} is now understood to be the collection of opponents' cdfs defined above and we replace each u_i with $u_{\tau(i)}$ and each Δu_i with $\Delta u_{\tau(i)}$.[23] Note

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[18] If there is a continuum of players of type t, we assume that the realized distribution of the error terms $(\eta_i)_{i \in t}$ in this type is given by f_t with probability one.

[19] It is equal to T unless $\tau(i)$ is a singleton, in which case $O(i) = T - \tau(i)$.

[20] That is, $a^{\hat{t}}(c)$ is the proportion of type-$\hat{t}$ agents who play actions less than or equal to c. If $\hat{t}$ is a singleton $\{j\}$, then $a^{\hat{t}}(c)$ equals one if j's action is no greater than c and zero otherwise.

[21] Agents of different types can also be identical.

[22] That is,

$$\left| a^t - \hat{a}^t \right| = \sup \left\{ k : \text{for all } a_i \in A_t, \text{ either } a^t(a_i + k) \leq \hat{a}^t(a_i) \text{ or } \hat{a}^t(a_i + k) \leq a^t(a_i) \right\}$$

[23] In A2, $(\underline{a}_1, ..., \underline{a}_I)$ and $(\overline{a}_1, ..., \overline{a}_I)$ become $(\underline{a}_i)_{i \in I}$ and $(\overline{a}_i)_{i \in I}$, respectively. Condition (b) in

that in this case, assumption **A5** now has bite even in finite action games, since in this case there exist pairs of opponents' action profiles that are different but arbitrarily close.

The generalizations of Theorems 1 and 2 are as follows:

Theorem 5 $G(\nu)$ *has an essentially unique strategy profile surviving iterative strict dominance in the limit as $\nu \to 0$. In this profile, all players of a given type play the same increasing, pure strategy. More precisely, there exists an increasing pure strategy profile $(s_t^*)_{t \in T}$ such that if, for each ν, s^ν is a strategy profile that survives iterative strict dominance in $G(\nu)$, then $\lim_{\nu \to 0} s_i^\nu(x_i) = s_{\tau(i)}^*(x_i)$ for almost all $x_i \in \mathbb{R}$.*

For any $\varepsilon > 0$ and $\nu > 0$, let $Q(\varepsilon, \nu)$ be the set of parameters θ for which the surviving strategy profiles in $G(\nu)$ do *not* all prescribe that players play ε-close to the same pure strategy Nash equilibrium of some complete information game whose payoff parameter is ε-close to θ. More precisely, $Q(\varepsilon, \nu)$ is the set of parameters θ for which there is *no* Nash equilibrium action profile $a \in \times_{t \in T} A_t$, of the complete information game with payoffs $(u_t(\cdot, \theta'))_{t \in T}$ for some $\theta' \in [\theta - \varepsilon, \theta + \varepsilon]$, such that for every strategy $(s_i^\nu)_{i \in I}$ surviving iterative strict dominance in $G(\nu)$, $\left| s_i^\nu(\theta) - a_{\tau(i)} \right| \leq \varepsilon$ for all i.

Theorem 6 *In $G(\nu)$ in the limit as $\nu \to 0$, for almost all payoff parameters θ, players play arbitrarily close to some pure strategy Nash equilibrium of the complete information game with payoffs $u_t(\cdot, \theta')$ for some θ' that is arbitrarily close to θ. More precisely, for any $\varepsilon > 0$ there is a $\overline{\nu} > 0$ such that for any $\nu < \overline{\nu}$, $Q(\varepsilon, \nu)$ is contained*

Assumption **A5** becomes: for each θ there is a K_2 such that for all $a_i, \widehat{a}_i, a_i,$ and $\widehat{a}_i,$

$$\left| \Delta u_{\tau(i)}(a_i, \widehat{a}_i, a_{-i}, \theta) - \Delta u_{\tau(i)}(a_i, \widehat{a}_i, \widehat{a}_{-i}, \theta) \right| \leq K_2 |a_i - \widehat{a}_i| \sum_{t \in O(i)} \left| a^t - \widehat{a}^t \right|$$

in a finite union of closed intervals of $\mathbb{R}$ whose measure is less than ε.[24]

8   Concluding Remarks

8.1   Contagion Arguments

Our limit uniqueness argument generalizes the "infection" arguments of Carlsson and van Damme [5] and Morris, Rob and Shin [22]. (A precursor to this literature is Rubinstein [27].) However, providing a general argument for the case of many actions requires a significant extension of that logic. The intuition is most closely related to results in Burdzy, Frankel, and Pauzner [3] and its extension, Frankel and Pauzner [9].[25] These papers study dynamic models with complete information. There is a continuum of players who switch between two actions. There are frictions, so that players change actions asynchronously. Instantaneous payoffs depend on a parameter that follows a Brownian motion. This payoff parameter can reach "dominance regions" in which either action is strictly dominant.

These papers show that a unique equilibrium survives iterative dominance. While the details are different,[26] there is an analogy. In both cases, players play against opponents in different but nearby "states": the value of the Brownian motion at the moment when the opponent picks her action in those papers and a player's payoff signal in our paper. This local interaction gives rise to a contagion effect that begins in the dominance regions and spreads throughout the state space. The whole state

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[24]As in Theorem 2, the number of intervals in the union is finite and independent of ν for any fixed ε. This number may grow without bound as ε shrinks to zero.

[25]These papers are further extended in Frankel [8] and Levin [17].

[26]In particular, the vertical axis in Burdzy, Frankel, and Pauzner [3] captures not the opponent's action but the population action distribution; the horizontal axis captures the current value of the Brownian motion rather than the payoff signal; and the curves in that paper slope downwards rather than upwards.

space is affected because the interaction structure is stationary: the probability of playing against an agent who sees a state a given distance from one's own state is independent of one's own state.[27] Using this property, a translation argument as in section 3 implies that the lowest and highest strategies surviving iterative dominance must coincide.

8.2 Supermodular Games

Our global game belongs to the class of supermodular games, first studied by Topkis [28] and further analyzed by Vives [30] and Milgrom and Roberts [19].[28] Arguments in that literature establish (1) the existence of a largest and smallest strategy profile surviving iterated deletion of strictly dominated strategies; (2) that the largest and smallest strategy profiles are themselves equilibria; and (3) that those largest and smallest strategies are monotonic in players' signals. In global games, an additional property holds: the largest and smallest strategy profiles coincide in the limit as the signal errors vanish. Thus, there is a unique, monotonic equilibrium in the limit.

In the light of this interpretation, it is natural to ask if there still exists exactly one monotonic equilibrium when the supermodularity assumption (**A1**) is weakened (see Athey [1]). The answer may be yes, if a player's best response rises when her opponents switch from one monotonic strategy to a higher monotonic strategy. This might be proved using something like the translation argument of section 3. (We have not checked this rigorously as it is beyond the scope of this paper). However,

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[27]In our paper, a player's signal asymptotically has no effect on her posterior belief that her opponent's signal differs from hers by a given amount. In Burdzy, Frankel, and Pauzner [3], the stationarity of Brownian motion implies that the payoff parameter a player sees when choosing her action has no effect on the probability that she will meet an opponent who will have chosen his action when the payoff parameter will have shifted by a given amount.

[28]To be more precise, the uniform prior game is a supermodular game and our continuity arguments establish that non-uniform prior games are close to the supermodular game when noise is small.

without **A1** we cannot show that iterative dominance yields a unique equilibrium; in particular, we cannot rule out the existence of other, non-monotonic equilibria.[29]

8.3   Noise-Independent Selection and Robustness

Our example in which noise independent selection fails is for a two player, four action symmetric payoff game. This example is minimal in the sense that noise independent selection must hold with two players and symmetric payoffs if there are fewer than four actions.   However, noise independent selection can fail in games with three players if payoffs are asymmetric, as shown by Carlsson [4].   One application is Corsetti, Dasgupta, Morris and Shin [7], who study models of currency attacks in which noise-independent selection can fail.

Our noise independent selection results are related to work on the robustness of equilibria to incomplete information (Kajii and Morris [15]). A Nash equilibrium of a complete information game is *robust to incomplete information* if every incomplete information game in which payoffs are almost always given by that complete information game has an equilibrium in which that Nash equilibrium is almost always played. Kajii and Morris showed that risk dominant equilibria of two player, two action games and, more generally, **p**-dominant equilibria of many player, many action games with $\sum_{i=1}^{I} p_i < 1$ are robust to incomplete information. Ui [29] has shown that potential maximizing action profiles are robust to incomplete information.[30] The

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[29]Morris and Shin [25] show that among binary action symmetric payoff continuum player games, a single crossing property on payoffs and a monotone likelihood ratio property on signals implies the existence of a unique monotonic equilibrium.   However, there is no guarantee that there do not exist non-monotonic equilibria. The bank run game of Goldstein and Pauzner [11] belongs to this class.   By assuming that noise is uniformly distributed, Goldstein and Pauzner [11] are able to show the existence of a unique equilibrium, which is monotonic, but their argument does not show that the game is dominance solvable.

[30]In the absence of strategic complementarities, the potential maximizing action profile actually satisfies a slightly weaker notion of robustness to incomplete information.

global games introduced by Carlsson and van Damme [5] and studied in this paper represent a different way of adding an intuitively small amount of incomplete information about payoffs. However, one can show that *if* a complete information game has a robust equilibrium, then that equilibrium must be the noise independent selection. The sufficient conditions for noise independent selection in this paper are in fact also sufficient for robustness to incomplete information (see Morris and Ui [26]).

Appendix

We first prove Theorems 5 and 6, for the continuum player extension of section 7. Our main Theorems 1 and 2 follow as special cases of these results.

Proof of THEOREM 5. The proof has two parts. In the first (Lemma 3), we consider a simplified game in which a player's prior over θ is uniform over some very large interval that includes $\left[\underline{\theta}, \overline{\theta}\right]$, and her payoff depends directly on her signal rather than on θ. Because of the prior, if a player's signal is not in a dominance region, then her posterior over the difference between her signal and those of other players is independent of her signal, so we can show that a unique (up to discontinuities), increasing strategy profile survives iterative strict dominance even without taking the signal error scale factor ν to zero. The argument generalizes the translation argument used in the intuition.

In the second part of the proof, we show that the original game "converges" to the simplified one as the signal errors shrink. That is, a player's posterior over the differences between her signal and those of other players becomes approximately independent of her own signal (Lemma 4). Moreover, in the limit it does not matter whether a player's payoffs depends on her signal or on θ since these become arbitrarily close. As a result, the strategy profiles surviving iterative dominance in the original and simplified games converge to each other (Lemma 5).

For each type $t \in O(i)$, let z_t be the realized c.d.f. of normalized differences between signals of type-t players and the signal of i: $z_t(c)$ is the proportion of players $j \in t$ for whom $\frac{x_j - x_i}{\nu} \leq c$.[31] Let Z_{-i} be the set of all vectors of the form $(z_t)_{t \in O(i)}$. Let $\widetilde{\pi}_{\tau(i)}\left(z \,|x_i, \nu\right)$ be player i's density over $z \in Z_{-i}$ in the game $G(\nu)$ given i's signal x_i and the scale factor ν.

The simplified game is defined as follows. Let the state θ be drawn *uniformly*

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[31] z_t depends on the realizations of players' signals and is thus a random variable. For example, if if t is a singleton type $\{j\}$, then $z_t(c)$ equals one if $\frac{x_j - x_i}{\nu} \leq c$ and zero else.

from some large interval that includes $[\underline{\theta} - \nu, \overline{\theta} + \nu]$ and let player i's payoff depend on her *signal* x_i instead of the state. Thus $u_{\tau(i)}(a_i, a_{-i}, x_i)$ is player i's payoff if action profile (a_i, a_{-i}) is chosen and she observes signal x_i. Note that with a uniform prior on states, player i's posterior $\widetilde{\pi}_{\tau(i)}(z | x_i, \nu)$ is independent of her signal x_i and the scale factor ν, so we can write it as $\pi_{\tau(i)}(z)$.

Lemma 3 shows that each game $G^*(\nu)$ has an essentially unique strategy profile surviving iterative strict dominance. In this profile, all players of a given type play essentially the same pure, increasing strategy.

Lemma 3 *For each $\nu > 0$ there exists a weakly increasing strategy profile $s^{*\nu}$ such that any profile s that survives iterative strict dominance in $G^*(\nu)$ must (a) be weakly increasing in x_i for all i and (b) agree with $s_i^{*\nu}$ except perhaps at the (at most countably many) signals x_i at which $s_i^{*\nu}(x_i)$ is discontinuous. Moreover, $s^{*\nu}$ prescribes the same strategy for all players of a given type.*

Proof. For any player i of type t, let Z_{-t} be the set of all possible vectors $z = (z_t)_{t \in O(i)}$ of normalized differences between i's opponents signals and her own signal. For any pure strategy profile $s = (s_t)_{t \in T}$ (which, as the notation indicates, prescribes the same strategy s_t for all players of type t), let $a_{-i}(s, x_i, z; \nu)$ denote the opposing action distribution that a player i faces if she sees the signal x_i, others play according to s, z is the vector of normalized signal differences, and ν is the scale factor. More precisely, $a_{-i}(s, x_i, z; \nu) = (a^t)_{t \in O(i)}$ where $a^t(c)$ is the proportion of players $j \in t$ for whom the prescribed action $s_t(x_i + \nu z_j)$ is no greater than c.

Let $BR_t(s, x_i)$ be the set of optimal actions for a player i of type t who sees signal x_i and whose opponents play according to the pure strategy $s = (s_{\hat{t}})_{\hat{t} \in T}$:

$$BR_t(s, x_i) = \arg\max_{a_t \in A_t} E_{\pi_t}\left[u_t\left(a_i, a_{-i}(s, x_i, z; \nu), x_i\right)\right]$$

where for any function $h(z)$, $E_{\pi_t}[h(z)] = \int_{z \in Z_{-i}} \pi_t(z) h(z) dz$. Note that by assumption **A4** and the theorem of the maximum, BR_t must be upper hemicontinuous in x_i,

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even if s is discontinuous. (Because of the noise in the signals, i's posterior over the distribution of opponents' actions changes continuously in her signal).

We first do iterated dominance to establish a lower bound on the actions players can choose for each signal. We define a sequence of strategy profiles $s^k \equiv \left(s_t^k\right)_{t \in T}$ for $k = 0, 1, \ldots$ as follows. Let s^0 be the constant profile in which all players play 0 for any signal. Let s^{k+1} be the smallest best response to s^k: $s_t^{k+1}(x_i) = \min BR_t(s^k, x_i)$. **A1** and **A3** imply that (i) $s_t^k(x_i)$ is weakly increasing in x_i for all (t, k) and (ii) $s_t^k(x_i)$ is weakly increasing in k for all (t, x_i). **A2** implies $s_t^k(\underline{\theta}) = 0$ and $s_t^k(\overline{\theta}) = 1$ for all types t and for $k \geq 1$. Let $\underline{s}$ be defined by $\underline{s}_t(x_i) = \lim_{k \to \infty} s_t^k(x_i)$. A player i seeing signal x_i must choose an action that is at least $\underline{s}_{\tau(i)}(x_i)$. By induction, $\underline{s}_t$ is weakly increasing. Since BR_t is upper hemicontinuous and (by an induction argument) $\min BR_t(\underline{s}, x_i)$ is weakly increasing in x_i, $\underline{s}_t(x_i)$ must be left continuous.

We next construct an upper bound on players' actions. For any $\lambda \in \mathbb{R}$, let $\widehat{s}$ be the right-continuous version of $\underline{s}$ (i.e., $\widehat{s}_t(x_i) = \lim_{\varepsilon \downarrow 0} \underline{s}_t(x_i + \varepsilon)$) and let $s^\lambda = (s_t^\lambda)_{t \in T}$ be the translation of $\widehat{s}$ to the left by λ: for all t and x_i, $s_t^\lambda(x_i) = \widehat{s}_t(x_i + \lambda)$. Let λ_0 be large enough that, for all t, $s_t^{\lambda_0}(\underline{\theta}) = 1$; since s^{λ_0} is weakly increasing, a player of type t with signal x_i will never play an action that is greater than $s_t^{\lambda_0}(x_i)$. Now let λ_k be the smallest number such that no type-t player who expects others to play according to $s^{\lambda_{k-1}}$ will ever play above $s_t^{\lambda_k}$:

$$\lambda_k = \inf\{\lambda : s_t^\lambda(x_i) \geq \max BR_t(s^{\lambda_{k-1}}, x_i) \ \forall t, x_i\}$$

A1 and **A3** imply that λ_k is weakly decreasing in k. Let $\lambda_\infty = \lim_{k \to \infty} \lambda_k$, and denote $\overline{s} = s^{\lambda_\infty}$. (Note that $\lambda_\infty \geq 0$ since the iterations cannot go beyond $\underline{s}$.)

By construction, a type-t player who sees signal x_i will never play an action that exceeds $\overline{s}_t$: $\overline{s}_t(x_i) \geq \max BR_t(\overline{s}, x_i)$ for all x_i. We will show by contradiction that $\lambda_\infty = 0$: $\overline{s}$ and $\underline{s}$ coincide. This means that the strategies in any profile that survives iterative dominance in $G^*(\nu)$ must be weakly increasing functions that agree with $\underline{s}$ at all points of continuity. This will prove the lemma since a monotonic function can have at most a countable number of discontinuities.

Claim 1 *There is a player i and a signal x_i^* such that $\overline{s}_i(x_i^* - \varepsilon) < \max BR_i(\overline{s}, x_i^* + \varepsilon)$ for all $\varepsilon > 0$.*

Proof Since $\overline{s}$ is the limit of the iterations from the left, for all $\varepsilon > 0$ there is a player i and a signal x_i such that $\overline{s}_i(x_i - \varepsilon) < \max BR_i(\overline{s}, x_i)$. (Otherwise let ε' be such that there is no i and x_i for which $\overline{s}_i(x_i - \varepsilon') < \max BR_i(\overline{s}, x_i)$; define $\lambda_\infty' = \lambda_\infty - \varepsilon'$. The limit of the iterations must be no greater than λ_∞', a contradiction.) Since the number of players is finite and each $\overline{s}_i$ is weakly increasing, there must be a particular player i such that for all $\varepsilon > 0$, $\overline{s}_i(x_i(\varepsilon) - \varepsilon) < \max BR_i(\overline{s}, x_i(\varepsilon))$ for some $x_i(\varepsilon)$. Define $x_i'(\varepsilon) = x_i(2\varepsilon) - \varepsilon$; we know that for all $\varepsilon > 0$, $\overline{s}_i(x_i'(\varepsilon) - \varepsilon) < \max BR_i(\overline{s}, x_i'(\varepsilon) + \varepsilon)$. Since for all ε, $x_i'(\varepsilon) \in [\underline{\theta}, \overline{\theta}]$, there is a convergent subsequence of $x_i'(\varepsilon)$ as $\varepsilon \to 0$; let x_i^* be the limit. For all $\varepsilon > 0$, $\overline{s}_i(x_i^* - \varepsilon) < \max BR_i(\overline{s}, x_i^* + \varepsilon)$. (Why: we can take ε' small enough that $\overline{s}_i(x_i'(\varepsilon') - \varepsilon/2) < \max BR_i(\overline{s}, x_i'(\varepsilon') + \varepsilon/2)$ and $|x_i'(\varepsilon') - x_i^*| < \varepsilon/2$; hence, $\overline{s}_i(x_i'(\varepsilon') - \varepsilon/2) \geq \overline{s}_i(x_i^* - \varepsilon)$ and $\max BR_i(\overline{s}, x_i'(\varepsilon') + \varepsilon/2) \leq \max BR_i(\overline{s}, x_i^* + \varepsilon)$.) Q.E.D.$_{\text{(Claim 1)}}$

We claim that if $\lambda_\infty > 0$, then for some $\varepsilon > 0$,

$$\max BR_i(\overline{s}, x_i^* + \varepsilon) \leq \max BR_i(\widehat{s}, x_i^* + \lambda_\infty - \varepsilon) \leq \widehat{s}_i(x_i^* + \lambda_\infty - \varepsilon) = \overline{s}_i(x_i^* - \varepsilon)$$

Only the first inequality is nontrivial. In the two cases (i.e., i getting signal $x_i^* + \varepsilon$ and expecting others to play $\overline{s}$ vs. i getting signal $x_i^* + \lambda_\infty - \varepsilon$ and expecting others to play $\widehat{s}$), the distributions of action profiles i expects to see become identical as $\varepsilon \to 0$. Let

$$\widehat{a}_i = \max BR_i(\widehat{s}, x_i^* + \lambda_\infty - \varepsilon)$$
$$\overline{a}_i = \max BR_i(\overline{s}, x_i^* + \varepsilon)$$

We must show that $\overline{a}_i \leq \widehat{a}_i$. We know that $\widehat{a}_i$ is strictly better than any higher action if i gets signal $x_i^* + \lambda_\infty - \varepsilon$ and expects others to play $\widehat{s}$: for all $a_i > \widehat{a}_i$, $E_{\pi_t}(\Delta u_{\tau(i)}(a_i, \widehat{a}_i, a_{-i}(\widehat{s}, \widehat{x}_i, z; \nu), \widehat{x}_i) < 0$ where $\widehat{x}_i = x_i^* + \lambda_\infty - \varepsilon$. We claim that $E_{\pi_t}(\Delta u_{\tau(i)}(a_i, \widehat{a}_i, a_{-i}(\overline{s}, \overline{x}_i, z; \nu), \overline{x}_i)) < 0$ where $\overline{x}_i = x_i + \varepsilon$. (This will imply $\overline{a}_i \leq \widehat{a}_i$.)

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Since the action distributions played by i's opponents in the two situations differ by the order of ε while the payoff parameter differs by $\lambda_\infty - 2\varepsilon$, assumptions **A3** and **A5** imply:

$$E_{\pi_t}(\Delta u_{\tau(i)}(a_i, \widehat{a}_i, a_{-i}(\overline{s}, \overline{x}_i, z; \nu), \overline{x}_i)) - E_{\pi_t}(\Delta u_{\tau(i)}(a_i, \widehat{a}_i, a_{-i}(\widehat{s}, \widehat{x}_i, z; \nu), \widehat{x}_i)$$

$$\leq -K_0(a_i - \widehat{a}_i)(\lambda_\infty - 2\varepsilon) + K_2(a_i - \widehat{a}_i)|T|o(\varepsilon)$$

$$\underset{\varepsilon \to 0}{\to} -K_0(a_i - \widehat{a}_i)\lambda_\infty$$

which is negative if $\lambda_\infty > 0$. ($|T|$ denotes the number of types in the game and $o(\varepsilon)$ denotes a term that is on the order of ε.) This establishes that $\lambda_\infty = 0$.

When all action sets are finite, assumption **A2** can be replaced by **A2′** by use of the following claim.

Claim 2 *Suppose all action sets are finite and assume A1, A2′, A3, A4, and A5. Let $\underline{a} = (\underline{a}_t)_{t \in T}$ and $\overline{a} = (\overline{a}_t)_{t \in T}$ be the unique Nash equilibrium strategy profiles of the underlying complete information game for payoff parameters $\theta < \underline{\theta}$ and for $\theta > \overline{\theta}$, respectively.*[32] *There is a constant K such that for any increasing pure equilibrium strategy profile s of $G^*(\nu)$ that assigns the same strategy to all players of a given type, $\underline{a}$ must be played for all signals below $\underline{\theta}' = \underline{\theta} - K\nu$ and $\overline{a}$ must be played for all signals above $\overline{\theta}' = \overline{\theta} + K\nu$.*

Proof. For any positive integer n, consider the signal vector given by $x_i = \underline{\theta} - n\nu$ for all i. Since $\underline{a}$ is the unique Nash equilibrium when payoffs equal $\underline{\theta} - n\nu$, the only way that s can prescribe something other than $\underline{a}$ at this signal vector is if some

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[32] As implied by the notation, these unique equilibria must prescribe the same action for all players of a given type. Else the players of a given type would be indifferent between two different actions. But then if all switched to (say) the higher of these actions and we then performed iterative best response in the complete information game, we would (by **A1**) converge to a different Nash equilibrium in which all players' actions were weakly higher and some were strictly higher. This contradicts the assumption that there is a unique Nash equilibrium.

player i is uncertain (expecting s to be played) about the action played by one of her opposing types: if, for some $t \in O(i)$, $s_t(x)$ takes on more than one value for signals $x \in [\underline{\theta} - (n+1)\nu, \underline{\theta} - (n-1)\nu]$.[33] Since s is monotonic and each action set is finite, this condition can hold for at most a finite set of positive integers n. In particular, if $x < \underline{\theta}' = \underline{\theta} - \nu \sum_{t \in T} |A_t|$ then $s_t(x) = \underline{a}_t$ for all t. Likewise, $\overline{a}$ must be played for signals above $\overline{\theta}' = \overline{\theta} + \nu \sum_{t \in T} |A_t|$. Q.E.D.$_{\text{(Claim 2)}}$

We now iterate from above to obtain an upper bound on the set of equilibrium strategies. For all t, let $\widehat{s}_t^0(x_i) = 1$ for all $x_i \in \mathbb{R}$ and let $s_t^{k+1}(x_i) = \max BR_t(s^k, x_i)$. Let $\widehat{s}$ be defined by $\widehat{s}_t(x_i) = \lim_{k \to \infty} \widehat{s}_t^k(x_i)$. A player i seeing signal x_i must choose an action that is at most $\widehat{s}_{\tau(i)}(x_i)$. By induction, $\widehat{s}_t$ is weakly increasing and since $\widehat{s}$ is a best response to itself, it is an equilibrium of $G^*(\nu)$. By Claim 2, under $\widehat{s}$ players must play $\underline{a}$ for signals below $\underline{\theta}'$ and $\overline{a}$ for signals above $\overline{\theta}'$.

For any $\lambda \in \mathbb{R}$, let s^λ denote the translation of the right-continuous version of $\underline{s}$ to the left by λ. Let λ_0 be large enough that, for all t and signals x_i, $s_t^{\lambda_0}(x_i) \geq \widehat{s}_t(x_i)$. ($\lambda_0 = \overline{\theta}' - \underline{\theta}'$ will suffice.) Players cannot choose actions that lie above s^{λ_0}. The rest of the proof proceeds as before: we iterate from the left using translations s^{λ_k} until we reach a limit, and prove (using identical arguments) that this limit must equal $\underline{s}$. Q.E.D.$_{\text{Lemma 3}}$

Lemma 4 shows that as $\nu \to 0$, players' posteriors over normalized signal differences $G(\nu)$, $\widetilde{\pi}_{\tau(i)}(z \,|x_i, \nu)$, converge to the posteriors in $G^*(\nu)$. For any probability measure μ on Z_{-i}, let $\Psi_\varepsilon(\mu)$ be the set of probability measures that differ from μ by no more than ε for any subset of Z_{-i}:

$$\Psi_\varepsilon(\mu) \equiv \left\{ \mu' \; : \; \sup_{S \subset Z_{-i}} |\mu(S) - \mu'(S)| \leq \varepsilon \right\}.$$

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[33]Because of the bounded supports, player i knows that all other players' signals will be within ν of $x_i = \underline{\theta} - n\nu$.

Lemma 4 *For any $\varepsilon > 0$ and compact interval $B = [b_0, b_1]$ such that $[b_0 - \nu, b_1 + \nu]$ is contained in the interior of the support of ϕ, there exists $\overline{\nu} > 0$ such that $\widetilde{\pi}_{\tau(i)} \left(\cdot \, | x_i, \nu \right) \in \Psi_\varepsilon \left(\pi_{\tau(i)} \left(\cdot \right) \right)$ for all $x_i \in B$ and all $\nu \leq \overline{\nu}$.*

Proof. Denote

$$\rho \left(\nu \right) = \sup_{x_i \in \mathbf{B}} \frac{\max\limits_{\theta' \in \left[-\frac{1}{2}, \frac{1}{2} \right]} \phi \left(x_i + \nu \theta' \right)}{\min\limits_{\theta' \in \left[-\frac{1}{2}, \frac{1}{2} \right]} \phi \left(x_i + \nu \theta' \right)}.$$

Since the support of ϕ includes $[b_0 - \nu, b_1 + \nu]$, ϕ has a strictly positive minimum value over this interval. Since ϕ is also continuous, $\max_{\theta' \in \left[-\frac{1}{2}, \frac{1}{2} \right]} \phi \left(x_i + \nu \theta' \right) - \min_{\theta' \in \left[-\frac{1}{2}, \frac{1}{2} \right]} \phi \left(x_i + \nu \theta' \right)$ converges to 0 as $\nu \to 0$ uniformly for $x_i \in [-b, b]$. Hence,

$$\rho \left(\nu \right) = \sup_{x_i \in \mathbf{B}} \left(1 + \frac{\max\limits_{\theta' \in \left[-\frac{1}{2}, \frac{1}{2} \right]} \phi \left(x_i + \nu \theta' \right) - \min\limits_{\theta' \in \left[-\frac{1}{2}, \frac{1}{2} \right]} \phi \left(x_i + \nu \theta' \right)}{\min\limits_{\theta' \in \left[-\frac{1}{2}, \frac{1}{2} \right]} \phi \left(x_i + \nu \theta' \right)} \right) \to 1 \text{ as } \nu \to 0.$$

Since the support of each f_i is contained in the interval $\left[-\frac{1}{2}, \frac{1}{2} \right]$, for all $x_i \in \mathbf{B}$,

$$\frac{\pi_i \left(z \right)}{\rho \left(\nu \right)} \leq \widetilde{\pi}_i \left(z \, | x_i, \nu \right) \leq \rho \left(\nu \right) \pi_i \left(z \right).$$

since

If t is a singleton type $\{j\}$, let $\Gamma_t(z_t, \theta, x_i; \nu) = f_t \left(\frac{x_j - \theta}{\nu} \right)$. If t is a continuum type, let $\Gamma_t(z_t, \theta, x_i; \nu)$ be the Dirac delta function [13, p. 276] that equals infinity if z_t is the c.d.f. of normalized signal differences between the signals of type-t players and the signal x_i if the distribution of type-t signal errors is f_t and the true parameter is θ. Since $z_t(c)$ is the proportion of players $j \in t$ for whom $\frac{x_j - x_i}{\nu} \leq c$ or, equivalently, for whom j's signal error $\frac{x_j - \theta}{\nu}$ is no greater than $c + \frac{x_i - \theta}{\nu}$, $\Gamma_t(z_t, \theta, x_i; \nu) = \infty$ iff $z_t'(c) = f_t(c + \frac{x_i - \theta}{\nu})$ for all c and $\Gamma_t(z_t, \theta, x_i; \nu) = 0$ otherwise. Letting $\theta' = -\frac{x_i - \theta}{\nu}$,

$\Gamma_t(z_t, \theta, x_i; \nu) = \Gamma_t(z_t, \theta', 0; 1)$. By Bayes's Rule,

$$\widetilde{\pi}_{\tau(i)}\left(z \mid x_i, \nu\right) = \frac{\Pr(z \& x_i)}{\Pr(x_i)} = \frac{\int\limits_{\theta=-\infty}^{\infty} \phi\left(\theta\right) f_i\left(\frac{x_i-\theta}{\nu}\right) \prod\limits_{t\in O(i)} \Gamma_t(z_t, \theta, x_i; \nu) d\theta}{\int\limits_{\theta=-\infty}^{\infty} \phi\left(\theta\right) f_i\left(\frac{x_i-\theta}{\nu}\right) d\theta} \tag{5}$$

$$= \frac{\int\limits_{\theta'=-\infty}^{\infty} \phi\left(x_i + \nu\theta'\right) f_i\left(-\theta'\right) \prod\limits_{t\in O(i)} \Gamma_t(z_t, \theta', 0; 1) d\theta'}{\int\limits_{\theta'=-\infty}^{\infty} \phi\left(x_i + \nu\theta'\right) f_i\left(-\theta'\right) d\theta'} \tag{6}$$

Thus, for any $S \subset Z_{-i}$,

$$\left| \int_{z\in S} \widetilde{\pi}_i\left(z \mid x_i, \nu\right) dz - \int_{z\in S} \pi_i\left(z\right) dz \right| \leq \left(\int_{z\in S} \pi_i\left(z\right) dz \right) \max\left(\rho\left(\nu\right) - 1, \frac{1}{\rho\left(\nu\right)} - 1 \right)$$

$$\leq \max\left(\rho\left(\nu\right) - 1, \frac{1}{\rho\left(\nu\right)} - 1 \right).$$

Q.E.D._{Lemma 4}

Lemma 5 uses the above results to show that as the signal noise shrinks, players' behavior in $G(\nu)$ converges to the unique outcome of $G^*(\nu)$. Write $\underline{s}^\nu$ ($\overline{s}^\nu$) for the left (right) continuous version of $s^{*\nu}$, the essentially unique equilibrium of the game $G^*(\nu)$. By Lemma 3, $\underline{s}^\nu$ and $\overline{s}^\nu$ each prescribes the same strategy for all players of a given type.

Lemma 5 *For any $\varepsilon > 0$, there exists $\overline{\nu} > 0$ such that for all $\nu \leq \overline{\nu}$ and any strategy profile s^ν of $G(\nu)$ surviving iterated deletion of strictly dominated strategies,*

$$\overline{s}_i^\nu\left(x_i + \varepsilon\right) \geq s_i^\nu\left(x_i\right) \geq \underline{s}_i^\nu\left(x_i - \varepsilon\right)$$

Proof. We first reiterate some definitions from the proof of Lemma 3. For any player i of type t, Z_{-t} is the set of all possible vectors $z = (z_t)_{t\in O(i)}$ of normalized differences between i's opponents signals and her own signal. For any pure strategy profile $s = (s_t)_{t\in T}$ (which prescribes the same strategy s_t for all players of type

t), $a_{-i}(s, x_i, z; \nu)$ is the opposing action distribution that player i faces if she sees the signal x_i, others play according to s, and z is the vector of normalized signal differences.

We begin with a claim.

Claim 3 *Let $s = (s_t)_{t \in T}$ be a weakly increasing strategy profile satisfying*

$$E_{\pi_t} \left[\Delta u_t \left(s_t \left(x_i \right), a_i, a_{-i}(s, x_i, z; \nu), x_i \right) \right] \geq 0$$

for all $i \in I$, $x_i \in \mathbb{R}$ and $a_i \leq s_t \left(x_i \right)$ (where t is the type of i).
Then, for any $\varepsilon > 0$ and for any compact interval $B \subset \mathbb{R}$, there exists $\delta > 0$, such that

$$E_{\pi'_t} \left[\Delta u_t \left(s_t \left(x_i \right), a_i, a_{-i}(s, x_i, z; \nu), x_i + \varepsilon \right) \right] > 0$$

for all $i \in I$, $\pi'_t \in \Psi_\delta \left(\pi_t \right)$, $x_i \in B$ and $a_i < s_t \left(x_i \right)$.

Proof. For all $\delta > 0$ and $\pi'_i \in \Psi_\delta \left(\pi_i \right)$,

$$E_{\pi'_t} \left[\Delta u_t \left(s_t \left(x_i \right), a_i, a_{-i}(s, x_i, z; \nu), x_i + \varepsilon \right) \right]$$

$$\geq \left\{ \begin{array}{l} (1 - \delta) E_{\pi_t} \left[\Delta u_t \left(s_t \left(x_i \right), a_i, a_{-i}(s, x_i, z; \nu), x_i + \varepsilon \right) \right] \\ + \delta \Delta u_t \left(s_t \left(x_i \right), a_i, \mathbf{0}_{-i}, x_i + \varepsilon \right) \end{array} \right\}$$

$$\geq \left\{ \begin{array}{l} (1 - \delta) E_{\pi_t} \left[\begin{array}{l} \Delta u_t \left(s_t \left(x_i \right), a_i, a_{-i}(s, x_i, z; \nu), x_i + \varepsilon \right) \\ - \Delta u_t \left(s_t \left(x_i \right), a_i, a_{-i}(s, x_i, z; \nu), x_i \right) \end{array} \right] dz \\ + (1 - \delta) E_{\pi_t} \left[\Delta u_t \left(s_t \left(x_i \right), a_i, a_{-i}(s, x_i, z; \nu), x_i \right) \right] \\ + \delta \Delta u_t \left(s_t \left(x_i \right), a_i, \mathbf{0}_{-i}, x_i + \varepsilon \right) \end{array} \right\}$$

By **A3**, we can find a constant $K_2 > 0$ such that for all x_i in the compact interval B, the first term is at least $(1 - \delta) K_2 \varepsilon \left(s_t \left(x_i \right) - a_i \right)$. The second term is at least 0, by the premise of the lemma. By **A5** there is a constant K_3 such that the third term is at least $-\delta K_3 \left(s_t \left(x_i \right) - a_i \right)$. Thus the sum is at least $\left((1 - \delta) K_2 \varepsilon - \delta K_3 \right) \left(s_t \left(x_i \right) - a_i \right)$.

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This expression must be positive (for all $a_i < s_t(x_i)$) if we choose δ such that $\frac{\delta}{1-\delta} <$ $\frac{K_2 \varepsilon}{K_3}$. Q.E.D.$_{\text{Claim 3}}$

By construction of $\underline{s}$, we know that for all $b \geq \varepsilon$, $i = 1, ..., I$ and $x_i \in \mathbb{R}$

$$E_{\pi_t}\left[\Delta u_t\left(\underline{s}_t(x_i - b), a_i, a_{-i}(s, x_i - b, z; \nu), x_i - b\right)\right] \geq 0$$

for all $a_i \leq \underline{s}_t(x_i - b)$. This implies (by Lemma 4 and Claim 3) that there exists $\tilde{\nu}$ such that for all $\nu \leq \tilde{\nu}$, $b \geq \varepsilon$ and $x_i \in \mathbb{R}$,

$$E_{\tilde{\pi}_t(\cdot|x_i, \nu)}\left[\Delta u_t\left(\underline{s}_t(x_i - b), a_i, a_{-i}(s, x_i - b, z; \nu), x_i - \frac{b}{2}\right)\right] \geq 0$$

for all $a_i \leq \underline{s}_t(x_i - b)$.[34] This implies (changing notation only) that

$$\int\limits_{z \in Z_{-i}} \int\limits_{\theta = -\infty}^{\infty} \tilde{\phi}_t(z, \theta | x_i, \nu) \Delta u_t\left(\underline{s}_t(x_i - b), a_i, a_{-i}(s, x_i - b, z; \nu), x_i - \frac{b}{2}\right) dz d\theta \geq 0$$

for all $a_i \leq \underline{s}_t(x_i - b)$, where

$$\tilde{\phi}_t(z, \theta | x_i, \nu) = \frac{\phi(\theta) f_i\left(\frac{x_i - \theta}{\nu}\right) \prod\limits_{\hat{t} \in O(i)} \Gamma_{\hat{t}}(z_{\hat{t}}, \theta, x_i; \nu)}{\int\limits_{\theta = -\infty}^{\infty} \phi(\theta) f_i\left(\frac{x_i - \theta}{\nu}\right) d\theta}$$

is the density of (z, θ) given x_i. Now if $\nu < \frac{b}{2}$ we have by assumption **A3** that

$$\int\limits_{z \in Z_{-i}} \int\limits_{\theta = -\infty}^{\infty} \tilde{\phi}_t(z, \theta | x_i, \nu) \Delta u_t\left(\underline{s}_t(x_i - b), a_i, a_{-i}(s, x_i - b, z; \nu), \theta\right) dz d\theta \geq 0 \qquad (7)$$

for all $a_i < \underline{s}_t(x_i - b)$. Set $\nu \leq \overline{\nu} = \min\left\{\frac{\varepsilon}{2}, \tilde{\nu}\right\}$. Consider the strategy profile s' where $s'_t(x_i) = \underline{s}_t(x_i - b)$. By equation 7, we know that, in $G(\nu)$, each player's best response to s' is always at least s'.

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[34] Assuming A2, Claim 3 implies this for $x_i - b \in [\underline{\theta}, \overline{\theta}]$. For $x_i - b$ not in this interval, the result holds since $\underline{s}_t(x_i - b)$ must be either 0 or 1, depending on which is dominant at the payoff parameter $x_i - b$. Assuming A2′ instead of A2, Claim 3 implies this for $x_i - b \in [\underline{\theta}' - \tilde{\nu}, \overline{\theta}' + \tilde{\nu}]$, where $\underline{\theta}'$ and $\overline{\theta}'$ are defined near the end of the proof of Lemma 3. Below $\underline{\theta}'$, $\underline{a}$ must be played; above $\overline{\theta}'$, $\overline{a}$ must be played. Thus, changing the distribution of z leaves the integral unchanged.

Since this is true for any $b \geq \varepsilon$, this ensures that iterated deletion of strictly dominated strategies (using translations of $\underline{s}$ as in the proof of lemma 3), cannot lead below $\underline{s}_t^\nu (x_i - \varepsilon)$. A symmetric argument (using a symmetric version of Claim 3, whose proof is analogous) gives the upper bound. Q.E.D.$_{\text{Lemma 5}}$

Finally, we show that as the signal errors shrink, the essentially unique surviving strategy profile in the simplified game $G^*(\nu)$ converges to a limit.

Lemma 6 *There is a weakly increasing strategy profile s, which prescribes the same strategy for all players of a given type, such that for any $\varepsilon > 0$ there is a $\overline{\nu} > 0$ such that for all $\nu < \overline{\nu}$ and any strategy profile s^ν surviving iterative strict dominance in $G^*(\nu)$, the maximum horizontal distance between s^ν and s is ε: i.e., $s_i^\nu(x_i + \varepsilon) > s_i(x_i) > s_i^\nu(x_i - \varepsilon)$ for all i and x_i.*

Proof. To prove this, we will show that the surviving profiles are a Cauchy sequence. Fix $\varepsilon > 0$ and consider any ν', ν'', such that $\overline{\nu} > \nu' > \nu''$, where $\overline{\nu}$ will be specified later. Let $s^{\nu'}$ and $s^{\nu''}$ survive iterative strict dominance in $G^*(\nu')$ and $G^*(\nu'')$, respectively. (Note that these are equilibria of the corresponding games.) We will show that the maximum horizontal distance between $s^{\nu'}$ and $s^{\nu''}$ is ε: i.e., $s_t^{\nu''}(x + \varepsilon) > s_t^{\nu'}(x) > s_t^{\nu''}(x - \varepsilon)$ for all types t and signals x.

We will transform $s^{\nu'}$ into a strategy profile $\widehat{s}$ and then do iterative dominance in $G^*(\nu'')$ using translations of $\widehat{s}$, and show that the limit (which bounds $s^{\nu''}$) is within ε of $s^{\nu'}$.

We defined $Q_1(\nu, \varepsilon')$ above to be such that in any surviving profile in $G^*(\nu)$, each player i who sees a signal that is *not* in $Q_1(\nu, \varepsilon')$ can bound the action of each opponent j within an interval of length ε'. By Claim 5 below, there is a $\overline{\nu} > 0$ such that if $\nu' < \overline{\nu}$, $Q_1^{\nu'} = Q_1 \left(\nu', \frac{\varepsilon}{2} \frac{K_0}{K_2 I} \right)$ is contained in a finite union of closed intervals of the form $[2n\nu', 2(n+1)\nu']$ (each of length $2\nu'$) for integer n; the measure of this union is less than $\frac{\varepsilon}{2}$.

We transform $s^{\nu'}$ into $\widehat{s}$ as follows. For signals x outside of $Q_1^{\nu'}$, let $\widehat{s}_t(x) = s_t^{\nu'}(x)$. For signals in any maximal interval[35] $[z - \delta, z + \delta]$ in $Q_1^{\nu'}$, we compress the segment of $s^{\nu'}$ horizontally by the ratio of ν'' and ν' and patch the remaining gaps with a constant action profile. More precisely, for each signal $x \in [z - \delta, z + \delta]$, let:

$$\widehat{s}_t(x) = \begin{cases} s_t^{\nu'}(z - \delta) \text{ if } x \leq z - \delta \frac{\nu''}{\nu'} \\ s_t^{\nu'}(z + (x - z) \frac{\nu'}{\nu''}) \text{ if } x \in \left[z - \delta \frac{\nu''}{\nu'}, z + \delta \frac{\nu''}{\nu'} \right] \\ s_t^{\nu'}(z + \delta) \text{ if } x \geq z + \delta \frac{\nu''}{\nu'} \end{cases}$$

We now perform iterative dominance in $G^*(\nu'')$ using translations of $\widehat{s}$ from the left, yielding an bound from the left on the profiles surviving iterative dominance in the game. Suppose the iterations stop at a horizontal distance λ to the left of $\widehat{s}$. Let this translation be s^λ. We first assume that both that the action space is $[0, 1]$ and the strategies surviving iterative dominance are continuous functions of players' signals, and the best-response correspondence is always single-valued. Later we consider the general case. Under these assumptions, there is a type t and a signal x such that $s_t^\lambda(x)$ equals $BR_t^{\nu''}(s^\lambda, x)$, the best response to s^λ in the game $G^*(\nu'')$ for a type-t player with signal x.

First suppose $x + \lambda \notin Q_1^{\nu'}$. This means that x corresponds to the part of s^λ that was not altered in the construction of $\widehat{s}$. In this case we show directly that the bound s^λ is within a horizontal distance of ε from $s^{\nu'}$. Suppose i got the signal $x + \lambda$ in the game $G^*(\nu')$ and expected her opponents to play according to $s^{\nu'}$. Then she could place each opponent j's action within an interval I_j of length $\frac{\varepsilon}{2} \frac{K_0}{K_2 I}$. By construction, if i gets signal x in $G^*(\nu'')$ and expects others to play profile s^λ, she can place each opponent's action within the same interval I_j. (By Claim 5 below, each interval in $Q_1^{\nu'}$ is of measure at least $2\nu'$ and $\nu'' < \nu'$. Thus, by construction of s^λ, the signals at which an opponent j plays actions outside of I_j can also be ruled out if i sees signal x.) This means that in the two cases (i.e., i getting signal $x + \lambda$ in $G^*(\nu')$ and

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expecting others to play $s^{\nu'}$ vs. i getting signal x in $G^*(\nu'')$ and expecting others to play s^λ), i expects approximately the same opponents' actions; more precisely, each action differs by no more than $\frac{\varepsilon}{2}\frac{K_0}{K_2 I}$, the length of I_j. By assumptions A3 and A5, the change in u_t following a unit change in any opposing type's action is no more than K_2 while the change in u_t following a unit change in θ (which in G^* equals the player's signal) is at least K_0. Thus, for $s_t^\lambda(x)$ to be a best response a player who expects s^λ, the difference λ in signals between the two cases must be less than ε.

Now suppose $x + \lambda \in Q_1^{\nu'}$. Denote the maximal interval in $Q_1^{\nu'}$ in which this signal lies by $[z - \delta, z + \delta]$. We can assume WLOG that $x + \lambda \in \left[z - \delta\frac{\nu''}{\nu'}, z + \delta\frac{\nu''}{\nu'}\right]$; otherwise, x must on a horizontal part of s^λ, but the iterations cannot have been blocked at such a point. We now consider the signal $y = z + (x + \lambda - z)\frac{\nu'}{\nu''}$. This is the reverse-translation of $x + \lambda$ and has the property that $s_t^\lambda(x) = s_t^{\nu'}(y)$. We will argue that the action distribution seen in each case is almost the same. A player seeing y and playing against $s^{\nu'}$ in $G^*(\nu')$ knows that her opponents' signals will be in the interval $Y = [y - \nu', y + \nu']$. We split Y into two parts. For opponents' signals in the part of Y that lies in $[z - \delta, z + \delta]$, the actions prescribed by $s^{\nu'}$ are, by construction, identical to the actions prescribed by s^λ at the corresponding signals in $G^*(\nu'')$. For opponents' signals lying outside, these signals lie in an interval of length at most ν' that is contained in the complement of $Q_1^{\nu'}$. We know that for each type t, the actions prescribed by $s_t^{\nu'}$ for such signals all lie in an interval I_t of size at most $\frac{\varepsilon}{2}\frac{K_0}{K_2 I}$. By construction of s^λ, for each player of type t, the actions prescribed by s^λ for the corresponding signals in $G^*(\nu'')$ are also in I_t. Thus, the above argument implies that $|x - y| < \varepsilon$: $s^{\nu''}$ cannot anywhere lie more than ε to the left of $s^{\nu'}$. This completes the proof for the continuous case.

We now explain how the proof changes in the general case. The construction and iterations are as before. However, we let $s^{\nu'}$ be the *right-continuous version* of the essentially unique surviving profile in $G^*(\nu')$. This ensures that its transformation $\widehat{s}$ is also right-continuous, so the limit s^λ is an upper bound on strategies chosen in

$G^*(\nu'')$. We also can no longer assume that $s_t^\lambda(x) = BR_t^{\nu''}(s^\lambda, x)$ for some t and x. We can say only that (a) s^λ lies above its highest best response: a type-t player who sees signal x will never play an action that exceeds the right continuous version of s_t^λ; and (b) s^λ cannot be translated to the right without falling below its highest best response somewhere: more precisely, as in Claim 1, there must be a type t and a signal x such that $s_t^\lambda(x - \varepsilon') < \max BR_t^{\nu''}(s^\lambda, x + \varepsilon')$ for all $\varepsilon' > 0$.[36]

For $\varepsilon' \approx 0$, let $x - \varepsilon'$ in the game $G^*(\nu'')$ with profile s^λ correspond to $y - \varepsilon'$ in $G^*(\nu')$ with profile $s^{\nu'}$: players with these signals find themselves at corresponding points on the two action profiles and thus are prescribed to take the same action. We claim that if $\lambda > \varepsilon$, then for ε' very small relative to λ,

$$\max BR_t^{\nu''}(s^\lambda, x + \varepsilon') \le \max BR_t^{\nu'}(s^{\nu'}, y - \varepsilon') \le s_t^{\nu'}(y - \varepsilon') = s_t^\lambda(x - \varepsilon')$$

Only the first inequality is nontrivial. Between the two cases (i.e., a player getting signal $x + \varepsilon'$ in $G^*(\nu'')$ and expecting others to play s^λ vs. a player getting signal $y - \varepsilon'$ in $G^*(\nu')$ and expecting others to play $s^{\nu'}$), the possible actions of each opponent differ by no more than $\frac{\varepsilon}{2}\frac{K_0}{K_2 I} + o(\varepsilon')$. Let

$$a_t' = \max BR_t^{\nu'}(s^{\nu'}, y - \varepsilon')$$
$$a_t^\lambda = \max BR_t^{\nu''}(s^\lambda, x + \varepsilon')$$

We must show that $a_t^\lambda \le a_t'$. By definition, a_t' is strictly better than any higher action if a type-t player i gets signal $y - \varepsilon'$ in $G^*(\nu')$ and expects others to play $s^{\nu'}$: for all $a_t > a_t'$, $E_{\pi_t}(\Delta u_t(a_t, a_t', a_{-i}(s^{\nu'}, y - \varepsilon', z; \nu'), y - \varepsilon') < 0$, where the expectation is based on the signal distribution in $G^*(\nu')$. We claim that a_t' is also strictly better than any higher action if a type-t player i gets signal $x + \varepsilon'$ in $G^*(\nu'')$ and expects others to play s^λ: for all $a_t > a_t'$, $E_{\pi_t}(\Delta u_t(a_t, a_t', a_{-i}(s^\lambda, x + \varepsilon', z; \nu''), x + \varepsilon')) < 0$. Since the possible actions of each of i's opponents differ by no more than $\frac{\varepsilon}{2}\frac{K_0}{K_2 I} + o(\varepsilon')$ in the two situations while the payoff parameter differs by at least $\lambda - 2\varepsilon' - \varepsilon/2$,

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[36] In the general case, the best response is a correspondence rather than a function.

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assumptions **A3** and **A5** imply:

$$E_{\pi_t}(\Delta u_t(a_t, a_t', a_{-i}(s^\lambda, x + \varepsilon', z; \nu'')), x + \varepsilon'))$$

$$- E_{\pi_t}(\Delta u_t(a_t, a_t', a_{-i}(s^{\nu'}, y - \varepsilon', z; \nu')), y - \varepsilon'))$$

$$\leq -K_0(a_t - a_t')(\lambda - 2\varepsilon' - \varepsilon/2) + K_2(a_t - a_t')I\left(\frac{\varepsilon}{2}\frac{K_0}{K_2 I} + o(\varepsilon')\right)$$

$$\underset{\varepsilon' \to 0}{\to} K_0(a_t - a_t')(\varepsilon - \lambda)$$

which is negative if $\lambda > \varepsilon$. This proves the lemma. Q.E.D.$_{\text{(Lemma 6)}}$.

Theorem 5 follows immediately from this result and Lemmas 3, 5 and 6.Q.E.D.$_{\text{(Theorem 5)}}$

The following three claims were used in the proofs of Lemma 6 and will be used in the proof of Theorem 6.

For any $\varepsilon > 0$ and $\nu > 0$, let $Q_0(\nu, \varepsilon)$ be the set of parameters θ for there is some strategy profile $s^{\nu*} = (s_t^{\nu*})_{t \in T}$ that survives iterative strict dominance in $G^*(\nu)$ and some strategy profile $s^\nu = (s_i^\nu)_{i \in I}$ that survives iterative strict dominance in $G(\nu)$ for which $|s_t^{\nu*}(\theta) - s_i^\nu(\theta)| > \varepsilon$ for some i of type t. (Unlike $s^{\nu*}$, s^ν has not been shown to assign the same strategy to all players of a given type.)

Claim 4 *For any $\varepsilon > 0$ there is a $\overline{\nu} > 0$ such that for all $\nu < \overline{\nu}$, $Q_0(\nu, \varepsilon)$ is contained in a union of at most $4|T|/\varepsilon$ closed intervals of $\mathbb{R}$; the total measure of the union is less than ε.*

Proof. Let $s^{\nu*}$ and s^ν be any strategies surviving iterative strict dominance in $G^*(\nu)$ and $G(\nu)$, respectively. By Lemma 5, for any $\varepsilon' > 0$, there exists $\overline{\nu} > 0$ such that for all $\nu \leq \overline{\nu}$,

$$\overline{s}_t^\nu(\theta + \varepsilon') \geq s_i^\nu(\theta) \geq \underline{s}_t^\nu(\theta - \varepsilon')$$

where t is i's type and $\overline{s}^\nu$ and $\underline{s}^\nu$ are, respectively, the right-continuous and left-continuous versions of the essentially unique surviving strategy profile of $G^*(\nu)$. In

addition, by Lemma 3, any $s^{\nu*}$ that survives in $G^*(\nu)$ must be weakly increasing, so

$$\overline{s}_t^\nu \left(\theta + \varepsilon'\right) \geq s_t^{\nu*}\left(\theta\right) \geq \underline{s}_t^\nu\left(\theta - \varepsilon'\right)$$

Thus, the absolute difference between $s_i^\nu\left(\theta\right)$ and $s_t^{\nu*}\left(\theta\right)$ is bounded by $\overline{s}_t^\nu\left(\theta + \varepsilon'\right) - \underline{s}_t^\nu\left(\theta - \varepsilon'\right)$. We now divide the real line into intervals $I_n = [n\varepsilon', (n+1)\varepsilon']$ for integral n. Let M_0 be the set of such intervals for which there is a $\theta \in I_n$ and a player i of some type t for whom, for some $s^{\nu*}$ and s^ν surviving iterative strict dominance in $G^*(\nu)$ and $G(\nu)$, $|s_i^\nu(\theta) - s_t^{\nu*}(\theta)| \geq \varepsilon$. Clearly, $Q_0(\nu, \varepsilon) \subseteq M_0$. By the preceding argument, $\overline{s}_t^\nu\left(\theta + \varepsilon'\right) - \underline{s}_t^\nu\left(\theta - \varepsilon'\right) \geq \varepsilon$, so $\overline{s}_t^\nu\left((n+2)\varepsilon'\right) - \underline{s}_t^\nu\left((n-1)\varepsilon'\right) \geq \varepsilon$. By definition of $\overline{s}^\nu$ and $\underline{s}^\nu$, $\underline{s}_t^\nu(z) \geq \overline{s}_t^\nu(z')$ for any $z > z'$, so

$$\overline{s}_t^\nu \left((n+2)\varepsilon'\right) - \overline{s}_t^\nu\left((n-2)\varepsilon'\right) \geq \varepsilon \tag{8}$$

Since $\overline{s}_t^\nu$ is weakly increasing and in $[0, 1]$ for each t, the set of integers n and types t for which (8) holds is at most $4|T|/\varepsilon$. Thus, M_0 is a finite set and the total measure of M_0 is at most $4|T|\varepsilon'/\varepsilon$. For this measure to be less than ε, it suffices to take $\varepsilon' < \varepsilon^2/4|T|$. Q.E.D.(Claim 4)

Claim 5 shows that when signal errors are small, for most signals players can closely approximate what other players will do. For any $\nu > 0$ and $\varepsilon > 0$, let $Q_1(\nu, \varepsilon)$ be the set of parameters θ such that after removing strategies that do not survive iterative strict dominance in $G^*(\nu)$, there are types $t \neq \widehat{t}$ such that conditional on a player of type t getting signal θ, the set of actions a type-$\widehat{t}$ player might take are not contained in an interval of length ε. Each player j who sees a signal that is *not* in $Q_1(\nu, \varepsilon)$ can bound the action of each opponent i within an interval of length ε. Claim 5 shows that $Q_1(\nu, \varepsilon)$ is of measure less than ε for small enough ν. This claim is also used in the proof of Lemma 6.

Claim 5 *For any $\varepsilon > 0$ there is a $\overline{\nu} > 0$ such that for all $\nu < \overline{\nu}$, $Q_1(\nu, \varepsilon)$ is contained in a union of at most $5|T|/\varepsilon$ intervals of the form $[2n\nu, (2n+2)\nu]$ for integral n; the total measure of the union is less than ε.*

Proof. Since each signal error is at most $\nu/2$ and since $\overline{s}^\nu$ and $\underline{s}^\nu$ are upper and lower bounds on the strategies that survive iterative strict dominance in $G^*(\nu)$, a necessary condition for θ to be in $Q_1(\nu, \varepsilon)$ is that for some type t, $\overline{s}_t^\nu(\theta+\nu)-\underline{s}_t^\nu(\theta-\nu) > \varepsilon$. Divide the real line into intervals $I_n = [2n\nu, (2n+2)\nu]$ for integral n. Let M_1 be the set of such intervals for which there is a $\theta \in I_n$ and a type t for which $\overline{s}_t^\nu(\theta + \nu) - \underline{s}_t^\nu(\theta - \nu) > \varepsilon$. Clearly, $Q_1(\nu, \varepsilon) \subseteq M_1$. By definition of $\overline{s}^\nu$ and $\underline{s}^\nu$, $\underline{s}_t^\nu(z) \geq \overline{s}_t^\nu(z')$ for any $z > z'$, so

$$\overline{s}_t^\nu\left((2n+4)\nu\right) - \overline{s}_t^\nu\left((2n-1)\nu\right) > \varepsilon \tag{9}$$

Since $\overline{s}_t^\nu$ is weakly increasing and in $[0,1]$ for each t, the set of integers n and types t for which (8) holds is at most $5\,|T|\,/\varepsilon$. Thus, M_1 is a finite union of intervals whose total measure is at most $5\,|T|\,\nu/\varepsilon$. For this measure to be less than ε, it suffices to take $\nu < \varepsilon^2/5\,|T|$. Q.E.D.$_{\text{(Claim 5)}}$

The next claim uses the prior one to show that for almost all payoff parameters θ, strategies that survive iterative strict dominance in $G^*(\nu)$ must prescribe at θ that players play close to some Nash equilibrium of some complete information game with payoff parameter close to θ. More precisely, let $Q_2(\varepsilon, \nu)$ be the set of parameters θ for which there is *no* action profile $a = (a_t)_{t\in T}$ with the following two properties: (1) for some $\theta' \in [\theta - \varepsilon, \theta + \varepsilon]$, the complete information game with payoffs $(u_t(, \theta'))_{t\in T}$ has a Nash equilibrium in which all players of type t play a_t; (2) for every strategy $s^{*\nu}$ surviving iterative strict dominance in $G^*(\nu)$, $|s_t^{*\nu}(\theta) - a_t| \leq \varepsilon$ for all types t.

Claim 6 *For any $\varepsilon > 0$ there is a $\overline{\nu} > 0$ such that for all $\nu < \overline{\nu}$, $Q_2(\nu, \varepsilon)$ is contained in a finite union of closed intervals of $\mathbb{R}$ whose measure is less than ε. The number of intervals in the union can depend on ε but not on ν.*

Proof.[37] Fix $\varepsilon > 0$. Recall that $Q_1(\nu, \eta)$ is the set of signals x such that, after removing strategies that do not survive iterative strict dominance in $G^*(\nu)$, there are

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[37] We thank Itzhak Gilboa for suggesting this proof.

types $t \neq \widehat{t}$ such that conditional on a type-t player getting signal x, the set of actions type-$\widehat{t}$ players choose are not contained in an interval of length η. By Claim 5, for any $\eta > 0$ there is a $\overline{\nu}_\eta > 0$ such that for all $\nu < \overline{\nu}_\eta$, $Q_1(\nu, \eta)$ is contained in a finite union of closed intervals whose measure is less than η. But for any η,

$$
\begin{aligned}
Q_2(\nu, \varepsilon) \; = \; & [Q_2(\nu, \varepsilon) \cap Q_1(\nu, \eta)] \\
& \cup \left[Q_2(\nu, \varepsilon) \cap Q_1^C(\nu, \eta) \right] \qquad (10)
\end{aligned}
$$

where $Q_1^C(\nu, \eta)$ is the complement of $Q_1(\nu, \eta)$. Every signal x in $Q_2(\nu, \varepsilon) \cap Q_1^C(\nu, \eta)$ has the property that on getting it, each player knows to within η what players of all types will do in $G^*(\nu)$ (since $x \in Q_1^C(\nu, \eta)$) yet there is no Nash equilibrium $(a_t)_{t \in T}$ of the complete information game with payoff parameter $\theta \in [x - \varepsilon, x + \varepsilon]$ such that each a_t is ε-close to the action prescribed for type t in the essentially unique equilibrium of $G^*(\nu)$ at the signal x.

We will show that for fixed $\varepsilon > 0$, there is an $\overline{\eta}_\varepsilon$ such that for $\eta \leq \overline{\eta}_\varepsilon$, the set $Q_2(\nu, \varepsilon) \cap Q_1^C(\nu, \eta)$ is *empty* for any ν. Thus, by setting $\eta = \min\{\overline{\eta}_\varepsilon, \varepsilon\}$ and $\nu = \overline{\nu}_\eta$, we ensure that (a) for all $\nu < \overline{\nu}$, $Q_1(\nu, \eta)$ (and thus $Q_2(\nu, \varepsilon) \cap Q_1(\nu, \eta)$) is contained in a finite union of closed intervals whose measure is less than ε; and (b) since $\eta \leq \overline{\eta}_\varepsilon$, the set $Q_2(\nu, \varepsilon) \cap Q_1^C(\nu, \eta)$ is empty. By (10), this will prove Claim 6.

Suppose instead that the set $Q_2(\nu, \varepsilon) \cap Q_1^C(\nu, \eta)$ is *nonempty* for arbitrarily small η, where ν can depend on η. This means that there are arbitrarily small positive η's such that for some θ, even though in $G^*(\nu)$ each player i with signal θ has the utility function $u_{\tau(i)}(, \theta)$ and best-responds to a belief that each opponent j's action is in some interval of length η that contains j's true action (while follows from $\theta \in Q_1^C(\nu, \eta)$), the action profile the players play in $G^*(\nu)$ is not ε-close to any Nash equilibrium of the game with payoff parameter $\theta' \in [\theta - \varepsilon, \theta + \varepsilon]$ (which is the meaning of $\theta \in Q_2(\nu, \varepsilon)$). Let $\eta_0 > \eta_1 > \dots$ be a sequence of such η's that converges to zero. For each η_k take some $\theta_k \in Q_2(\nu, \varepsilon) \cap Q_1^C(\nu, \eta_k)$ and consider the action profile $\left(a_t^k\right)_{t \in T} = (s_t^{*\nu}(\theta_k))_{t \in T}$. By construction, each of these action profiles differs by at least ε from any Nash equilibrium. Since each type's action space is compact and each θ_k must lie in the

compact region $[\underline{\theta}, \overline{\theta}]$,[38] there is a subsequence of the η_k's such that the corresponding sequence (a^k, θ_k) converges to some limit $(a^\infty, \theta_\infty)$. By continuity of the payoff functions (axiom **A4**), each action a_t^∞ is a best response if a type-t player knows her opponents are playing according to a^∞ and the payoff parameter is θ_∞. Thus, a^∞ is a Nash equilibrium of the underlying complete information game when the payoff parameter is θ_∞. But then for high enough k, a^k is ε-close to the Nash equilibrium a^∞ of the complete information game with payoff parameter θ_∞, which itself must be ε-close to θ_k. Thus, (a^k, θ_k) does not satisfy the condition assumed of it. Q.E.D.$_{\text{(Claim 6)}}$

Proof of THEOREM 6. Theorem 6 follows immediately from claims 4, 5 and 6, since $Q(\varepsilon, \nu) \subset Q_0(\varepsilon/2, \nu) \cup Q_2(\varepsilon/2, \nu)$. Q.E.D.$_{\text{(Theorem 6)}}$

Proof of THEOREM 3. Let the actions be 0, 1/3, 2/3, and 1, so that a strategy profile takes the form

$$
s(x) = \begin{cases}
0, \text{ if } x < c_1 \\
\frac{1}{3}, \text{ if } c_1 \le x < c_2 \\
\frac{2}{3}, \text{ if } c_2 \le x < c_3 \\
1, \text{ if } c_3 \le x
\end{cases}
$$

A player observing signal c_1 will assign probability $\frac{1}{2}$ to her opponent choosing action 0, some probabilities λ and μ to her opponent choosing actions $\frac{1}{3}$ and $\frac{2}{3}$, respectively, and probability $\frac{1}{2} - \lambda - \mu$ to her opponent choosing action 1. A player observing signal c_2 will assign probability $\frac{1}{2} - \lambda$ to her opponent choosing action 0, probability λ to her opponent choosing action $\frac{1}{3}$, some probability η to her opponent choosing $\frac{2}{3}$, and probability $\frac{1}{2} - \eta$ to her opponent choosing action 1. A player seeing signal

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[38]If the action spaces are finite, axiom A2 is replaced by the weaker A2$'$ so θ_k may not lie in $[\underline{\theta}, \overline{\theta}]$. But by Claim 2, since any strategy profile surviving iterative strict dominance in $G^*(\nu)$ is an equilibrium of $G^*(\nu)$, it must prescribe that agents play Nash equilibria of the underlying complete information game for any $\theta \notin [\underline{\theta} - K\nu, \overline{\theta} + K\nu]$. Since each θ_k is in $Q_2(\nu, \varepsilon)$, each θ_k must lie in a compact interval (in $[\underline{\theta} - K\nu, \overline{\theta} + K\nu]$), which is all the proof requires.

c_3 will assign probability $\frac{1}{2}$ to her opponent playing 1, η to her opponent choosing $\frac{2}{3}$, some probability ν to her opponent choosing $\frac{1}{3}$, and probability $1 - \eta - \nu$ to her opponent choosing 0. Hence, each profile gives rise to four unknowns, λ, μ, η, and ν, as claimed.

We now present a specific counterexample with four actions, in which the equilibrium selected in the limit depends on the noise. Let $I = 2$, $A_1 = A_2 = \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}$ and let $G(\nu)$ and $\widehat{G}(\nu)$ be two games satisfying the assumptions of Section 2; these two games are identical except that in $G(\nu)$, η_1 and η_2 are distributed according to the density

$$f(\eta) = 1$$

on the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$; in $\widehat{G}(\nu)$, η_1 and η_2 are distributed according to the density

$$\widehat{f}(\eta) = 2 - 4|\eta|$$

on the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$. Note that under a uniform prior on θ, the resulting symmetric distributions of $z = \eta_1 - \eta_2$ have support $[-1, 1]$ and densities

$$\pi(z) = 1 - |z|$$

$$\widehat{\pi}(z) = \begin{cases} 2(1+z)^2, & \text{if } -1 \le z \le -\frac{1}{2} \\ 1 - 2z^2, & \text{if } -\frac{1}{2} \le z \le \frac{1}{2} \\ 2(1-z)^2, & \text{if } \frac{1}{2} \le z \le 1 \end{cases}$$

Assume that $u(\cdot, \theta^*) = g^*(\cdot)$, where g^* is given by the following symmetric matrix:

g^*	0	$\frac{1}{3}$	$\frac{2}{3}$	1
0	$2000, 2000$	$1936, 1656$	$1144, 1056$	$391, 254$
$\frac{1}{3}$	$1656, 1936$	$2000, 2000$	$1600, 1800$	$1245, 1000$
$\frac{2}{3}$	$1056, 1144$	$1800, 1600$	$2000, 2000$	$1660, 2160$
1	$254, 391$	$1000, 1245$	$2160, 1660$	$2000, 2000$

One may verify that this is a game of strategic complementarities, with payoffs quasiconcave in own actions, since $g_i^* \left(a_i + \frac{1}{3}, a_j \right) - g_i^* \left(a_i, a_j \right)$ is strictly decreasing in a_i and strictly increasing in a_j.

Let strategy profile $\widehat{s}[k]$ be defined by:

$$\widehat{s}_i[k](x_i) = \begin{cases} 0, \text{ if } x_i < k - \frac{1}{4} \\ \frac{1}{3}, \text{ if } k - \frac{1}{4} \le x_i < k \\ \frac{2}{3}, \text{ if } k \le x_i < k + \frac{1}{4} \\ 1, \text{ if } k + \frac{1}{4} \le x_i \end{cases}$$

Lemma 7 *Let the interaction structure be given by $\widehat{\pi}$. There exists $\widehat{\varepsilon} > 0$ and $\widehat{\delta} > 0$ such that if $\theta \in \left[\theta^* - \widehat{\delta}, \theta^* + \widehat{\delta} \right]$ and payoffs are always given by $u(\cdot, \theta^*)$, the best response to strategy profile $\widehat{s}[k]$ is less than or equal to $\widehat{s}[k + \widehat{\varepsilon}]$.*

PROOF. If player 1 observes x_1, she believes that $x_2 - x_1$ is distributed according to $\widehat{\pi}$. If she believes that her opponent is following strategy $\widehat{s}_2[k]$, her conjectures over her opponents' actions are the following:

		$a_2 = 0$	$a_2 = \frac{1}{3}$	$a_2 = \frac{2}{3}$	$a_2 = 1$
Player 1's Signal	$k - \frac{1}{4}$	$\frac{1}{2}$	$\frac{23}{96}$	$\frac{17}{96}$	$\frac{1}{12}$
	k	$\frac{25}{96}$	$\frac{23}{96}$	$\frac{23}{96}$	$\frac{25}{96}$
	$k + \frac{1}{4}$	$\frac{1}{12}$	$\frac{17}{96}$	$\frac{23}{96}$	$\frac{1}{2}$

One may verify that if player 1 observes signal $k - \frac{1}{4}$ and has payoffs given by $u(\cdot, \theta^*)$, then she strictly prefers action 0 to action $\frac{1}{3}$ (since $\frac{1}{2}(-344) + \frac{23}{96}(64) + \frac{17}{96}(456) + \frac{1}{12}(854) = -\frac{19}{4} < 0$). Similarly, if player 1 observes signal k and has payoffs given by $u(\cdot, \theta^*)$, she strictly prefers action $\frac{1}{3}$ to action $\frac{2}{3}$ (since $\frac{25}{96}(-600) + \frac{23}{96}(-200) + \frac{23}{96}(400) + \frac{25}{96}(415) = -\frac{25}{96} < 0$) and if player 1 observes signal $k + \frac{1}{4}$ and has payoffs given by $u(\cdot, \theta^*)$, she strictly prefers action $\frac{2}{3}$ to action 1. By continuity, these strict preferences will be maintained for signals in a small neighborhoods of those cutoff points and for payoffs in a small neighborhood of $u(\cdot, \theta^*)$. ∎

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Now consider the game $\widehat{G}^D(\nu, \theta^*)$, which is like $\widehat{G}^*(\nu)$, except that the payoff functions are replaced by:

$$u_i^D(a, \theta) = \begin{cases} u_i(a, \underline{\theta}), & \text{if } \theta \leq \underline{\theta} \\ u_i\left(a, \theta^* + \widehat{\delta}\right), & \text{if } \underline{\theta} \leq \theta \leq \theta^* + \widehat{\delta} \\ u_i(a, \overline{\theta}), & \text{if } \theta \geq \theta^* + \widehat{\delta} \end{cases}$$

Corollary 1 *In the game $\widehat{G}^D(\nu, \theta^*)$, any strategy s satisfying iterated deletion of strictly dominated strategies satisfies $s \leq \widehat{s}\left[\theta^* + \widehat{\delta} - \nu\right]$; thus $s_i(x_i) = 0$ for all $x_i \leq \theta^* + \widehat{\delta} - 2\nu$.*

PROOF. By induction, verify that if strategy profile s survives $k + 1$ rounds of deletion of strictly dominated strategies, then $s \leq \widehat{s}\left[\max\left(\underline{\theta} - \nu + k\widehat{\varepsilon}, \theta^* + \widehat{\delta} - \nu\right)\right]$. ∎

Now we have:

Lemma 8 *If s^ν is the essentially unique equilibrium of $\widehat{G}^*(\nu)$, then $s_i^\nu(x_i) = 0$ for all $x_i \leq \theta^* + \widehat{\delta} - \nu$.*

But now let strategy profile $s[k]$ be defined by:

$$s_i[k](x_i) = \begin{cases} 0, & \text{if } x_i < k - \frac{7}{25} \\ \frac{1}{3}, & \text{if } k - \frac{7}{25} \leq x_i < k \\ \frac{2}{3}, & \text{if } k \leq x_i < k + \frac{7}{25} \\ 1, & \text{if } \frac{7}{25} \leq k + x_i \end{cases}$$

Lemma 9 *Let the interaction structure be given by π. There exists $\varepsilon > 0$ and $\delta > 0$ such that if $\theta \in [\theta^* - \delta, \theta^* + \delta]$ and payoffs were always given by $u(\cdot, \theta^*)$, the best response to strategy profile $s[k]$ is more than or equal to $s[k - \varepsilon]$.*

PROOF. If player 1 observes x_1, she believes that $x_2 - x_1$ is distributed according to π. If she believes that her opponent is following strategy $\widehat{s}_2[k]$, her conjectures

over her opponents' actions are the following:

	$a_2 = 0$	$a_2 = \frac{1}{3}$	$a_2 = \frac{2}{3}$	$a_2 = 1$
$k - \frac{7}{25}$	$\frac{1}{2}$	$\frac{301}{1250}$	$\frac{203}{1250}$	$\frac{121}{1250}$
k	$\frac{162}{625}$	$\frac{301}{1250}$	$\frac{301}{1250}$	$\frac{162}{625}$
$k + \frac{7}{25}$	$\frac{121}{1250}$	$\frac{203}{1250}$	$\frac{301}{1250}$	$\frac{1}{2}$

Player 1's Signal

One may verify that if player 1 observes signal $k - \frac{7}{25}$ and her payoffs are given by $u\left(\cdot, \theta^*\right)$, then she strictly prefers action $\frac{1}{3}$ to action 0 (since $\frac{1}{2}\left(-344\right) + \frac{301}{1250}\left(64\right) + \frac{203}{1250}\left(456\right) + \frac{121}{1250}\left(854\right) = \frac{83}{625} > 0$). Similarly, if player 1 observes signal k and her payoffs are given by $u\left(\cdot, \theta^*\right)$, she strictly prefers action $\frac{2}{3}$ to action $\frac{1}{3}$ and if player 1 observes signal $k + \frac{7}{25}$ and her payoffs are given by $u\left(\cdot, \theta^*\right)$, she strictly prefers action 1 to action $\frac{2}{3}$. By continuity, these strict preferences will be maintained for signals in a small neighborhoods of those cutoff points and for payoffs in a small neighborhood of $u\left(\cdot, \theta^*\right)$. ∎

But now mimicking the above argument, we have:

Lemma 10 *If s^ν is the essentially unique equilibrium of $G^*\left(\nu\right)$, then $s_i^\nu\left(x_i\right) = 1$ for all $x_i \geq \theta^* - \delta + \nu$.*

Q.E.D.$_{\text{Theorem 3}}$

Proof of LEMMA 1. Consider any player i. In the continuous action space case, for any $\delta \in \left[0, c_i\right]$,

$$0 > v\left(a_i^* + \delta, a_{-i}^*\right) - v\left(a_i^*, a_{-i}^*\right) \geq \mu_i(a_i^*)\left(g_i\left(a_i^* + \delta, a_{-i}^*\right) - g_i\left(a_i^*, a_{-i}^*\right)\right),$$

so $g_i\left(a_i^* + \delta, a_{-i}^*\right) < g_i\left(a_i^*, a_{-i}^*\right)$. By own-action quasiconcavity, $g_i\left(a_i^*, a_{-i}^*\right)$ exceeds $g_i\left(a_i, a_{-i}^*\right)$ for any $a_i > a_i^*$. An analogous argument shows the same for any $a_i < a_i^*$. A similar proof applies in the case of finite actions. Q.E.D.$_{\text{Lemma 1}}$

Proof of THEOREM 4. For each $i \in I$, let $\underline{g}_i\left(a_i, a_{-i}\right)$ be a payoff function with the property that action 0 is strictly dominant. Fix some $\theta^* \in \mathbb{R}$ and consider the

global game $G^{**}(\nu, \theta^*)$ with uniform prior, some noise structure $(f_i)_{i \in I}$, and payoff functions

$$\underline{u}_i(a_i, a_{-i}, x_i) = \begin{cases} u_i(a_i, a_{-i}, \theta^*), & \text{if } x_i \geq \theta^* \\ \underline{g}_i(a_i, a_{-i}), & \text{if } x_i < \theta^* \end{cases}$$

Suppose that action profile $a^* = (a_i^*)_{i \in I}$ is an LP-maximizer of the complete informa-tion game $g = (u_i(\cdot, \theta^*))_{i \in I}$ and that this game is own-action quasiconcave.

For now, fix $\nu = 1$. We will be interested in left-continuous, weakly increasing strategy profiles in which no player i ever chooses an action above a_i^*. (The same ar-gument works for right-continuous strategies.) Any such strategy can be represented by a function $\zeta_i : [0, a_i^*] \to \mathbb{R}$ where $\zeta_i(a_i)$ is the highest signal at which player i plays an action less than or equal to a_i.

We wish to define the unique left-continuous best response to strategy profile ζ in the game $G^{**}(\nu, \theta^*)$. Note that the assumption that a^* is an LP-maximizer (and thus, by lemma 1, a strict Nash equilibrium) and the strategic complementarities assumption imply that the best response to ζ will itself involve each player i choosing an action less than or equal to a_i^*. We write $\beta(\zeta) = (\beta_i(\zeta))_{i \in I}$ for this best response.

To give an explicit expression for $\beta_i(\zeta)$, first write $s_i^{\zeta_i}$ for player i's strategy written in standard notation, i.e., $s_i^{\zeta_i}(x_i) = \min\{a_i : \zeta_i(a_i) \geq x_i\}$. (It is correct to take the min because of left continuity.) For any player i, let (x_i, x_{-i}) denote the vector of signal realizations. Let X_{-i} be the space of all signal vectors x_{-i}. Write $s_{-i}^{\zeta_{-i}}(x_{-i}) = \left(s_j^{\zeta_j}(x_j)\right)_{j \neq i}$ and $s^\zeta(x) = \left(s_j^{\zeta_j}(x_j)\right)_{j \in I}$.

If player i observes $x_i < \theta^*$, action 0 is dominant. If she observes $x_i \geq \theta^*$, her payoff to choosing action a_i, if she believes her opponents are following strategies ζ_{-i}, is

$$\int_{\theta \in \mathbb{R}} \int_{x_{-i} \in X_{-i}} g_i\left(\left(a_i, s_{-i}^{\zeta_{-i}}(x_{-i})\right), x_i\right) \left(\prod_{j \neq i} f_j(x_j - \theta)\right) dx_{-i} d\theta.$$

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Thus an action less than or equal to a_i is a best response if

$$\min\left\{\operatorname*{arg\,max}_{a_i'\in A_i}\int\limits_{\theta\in\mathbb{R}}\int\limits_{x_{-i}\in X_{-i}}g_i\left(\left(a_i',s_{-i}^{\zeta_{-i}}(x_{-i})\right),x_i\right)\left(\prod_{j\neq i}f_j(x_j-\theta)\right)dx_{-i}d\theta\right\}\leq a_i.$$

Recall that $\zeta_i(a_i)$ is the largest value of x_i at which an action less than or equal to a_i is played under i's strategy. Thus $\beta_i(\zeta)(a_i)$ is the maximum of θ^* and

$$\max\left\{x_i:\min\left\{\operatorname*{arg\,max}_{a_i'\in A_i}\int\limits_{\theta\in\mathbb{R}}\int\limits_{x_{-i}\in X_{-i}}g_i\left(\left(a_i',s_{-i}^{\zeta_{-i}}(x_{-i})\right),x_i\right)\left(\prod_{j\neq i}f_j(x_j-\theta)\right)dx_{-i}d\theta\right\}\leq a_i\right\}.$$

Now define:[39]

$$V(\zeta)=\int\limits_{\theta}\int\limits_{a_1=0}^{a_1^*}\cdots\int\limits_{a_I=0}^{a_I^*}(v(a)-v(a^*))\,dF_I(\zeta_I(a_I)-\theta)\cdots dF_1(\zeta_1(a_1)-\theta)d\theta$$

Intuitively, $V(\zeta)$ is the expected value of $v(a)-v(a^*)$, conditional on $\theta\geq\theta^*-\frac{1}{2}$. The expectation is taken with respect to an improper prior, so this expression will only be well defined if each player plays a_i^* for high enough signals; i.e., if $\zeta_i(a_i^*)$ is finite for all i. Otherwise $V(\zeta)$ will equal $-\infty$, since $v(a)<v(a^*)$ for all $a\neq a^*$.

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[39] In order to accommodate action sets that can include both intervals and points, the integrals are interpreted as follows. Let $A_i=\cup_{m=1}^M[b_m,c_m]$ where $M\geq 1$ can be infinity and the intervals are disjoint. (Isolated points are represented by setting $b_m=c_m$.) We define $\int_{a_i=0}^{a_i^*}f(a_i)dg(a_i)$ to equal

$$\sum_{m=1}^M\int_{a_i=b_m}^{c_m}f(a_i)dg(a_i)+\sum_{m=1}^{M-1}f(b_{m+1})\left[g(b_{m+1})-g(c_m)\right]$$

One can verify that the standard integration by parts formula holds using this definition:

$$\int_{a_i=0}^{a_i^*}f(a_i)dg(a_i)=f(a_i^*)g(a_i^*)-f(0)g(0)-\int_{a_i=0}^{a_i^*}g(a_i)df(a_i)$$

Now consider the sequence ζ^0, ζ^1, ..., where $\zeta_i^0(a_i) = \theta^*$ for all i and $a_i \in [0, a_i^*]$, and $\zeta^n = \beta\left(\zeta^{n-1}\right)$ for all $n > 0$. An induction argument shows that this is an increasing sequence: $\zeta_i^n(a_i) \geq \zeta_i^{n-1}(a_i)$ for all n, i, and a_i. Moreover, $V(\zeta^0)$ is finite and negative. We will show that $V(\zeta^n)$ is increasing in n. Thus, $V\left(\lim_{n\to\infty} \zeta^n\right) \neq -\infty$. This implies that in the limiting strategy profile $\lim_{n\to\infty} \zeta^n$, each player i plays a_i^* if her signal is high enough.

Let $\zeta = \zeta^n$ for any $n \geq 0$. Define $dF^i(a_{-i}|\theta)$ to be

$$dF_I(\zeta_I(a_I) - \theta) \cdots dF_{i+1}(\zeta_{i+1}(a_{i+1}) - \theta) dF_{i-1}(\beta(\zeta)_{i-1}(a_{i-1}) - \theta) \cdots dF_1(\beta(\zeta)_1(a_1) - \theta)$$

This is the probability of the action profile a_{-i} at the state θ if players $i - 1$ and under play according to $\beta(\zeta)$ while players $i + 1$ and above play according to ζ. We separate $V(\beta(\zeta)) - V(\zeta)$ into a telescopic sum and then integrate each summand by parts:

$$V(\beta(\zeta)) - V(\zeta) = \sum_{i=1}^{I} \int_\theta \int_{a_{-i}} \left\{ \int_{a_i=0}^{a_i^*} (v(a_i, a_{-i}) - v(a^*)) \, d \begin{bmatrix} F_i(\beta(\zeta)_i(a_i) - \theta) \\ -F_i(\zeta_i(a_i) - \theta) \end{bmatrix} \right\} dF^i(a_{-i}|\theta) d\theta$$

$$= \sum_{i=1}^{I} \int_\theta \int_{a_{-i}} \left\{ (v(a_i, a_{-i}) - v(a^*)) \begin{bmatrix} F_i(\beta(\zeta)_i(a_i) - \theta) \\ -F_i(\zeta_i(a_i) - \theta) \end{bmatrix} \right\} \bigg|_{a_i=0}^{|a_i^*} dF^i(a_{-i}|\theta) d\theta$$

$$- \sum_{i=1}^{I} \int_\theta \int_{a_{-i}} \left\{ \int_{a_i=0}^{a_i^*} \begin{bmatrix} F_i(\beta(\zeta)_i(a_i) - \theta) \\ -F_i(\zeta_i(a_i) - \theta) \end{bmatrix} (v(a_i, a_{-i}) - v(a_i - da_i, a_{-i})) \right\} dF^i(a_{-i}|\theta) d\theta$$

Since no action above a_i^* is played in either strategy profile, $\beta(\zeta)_i(a_i^*) = \zeta_i(a_i^*) = \infty$, so

$$\left\{ (v(a_i, a_{-i}) - v(a^*)) \begin{bmatrix} F_i(\beta(\zeta)_i(a_i) - \theta) \\ -F_i(\zeta_i(a_i) - \theta) \end{bmatrix} \right\} \bigg|_{a_i=0}^{|a_i^*} = -(v(0, a_{-i}) - v(a^*)) \begin{bmatrix} F_i(\beta(\zeta)_i(a_i) - \theta) \\ -F_i(\zeta_i(a_i) - \theta) \end{bmatrix} \geq 0$$

since a^* maximizes v and since $\beta(\zeta)_i(0) \geq \zeta_i(0)$. Thus,

$$V(\beta(\zeta)) - V(\zeta) \geq -\sum_{i=1}^{I} \int_{\theta} \int_{a_{-i}} \left\{ \int_{a_i=0}^{a_i^*} \left[\begin{array}{c} F_i(\beta(\zeta)_i(a_i) - \theta) \\ -F_i(\zeta_i(a_i) - \theta) \end{array} \right] [v(a_i, a_{-i}) - v(a_i - da_i, a_{-i})] \right\} dF^i(a_{-i}|\theta) d\theta$$

But for any a_i and θ, $F_i(\beta(\zeta)_i(a_i) - \theta) - F_i(\zeta_i(a_i) - \theta)$ is just the probability that a signal is observed between $\zeta_i(a_i)$ and $\beta(\zeta)_i(a_i)$ - the interval where under ζ player i plays more than a_i and her best response to ζ is to play something no greater than a_i. If instead players 1 through $i-1$ play according to $\beta(\zeta)$, player i's best response in this interval is still no greater than a_i by strategic complementarities. Therefore, by own-action quasiconcavity, player i's payoff, conditional on getting a signal in this interval, must be weakly decreasing in her own action: for $a_i' \in A_i$, $a_i' \leq a_i$,

$$\int_{\theta} \int_{a_{-i}} [g_i(a_i, a_{-i}) - g_i(a_i', a_{-i})] \left[\begin{array}{c} F_i(\beta(\zeta)_i(a_i) - \theta) \\ -F_i(\zeta_i(a_i) - \theta) \end{array} \right] dF^i(a_{-i}|\theta) d\theta \leq 0$$

Since $a_i \leq a_i^*$, $v(a_i, a_{-i}) - v(a_i - da_i, a_{-i}) \leq g_i(a_i, a_{-i}) - g_i(a_i - da_i, a_{-i})$ by the LP-maximizer condition. Thus:

$$V(\beta(\zeta)) - V(\zeta)$$

$$\geq -\sum_{i=1}^{I} \int_{\theta} \int_{a_{-i}} \int_{a_i=0}^{a_i^*} [v(a_i, a_{-i}) - v(a_i - da_i, a_{-i})] \left[\begin{array}{c} F_i(\beta(\zeta)_i(a_i) - \theta) \\ -F_i(\zeta_i(a_i) - \theta) \end{array} \right] dF^i(a_{-i}|\theta) d\theta$$

$$\geq -\sum_{i=1}^{I} \int_{\theta} \int_{a_{-i}} \int_{a_i=0}^{a_i^*} \mu_i(a_i) [g_i(a_i, a_{-i}) - g_i(a_i - da_i, a_{-i})] \left[\begin{array}{c} F_i(\beta(\zeta)_i(a_i) - \theta) \\ -F_i(\zeta_i(a_i) - \theta) \end{array} \right] dF^i(a_{-i}|\theta) d\theta$$

$$\geq 0$$

as claimed.

This implies that for all i and $a_i \in [0, a_i^*]$, $\zeta_i^n(a_i)$ converges to some finite upper bound $\zeta_i^\infty(a_i)$ as n grows. Let $\widehat{s}^1$ be this upper bound written in standard notation (i.e., $\widehat{s}_i^1(x_i) = \min\{a_i : \zeta_i^\infty(a_i) \geq x_i\}$). This is the smallest strategy profile surviving

iterated deletion of strictly dominated strategies in $G^{**}(1, \theta^*)$; moreover, there exists a $c > 0$ such that $\widehat{s}_i^1(x_i) = a_i^*$ for all i and $x_i \geq \theta^* + c$.

Changing ν is equivalent to relabeling the game $G^{**}(\nu, \theta^*)$. Thus if we write $\widehat{s}^\nu$ for the unique strategy profile surviving iterated deletion of strictly dominated strategies in $G^{**}(\nu, \theta^*)$, we have $\widehat{s}^\nu(x_i) = \widehat{s}^1\left(\frac{x_i - \theta^*}{\nu}\right)$. This in turn implies that $\widehat{s}_i^\nu(x_i) = a_i^*$ for all $x_i \geq \theta^* + \nu c$.

But now if s^ν is the essentially unique equilibrium of $G^*(\nu)$, we have that $s^\nu \geq \widehat{s}^\nu$ (this is true because the game $G^*(\nu)$ has everywhere higher best responses than the game $G^{**}(\nu, \theta^*)$). So we have:

Lemma 11 *For all $\varepsilon > 0$, there exists $\overline{\nu}$ such that for all $\nu \leq \overline{\nu}$, $s_i^\nu(\theta^* + \varepsilon) \geq a_i^*$.*

A symmetric construction gives:

Lemma 12 *For all $\varepsilon > 0$, there exists $\overline{\nu}$ such that for all $\nu \leq \overline{\nu}$, $s_i^\nu(\theta^* - \varepsilon) \leq a_i^*$.*

These two lemmas imply that if s^* is the left continuous limit of s^ν as $\nu \to 0$, then $s_i^*(\theta) \leq a_i^*$ if $\theta < \theta^*$ and $s_i^*(\theta) \geq a_i^*$ if $\theta > \theta^*$. By left continuity, $s_i^*(\theta^*) = a_i^*$. This proves the theorem. Q.E.D.$_{\text{Theorem 4}}$

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