

# **RESOLVING INDETERMINACY IN DYNAMIC SETTINGS: THE ROLE OF SHOCKS<sup>\*</sup>**

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## **ABSTRACT**

We introduce exogenous shocks in a standard dynamic model that otherwise would have multiple equilibria. The shocks take the form of a Brownian motion that affects the desirability of different actions. The multiplicity disappears: agents' actions depend only on payoff-relevant factors. There is no role for multiple, self-fulfilling prophecies or sunspots.

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# I. INTRODUCTION

It has long been agreed that expectations play a crucial role in determining economic outcomes. Many have also argued that expectations need not depend only on observable, payoff-relevant factors. Perhaps the most celebrated example is John Maynard Keynes's view that economic fluctuations are driven by the 'animal spirits' of entrepreneurs. In models with rational agents, this idea has been associated with the phenomenon of multiple rational expectations equilibria: cases in which more than one prophecy is self-fulfilling.

This paper shows that the phenomenon of multiple equilibria can be fragile to the introduction of aggregate shocks. We consider a standard dynamic model with multiple rational expectations equilibria. When we introduce shocks, the multiplicity disappears: regardless of where the economy begins, there is a unique path that it must follow. This means that agents' expectations are uniquely determined by payoff-relevant factors. Extraneous variables ('sunspots') or nonobservables ('investor sentiment') cannot have a real effect.

Our model is a simplification of Matsuyama (1991). Agents can work in either of two export sectors. One (cottage production or agriculture) has constant returns while the other (manufacturing) has increasing returns that are external to the agent. Consequently, an agent's relative payoff from working in the increasing returns sector is higher if more people do so. This relative payoff is also increasing in an exogenous parameter, which may reflect energy costs, weather, technology, or terms of trade. There are frictions: agents cannot switch sectors at will, but rather must wait for random opportunities to arrive.

In line with the findings of Matsuyama, we find multiple equilibria when the exogenous parameter is unchanging. For a range of initial conditions, if all agents believe that either sector will grow, then they will move there and thus make the prophecy self-fulfilling. We then consider what happens if the parameter changes over time according to a Brownian motion. This stochastic process has three key properties. The first is *persistence*: the parameter's current value gives some indication of where it will be in the near future. The second property is *frequency*: changes in the parameter come often. Third, we assume the existence of *dominance regions*. There is a chance that, in the perhaps very distant future, the parameter will reach values at which working in either sector is a dominant choice; i.e., at which an agent will move into the sector regardless of what others do. For example, there may be a chance that the soil will become so depleted that farmers will move to the manufac-

turing sector regardless of the choices of other agents. Conversely, higher energy costs could eventually make cottage production or agriculture a dominant choice (if manufacturing is more energy intensive).

It may be very unlikely that one of the sectors will ever become a dominant choice. However, the mere possibility that this *may* happen can have large effects. To see this, let us take the price of oil as the exogenous parameter and suppose that cottage production becomes a dominant choice if oil reaches \$1000 per barrel. Suppose the price of oil is \$999. If oil had a fixed price, we would not be able to draw any firm conclusions: since cottage production is not a *dominant* choice at \$999, if all agents were initially in manufacturing they might simply stay there forever. But this is not the case if there are shocks to the price of oil. With shocks, an agent cannot be sure that all others will stay in manufacturing until she gets another chance to change sectors, since the price of oil could easily reach \$1000 during this period, drawing agents into cottage production. Knowing this (and since the price of oil is already quite high), at \$999 agents will choose cottage production. But the same argument can be repeated: at \$998, they will choose cottage production because they know other agents will do so if the price reaches \$999. And so on.

The same argument can be applied for low prices of oil. Suppose that manufacturing becomes dominant if oil drops below \$1 per barrel. Then at \$1.01 agents will move out of cottage production, because of the chance that the price will drop below \$1 before they get another chance to change sectors. But this means that at \$1.02 they will enter manufacturing. And so on.

If we continue this line of reasoning *ad infinitum*, we end up with two thresholds, say \$10 and \$50. When oil costs less than \$10, agents choose manufacturing regardless of the current sizes of the two sectors. Above \$50, they always choose cottage production. In the intermediate range between the two cutoffs, their choice will depend on the current sizes of the two sectors. Agents will choose cottage production if this sector is sufficiently large; otherwise, they will choose manufacturing.

Importantly, throughout the three regions, there is no room for multiple, self-fulfilling prophecies. Agents' choices are uniquely determined by the current state (which includes both the payoff parameter and the sectoral distribution of agents). Only observable, payoff-relevant factors determine the evolution of the economy.

These results hold for shocks of any size. In particular, the shocks can be arbitrarily small. This case is perhaps the most surprising, since it reveals a discontinuity: there are multiple equilibria in a fixed environment, but not in a slightly stochastic one.

The rest of this paper is organized as follows. In section II we present the model. Section III analyzes the benchmark case of an unchanging environment. In section IV we show how things change with shocks. Section V discusses related literature. Concluding remarks appear in Section VI. More technical proofs are relegated to an appendix.

## II. THE MODEL

We consider a simplified version of the model of Matsuyama (1991). There is a small, open economy with a continuum of self-employed agents. Each agent can work in either of two sectors, one (C) with constant returns and the other (X) with increasing returns that are external to the agent. Time  $t$  is continuous. The economy has frictions: an agent's opportunities to (costlessly) switch sectors arrive as a Poisson process. Each agent receives such opportunities with an arrival rate  $\delta$ .

Agents are risk-neutral and live forever. The utility of an agent equals the integral of her lifetime production, discounted at the rate  $\theta$ . An agent in sector C produces a constant output flow whose value (at world prices) is normalized to one. An agent in sector X produces a variable amount whose value  $\pi(L^X, z)$  depends positively on the proportion  $L^X$  of agents in the sector and on an exogenous parameter  $z$ .<sup>2,3</sup> Both  $L^X$  and  $z$  are commonly observed. The parameter  $z$  can be interpreted as the state of technology, weather, or the relative price of good X on the world market. If we interpret  $\pi$  as the agent's profits,  $z$  may also reflect the world prices of nonlabor inputs in the two sectors. We assume that if  $z$  is large enough, an agent will move into sector X regardless of what other agents do. I.e., if  $z$  exceeds some  $\bar{z}$ ,  $\pi(0, z) > 1$ . For small enough  $z$ , C becomes a dominant choice: if  $z$  is below some  $\underline{z}$ ,  $\pi(1, z) < 1$ . We also assume that  $\pi$  is continuously differentiable in both arguments.

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<sup>2</sup> Since the goods in both sectors are traded and the country is small, we take prices as exogenous. In particular, a larger X sector does not lower the relative price of goods produced in that sector.

<sup>3</sup> In Matsuyama's version, agents' payoffs also depend on a random taste parameter. This is essentially the only difference between Matsuyama's model and our case of constant  $z$ .

### III. AN UNCHANGING ENVIRONMENT

We first analyze the benchmark case in which the environment does not change ( $z$  is constant over time). For  $z$  between  $\underline{z}$  and  $\bar{z}$ , both all-X and all-C are steady state equilibria. That is, if all agents are initially in one sector, it is an equilibrium for them to stay there. However, whether a given steady state can be reached depends on the initial value of  $L^X$ .

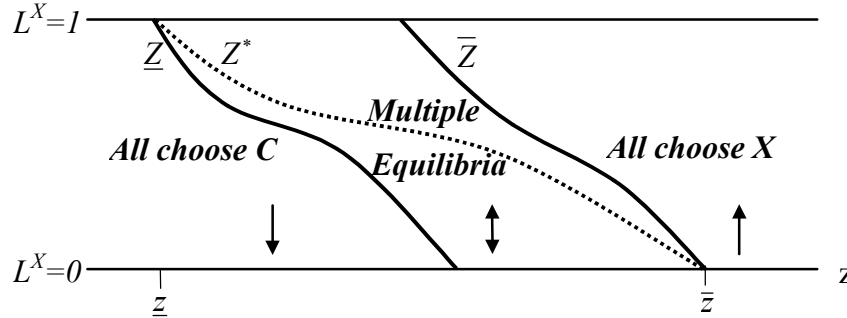


Figure I: A World Without Shocks (Proposition 1)

Figure I shows the set of long run outcomes for each  $z$  and for each initial value of  $L^X$ . The size of the X sector is measured on the vertical axis; the parameter  $z$  appears on the horizontal axis. In the rightmost region, all agents choose the X sector when they get the chance. This means that the economy converges to all-X. In the leftmost region, everyone chooses C at his first opportunity, so all-C is the only long run outcome.

In the region between  $\underline{z}$  and  $\bar{z}$ , there are multiple equilibria. All agents may choose X; then the X sector will grow, raising the wage there and making it indeed optimal to choose X. Or they may all choose C, lowering the wage in the shrinking X sector and making C the best choice. These results are summarized in Proposition 1.

**PROPOSITION 1.** There are decreasing functions  $\underline{Z}(L^X) < \bar{Z}(L^X)$  such that if  $z > \bar{Z}(L^X)$ , there is a unique equilibrium, in which agents always choose X. If  $z < \underline{Z}(L^X)$ , agents always choose C. For  $z$  between  $\underline{Z}(L^X)$  and  $\bar{Z}(L^X)$  there are multiple equilibria; both all-X and all-C are long run outcomes.

**Proof:** see appendix.

Note that  $\bar{Z}(0) = \bar{z}$ : if all agents are in the C sector, it is an equilibrium to remain as long as X is not a dominant choice. Likewise,  $\underline{Z}(1) = \underline{z}$ .

The dotted curve  $Z^*$  in Figure I is the myopic indifference line, given by

$\pi(L^X, z) = 1$ .<sup>4</sup> On this curve, the current wages in the two sectors are equal. As agents become more impatient relative to the speed at which they can change sectors (i.e., as  $\theta/\delta$  grows), they put more weight on the current wage and less weight on their expectations for the future. Hence, the curves  $\underline{Z}$  and  $\bar{Z}$  both converge to  $Z^*$ : in the limit of complete myopia, the equilibrium becomes unique. On the other hand, as agents become relatively more patient (as  $\theta/\delta$  shrinks), the area of multiplicity grows.

#### IV. A WORLD WITH SHOCKS

We now examine what happens if  $z$  changes randomly. We assume that  $z$  follows a Brownian motion. This is essentially the continuous time version of a random walk. It is characterized by a variance  $\sigma^2$  and a trend  $\mu$ . The variance measures the size of the random component; i.e., how fast  $z$  spreads out. The trend captures the deterministic part of  $z$ ; i.e., how its mean changes over time.<sup>5</sup> For example, a positive trend might reflect steady improvements in the technology of sector X. For now we assume that the trend  $\mu$  is a constant. We later relax this assumption in analyzing the limiting cases of small noise and small frictions.

**THEOREM 1.** With shocks, the equilibrium is unique. There is a decreasing function  $Z(L^X)$  such that agents choose X whenever  $z > Z(L^X)$  and C whenever  $z < Z(L^X)$ .

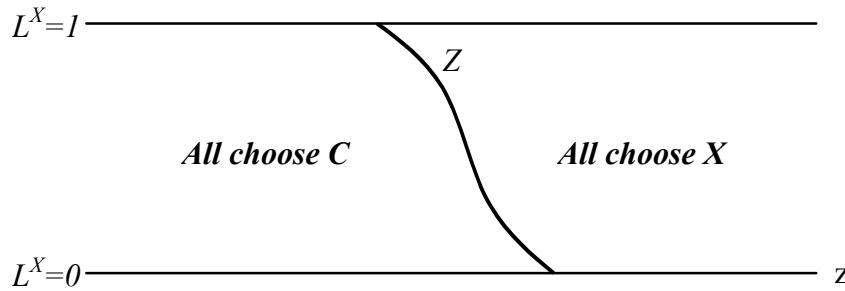


Figure II: A World With Shocks (Theorem 1).

<sup>4</sup> This curve is downwards sloping since  $\pi$  is increasing in both arguments.

<sup>5</sup> More precisely, the change in  $z$  over a brief period of length  $\varepsilon$  is normal with variance  $\sigma^2 \cdot \varepsilon$  and mean  $\mu \cdot \varepsilon$ .

Theorem 1 shows that there is a unique division line  $Z$  (Figure II). This division line is downwards sloping, so it divides the  $z$  axis into three regions. If  $z$  exceeds  $Z(0)$ , agents must choose the X sector. If it is below  $Z(1)$ , they all pick C. In the intermediate region, there is history dependence: their choice depends on the current size of the X sector. They move into the X sector if it is sufficiently large; otherwise, they choose C.

**Proof of Theorem 1** Recall our assumption that if  $z$  is sufficiently high, the X sector is a dominant choice, while C is dominant if  $z$  is low enough. These ‘dominance regions’ may be far from the current value of  $z$ , making it very improbable that  $z$  will reach one of these regions before an agent changes sectors again. However, the mere existence of such regions starts an iterative contagion effect that spreads throughout the parameter space.

Let  $Z_0$  be the boundary of the region where the X sector is a dominant choice (Figure III). We know that if an agent receives an opportunity to switch sectors when the current state is to the right of  $Z_0$ , she will choose sector X. Note that  $Z_0$  is the curve on which an agent is indifferent between the two sectors on the worst-case belief for sector X. This is simply the belief that in the future, each agent will choose the C sector when she gets the chance.

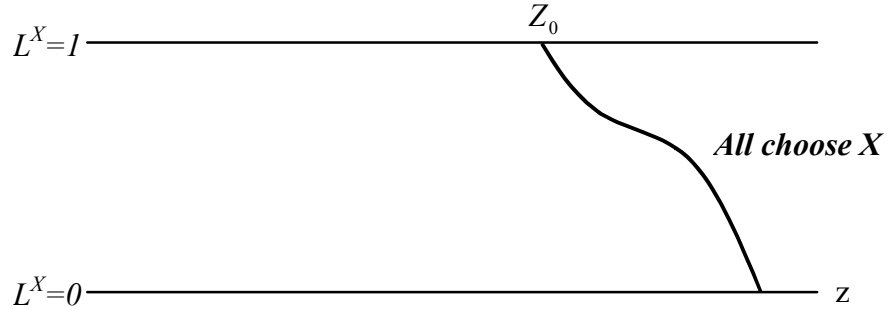


Figure III

Knowing that other agents must pick X to the right of  $Z_0$ , an agent actually wants to pick X slightly to the left of the curve as well. Why? On  $Z_0$  she was indifferent in the worst case: if all agents who choose after her were to pick C under *any* circumstances. But now she knows that they will actually choose X when they are to the right of  $Z_0$ . Since  $z$  changes stochastically, it will spend some time to the right of  $Z_0$  while the agent is committed to her choice. At such times, other agents who choose sectors will pick X. Since this raises her

assessment of the future size of the X sector, the agent is no longer indifferent on  $Z_0$ ; she strictly prefers X. Therefore, there is a new boundary  $Z_1$ , to the left of  $Z_0$ , such that agents must choose the X sector when to the right of  $Z_1$  (Figure IV). Note that  $Z_1$  is the curve on which an agent is indifferent between the two sectors on the worst case belief consistent with agents choosing X to the right of  $Z_0$ . This is simply the belief that all future agents will play according to  $Z_0$ : that they will choose C to the left of  $Z_0$  and X to the right.

This reasoning can be repeated, giving curves  $Z_2, Z_3$ , and so on *ad infinitum*. Let  $Z_\infty$  be the limit of this sequence (Figure IV). We know that agents must choose the X sector when to the right of  $Z_\infty$ . We cannot yet say what they will do when to the left.

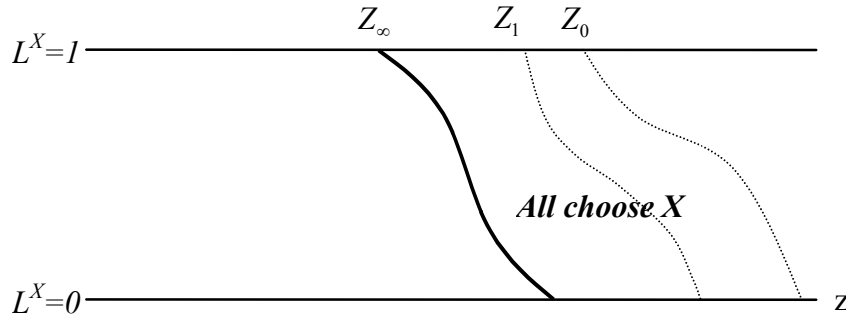


Figure IV

Note that  $Z_\infty$  is actually an equilibrium: if an agent expects all others to play according to  $Z_\infty$ , then it is optimal for her to do so as well. This is because  $Z_\infty$  is the limit of the iterative process, and on each  $Z_n$  an agent is indifferent between the two sectors if she expects all future agents to play according to  $Z_{n-1}$ .

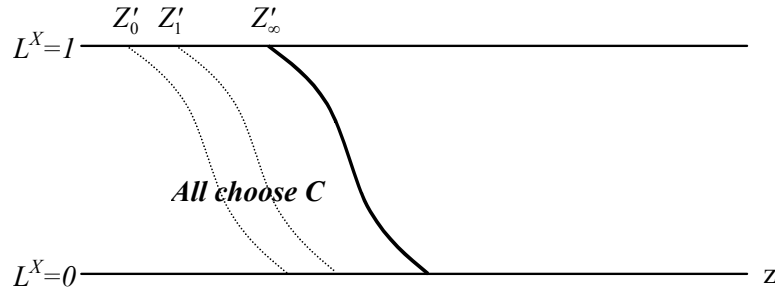


Figure V

We now start another iteration from the left side (Figure V). This iteration is somewhat different: we use translations of  $Z_\infty$ . (The reason will soon be apparent.) We begin with

a translation  $Z'_0$  of  $Z_\infty$  that is far enough over that the C sector is a dominant choice anywhere to the left of  $Z'_0$ . We then construct  $Z'_1$  as the rightmost translation of  $Z'_0$  such that an agent must choose C to the left of  $Z'_1$  if she believes that other agents will play according to  $Z'_0$ . Let  $Z'_\infty$  be the limit; agents must choose C when to the left of  $Z'_\infty$ .

What does it mean that the limit is  $Z'_\infty$ ?  $Z'_\infty$  is not necessarily an equilibrium, since we limited ourselves to translations of  $Z_\infty$ . However, if an agent expects all others to play according to  $Z'_\infty$ , then there must be at least one point A on  $Z'_\infty$  where she is indifferent between the sectors. Otherwise, if she strictly preferred the C sector everywhere on  $Z'_\infty$ , then the iterations would not have stopped at  $Z'_\infty$ .<sup>6</sup> Let B be the point on  $Z_\infty$  that is at the same height as A (Figure VI).

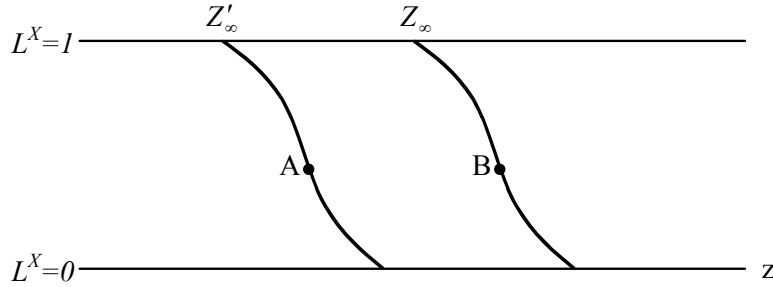


Figure VI

To show that the equilibrium is unique, we need to establish that the points A and B (and hence the curves  $Z_\infty$  and  $Z'_\infty$ ) must coincide. This implies that an agent's choice is uniquely determined by the current state, so that there is no scope for multiple outcomes that depend on agents' expectations.

The reasoning is as follows. Let us compare two players, one ('A') choosing at point A and believing that others will play according to  $Z'_\infty$  and the other ('B') choosing at B and expecting others to play according to  $Z_\infty$ . Since  $Z_\infty$  and  $Z'_\infty$  have the same shape, A and B expect the state  $(L^X, z)$  to have the same *relative dynamics*. That is, they expect the changes in

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<sup>6</sup> This argument implicitly assumes that payoffs are continuous. This holds since the behavior of the system  $(L^X_t, z_t)$  changes continuously as either the starting point  $(L^X_0, z_0)$  or the division line  $Z_\infty$  is moved (see Lemma 2 in BFP (1997b)). Since  $\pi$  is also continuous in its two arguments, the relative payoff to choosing X is a continuous function of  $(L^X_0, z_0)$  and of the division line. So as the division line is shifted to the right, the payoffs at all points on the line must change continuously. Hence, if there were no point of indifference, we could continue the iteration further.

the state, relative to its starting point (A or B), to have the same distribution. Why? First, the changes in  $z$  follow the same distribution<sup>7</sup> by our assumption that the trend of  $z$  is a constant. But for any given path of changes in  $z$ , the resulting path of  $L^X$  is the same for A as for B. It is the unique solution to the dynamical system illustrated in Figure VII.<sup>8</sup> When to the right of the curve ( $Z_\infty$  or  $Z'_\infty$ ),  $L^X$  rises at the rate  $\dot{L}^X = \delta(1 - L^X)$ : every agent who is still in C leaves at her first chance, there are  $1 - L^X$  such agents, and chances to leave arrive at the rate  $\delta$ . When to the left of the curve, agents switch from X to C. The proportion of X workers is  $L^X$ , so  $\dot{L}^X = -\delta L^X$ .

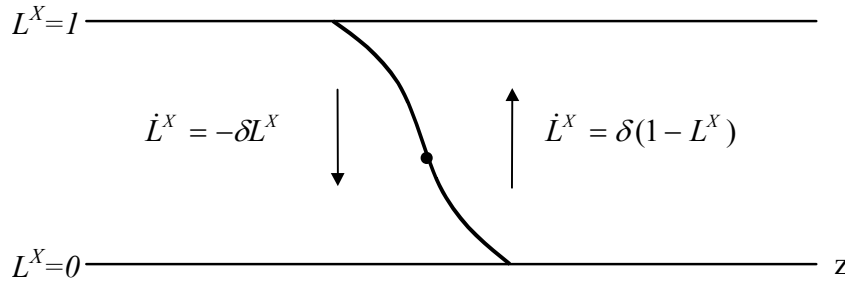


Figure VII

Hence, for each path of changes in  $z$ , agents A and B expect the same path of  $L^X$ . If A and B were different, B would expect uniformly higher values of  $z$  than A. Since the relative payoff to being in the X sector is increasing in  $z$ , B's payoff from choosing X would be higher than A's. But this cannot be, since both A and B are indifferent between the two sectors. Therefore, the curves coincide and the equilibrium is unique.

We conclude by showing that the division curve  $Z_\infty$  is downwards sloping. We do this by induction. On the assumption that all agents will choose the C sector, an agent's relative wage in the X sector is increasing in the initial values of both  $z$  and  $L^X$ . Hence,  $Z_0$  is downwards sloping. Now, on the assumption that all agents choose according to the *downwards sloping*  $Z_{n-1}$ , the relative wage is again increasing in both the initial values of  $z$  and  $L^X$ . To see why, consider any given path of changes in  $z$ . If we raise the initial value of either  $z$  or  $L^X$ , this can only lead us to spend more time to the right of  $Z_{n-1}$  (since it is downwards sloping). This guarantees that the X sector wage will remain higher forever. Thus,  $Z_n$  must also be

<sup>7</sup> More precisely, if the current value is  $z_t$ , the distribution of paths of changes  $(z_v - z_t)_{v>t}$  is independent of  $z_t$ .

<sup>8</sup> The fact that the path of  $L^X$  is unique given a path of  $z$ , while intuitively obvious, requires a rather technical proof that is deferred to the appendix (Lemma 1).

downwards sloping.

Q.E.D.

### Limiting Cases: Small Shocks or Small Frictions

While Theorem 1 shows that there is a unique equilibrium, the proof does not show how to calculate the division line  $Z_\infty$ . This problem becomes tractable in two limiting cases: when either the shocks or the frictions shrink to zero. Another benefit of examining these cases is that we do not need to assume a constant trend.

We now let the trend depend on  $t$  and  $z$ . The dependence on time  $t$  permits us to capture phenomena such as seasonality. Properties such as mean reversion can be captured through the dependence on  $z$ . For example, if  $\mu = -cz$  for some positive  $c$ , the trend always pulls  $z$  towards 0. We will see that the equilibrium is unique even if  $c$  is arbitrarily large relative to the variance  $\sigma^2$ . This means that while the iterative procedure is driven by the persistence of the shocks, arbitrarily little persistence is sufficient when shocks or frictions are small.

We first consider the case of small shocks. Suppose that  $z$  has variance  $\sigma^2$  and trend  $\lambda \cdot \mu(t, z)$ . Assume that  $\mu(t, z)$  is continuously differentiable and that for any given  $z$ ,  $\mu(t, z)$  is a bounded function of  $t$ . Theorem 2 shows that in the limit as  $\sigma^2$  and  $\lambda$  shrink, there is again a unique division line. Importantly, the relative rate at which  $\sigma^2$  and  $\lambda$  shrink does not matter, so that the trend can become very large relative to the variance.

The division line  $Z$  in this case is given by the following formula. For any  $L^X$ ,  $Z(L^X)$  is the value of  $z$  at which the weighted average of the relative wage equals zero:

$$(1) \quad \int_{\ell=0}^1 w_\ell [\pi(\ell, Z(L^X)) - 1] d\ell = 0$$

where the weight  $w_\ell$  equals  $[\ell/L^X]^{\theta/\delta}$  if  $\ell \leq L^X$  and  $[(1-\ell)/(1-L^X)]^{\theta/\delta}$  if  $\ell \geq L^X$ . Note that the integral takes into account the wage differential for all possible proportions of agents in the X sector. The weights are single peaked at  $L^X$ , so that the current relative wage has the most weight.

**THEOREM 2.** In the limit as the shocks shrink ( $\lambda \rightarrow 0$  and  $\sigma^2 \rightarrow 0$ ), agents choose X whenever  $z > Z(L^X)$  and C whenever  $z < Z(L^X)$ , where  $Z(L^X)$  is given by (1).

**Proof:** see appendix.

Theorem 2 can be interpreted as a negative robustness result. While there can be multiple equilibria in a completely static world, the introduction of very small shocks leads to uniqueness. These shocks must satisfy fairly mild assumptions. They must come frequently.<sup>9</sup> They must have some persistence, but there can be arbitrarily little since  $\lambda$  can become very large relative to  $\sigma^2$ . The shocks must have the potential to make either sector a dominant choice, but with small shocks this event is very remote.

Theorem 3 concerns the case of shrinking frictions. Again, a unique outcome is obtained (Figure VIII). However, this outcome has a new feature: there is no history dependence. The division line is vertical at  $z^*$ , defined by

$$\int_{\ell=0}^1 \pi(\ell, z^*) d\ell = 1$$

$z^*$  is the value of  $z$  at which the wage in the two sectors is the same, on average, if  $L^X$  is thought of as uniformly distributed between 0 and 1.

Suppose that  $z$  has variance  $\sigma^2$  and trend  $\mu(t, z)$ , where  $\mu(t, z)$  satisfies the assumptions of Theorem 2.

**THEOREM 3.** In the limit as frictions shrink (as  $\delta \rightarrow \infty$ ), agents choose X whenever  $z > z^*$  and C whenever  $z < z^*$ .

**Proof:** see appendix.

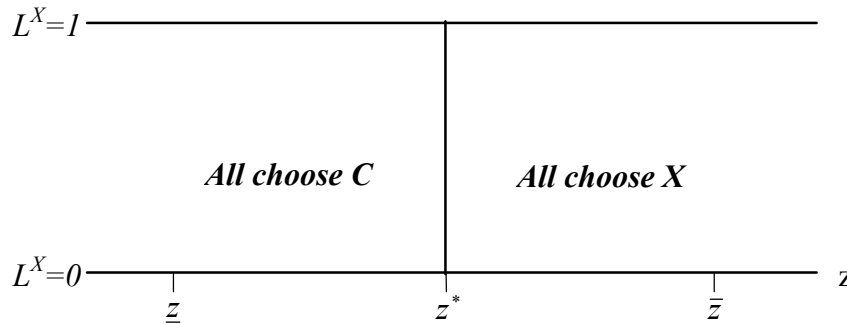


Figure VIII: Small Frictions (Theorem 3).

**Remark.** The division line becomes vertical at  $z^*$  also in the case of Theorem 2 (small shocks) as agents become very patient ( $\theta \rightarrow 0$ ), since then the weights  $w_\ell$  converge to 1.

<sup>9</sup> More precisely, there need only be a component that changes frequently; see section VI.

## V. RELATED LITERATURE

Multiple equilibria often arise in models with strategic complementarities, when an agent's incentive to take an action is stronger if others do so. In our model, these complementarities come from sector-specific increasing returns, as in Chacoliades (1978), Ethier (1982), Helpman and Krugman (1985), Krugman (1991) and Matsuyama (1991). Other examples of strategic complementarities include trade frictions (Diamond 1982), knowledge spillovers (Romer 1986), and public goods such as infrastructure (Murphy, Shleifer, and Vishny 1989).<sup>10</sup>

It is relatively straightforward to see how strategic complementarities can give rise to multiple *steady state* equilibria. There may be more than one state such that, if the economy starts there, it is an equilibrium to remain. However, a more interesting question is whether for a *given* initial condition there is more than one equilibrium path. Only if this is so can we say that extraneous factors (such as sunspots) can influence the economy.

For this question to be nontrivial, the economy must have some frictions that prevent it from jumping among steady states. For example, agents may have to search for jobs or trading opportunities. Or there might be state variables, such as capital, that can change only gradually. Without frictions, the dynamic model would be just a sequence of disconnected static models, so multiple steady states would translate automatically into multiple dynamic paths. Dynamic models with frictions have been studied by many authors, including Benhabib and Farmer (1994), Diamond and Fudenberg (1989), Drazen (1988), Drugeon and Wigniolle (1996), Krugman (1991), Matsuyama (1991), Weil (1989), and Zilibotti (1995). These models all find that from given initial conditions, there can be multiple equilibria.

The above models assume a fixed environment. Our model differs in that it has shocks, which lead to a unique equilibrium. One property of our shocks that is crucial for this result is that the shocks have the potential to make any action a dominant choice. Without this assumption, one can still have multiple equilibria, as shown, e.g., by Benhabib and Farmer (1996) and Farmer and Guo (1994).

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<sup>10</sup> See also Ball and Romer (1991), Bryant (1983), Cooper and John (1988), Diamond (1990), Gali (1994), Romer (1987), and Shleifer (1986). Caballero and Lyons (1992) present evidence for the empirical importance of strategic complementarities.

Some of our results use mathematical tools from Burdzy, Frankel, and Pauzner (henceforth, BFP) (1997b). These tools were applied in BFP (1997a) to models of pairwise random matching in games with two actions. Support was found for the risk-dominance selection criterion of Harsanyi and Selten (1988). The current paper uses the tools of BFP (1997b) only to analyze the limiting cases of small frictions and small shocks (Theorems 2 and 3).<sup>11</sup> Theorem 1, which permits shocks and frictions of arbitrary sizes, cannot be proved using these tools. Instead we use a new approach, which has the side benefit of being simpler and more intuitive.

## VI. CONCLUDING REMARKS

### Coordinating Agents' Expectations

When a model has multiple equilibria, it is unclear how agents' expectations become coordinated. If we as economists cannot predict what will happen, how do the agents know? Instead, they may differ in their predictions or simply be confused. If so, the outcome may not coincide with any equilibrium. Our model shows how exogenous shocks can cause agents to coordinate their expectations on a particular outcome. This solves the coordination problem, but at a price. While our agents need not be able to guess which equilibrium will be played, they must be able make complicated calculations and trust others to do the same. More importantly, they must agree on fine details of the economy, including the structure of the exogenous shocks.

### Critical Properties of the Shocks

Our shocks have three key properties. The first is the existence of *dominance regions*: extreme values of the random parameter at which a given choice is optimal regardless of what other agents do. We need such regions to start our iterative process that determines how agents behave throughout the parameter space.

The two other key features of the shocks come from our assumption that  $z$  follows a Brownian motion. One is *persistence*: the value of the parameter at one point in time is positively correlated with its future values. As a result, an agent who chooses sectors at

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<sup>11</sup> Even in the limiting cases, the current paper differs technically from BFP (1997a) in that an agent's payoff can depend nonlinearly on the sizes of the two sectors. In BFP, the assumption of pairwise interactions implies linearity.

some value of  $z$  cares about what others will do at nearby values of  $z$ . The other property is that the shocks come *frequently*. (With Brownian motion, changes in  $z$  occur continually.) These two properties imply that if the random parameter is close to a region where we know how agents behave, the probability is high that it will soon enter that region, at least temporarily. Hence, an agent who chooses sectors when just to the left of an area where all others choose the X sector must expect a nontrivial proportion of others to pick X in the near future.

To see this more clearly, consider an agent who chooses sectors while just to the left of the downwards sloping division curve  $Z$ . If shocks are infrequent, then while waiting for a shock all agents will choose C (see the dynamics in Figure VII). During this time, the state moves further away from  $Z$ . It may even move far enough that a shock, when it comes, will not be strong enough to move the state into the X region. Hence, the agent need not expect any others to choose the X sector in the foreseeable future. With frequent shocks, this cannot happen:  $L^X$  does not have time to change before the shocks move the state to the right, into the C region.<sup>12</sup>

## Discontinuous Stochastic Processes

Brownian motion has continuous sample paths. One may also wonder about parameters that are usually continuous but jump occasionally, such as the price of oil. To model this, consider a process that is the *sum* of a Brownian motion and a process with discrete jumps that occur at random times. All of our results hold for any such process.<sup>13</sup> This is because the continuous component ensures the key properties (persistence and frequency) discussed above.

## Other Dynamics

We assume agents receive chances to switch according to Poisson processes. This leads to dynamics that are particularly easy to analyze. However, there are other plausible

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<sup>12</sup> This is because the change in  $z$  over a short time interval  $\varepsilon$  has a large random component: its standard deviation is of order  $\sqrt{\varepsilon}$ . (Its variance must be of order  $\varepsilon$  for the variance of changes in  $z$  over a fixed, longer interval to be nontrivial; this is just a consequence of  $z$  having independent increments.) Since  $L^X$  changes approximately linearly with time, its effect is of order only  $\varepsilon$  and thus is swamped by the shocks. Hence, the shocks govern the short run behavior of the system.

<sup>13</sup> For Theorem 2, we must explain how the shocks go to zero. This can happen in two ways: the discrete jumps may become less and less frequent, or they may retain their frequency but become smaller and smaller. Theorem 2 holds in both cases.

dynamics. For example, Krugman (1991) assumes that agents can switch sectors at any time, but at a cost that is increasing in the overall switching rate. The analysis of how other dynamics perform in the presence of exogenous shocks remains an interesting open issue.

## APPENDIX

**Proof of Proposition 1:** Let us take the initial proportion  $L_0^X$  of X workers as given. When it is an equilibrium for all agents in the C sector to move to X? It suffices to check that an agent who chooses at time zero gains from doing so if she expects all other agents to follow. This is because the growth of the X sector raises the relative wage in X, thereby strengthening the incentive to choose X. Under these expectations,  $L^X$  grows at the rate  $\dot{L}^X = \delta(1 - L^X)$ : every agent who is still in C leaves at her first chance, there are  $1 - L^X$  such agents, and chances to leave arrive at the rate  $\delta$ . Therefore, choosing X rather than C raises an agent's payoff by the amount:

$$\underline{U}(L_0^X, z) = \int_{t=0}^{\infty} e^{-(\theta+\delta)t} (\pi(L_t^\uparrow, z) - 1) dt$$

where  $L_t^\uparrow = 1 - (1 - L_0^X)e^{-\delta t}$ . (Note that the discount rate is the product of  $e^{-\theta t}$ , the agent's pure discount rate, and  $e^{-\delta t}$ , the probability that she has not received another switching opportunity.) Moving to X is an equilibrium iff  $\underline{U} \geq 0$ . Since  $\underline{U}$  is increasing in both arguments, there is decreasing function  $\underline{Z}(L^X)$  such that moving to X is an equilibrium whenever  $z \geq \underline{Z}(L^X)$ . ( $\underline{Z}$  satisfies  $\underline{U}(L^X, \underline{Z}(L^X)) = 0$ .) A similar argument shows that moving to C is an equilibrium whenever

$$\overline{U}(L_0^X, z) = \int_{t=0}^{\infty} e^{-(\theta+\delta)t} (\pi(L_t^\downarrow, z) - 1) dt \leq 0$$

where  $L_t^\downarrow = L_0^X e^{-\delta t}$ . Define  $\overline{Z}$  by  $\overline{U}(L^X, \overline{Z}(L^X)) = 0$ . Since  $L_t^\downarrow$  is always below  $L_0^X$  and  $L_t^\uparrow$  is always above, whenever  $\overline{U}$  (which is proportional to a weighted average of  $\pi(L_t^\downarrow, z) - 1$  for all  $t \geq 0$ ) equals zero,  $\underline{U}$  (which is proportional to a weighted average of  $\pi(L_t^\uparrow, z) - 1$ ) must be positive. This implies that  $\underline{Z}(L^X) < \overline{Z}(L^X)$ . Q.E.D.

**LEMMA 1 (used in proof of Theorem 1)** Suppose agents choose according to  $Z_\infty$  or  $Z'_\infty$ .

For almost every path of  $z$ , there is a unique path of  $L^X$ .

**Proof:** Let the curve according to which agents choose be given by  $z = Z(L^X)$ . By Theorem 1 in Burdzy, Frankel, and Pauzner (1997a), there is a unique such path of  $L^X$  if  $Z$  is a Lipschitz function: if there is a finite constant  $c$  such that for any  $\ell$  and  $\ell'$ ,  $|Z(\ell) - Z(\ell')| < c|\ell - \ell'|$ . Every curve  $Z_n$  is contained in the compact set  $(L^X, z) \in [0, 1] \times [\underline{z}, \overline{z}]$ . Hence, since  $\pi$  is continuously differentiable and strictly increasing in both arguments, there are finite, positive constants  $a$  and  $b$  such that  $\pi_{L^X} < a$  and  $\pi_z > b$  at all points on each  $Z_n$ . We

will show by induction that all curves  $Z_n$  (and hence the limit  $Z_\infty$ ) must be Lipschitz with constant  $c = a/b < \infty$ .

To see why this holds for  $Z_0$ , consider two distinct points on  $Z_0$ ,  $(\ell, z)$  and  $(\ell', z')$ , where  $\ell' > \ell$  and  $z' < z$ . We will compare payoffs at the two points path by path. That is, for any path of the Brownian motion  $(z_v)_{v \geq t}$  starting at  $z_t = z$ , we will compare the payoff at  $(\ell, z)$  to the payoff at  $(\ell', z')$  when the Brownian motion follows the path  $(z_v + z' - z)_{v \geq t}$ . Since  $Z_0$  is computed assuming agents always choose C, the difference in future values of  $L^X$  is no greater than  $\ell' - \ell$ . (In particular, it equals  $(\ell' - \ell)e^{-\delta(v-t)}$ .) The difference in the payoff parameter is constant at  $z' - z$ . Hence, the difference in X sector payoffs at any future date when starting at  $(\ell, z)$  vs.  $(\ell', z')$  is always greater than  $-(\ell' - \ell)a + (z - z')b$ . So the only way an agent can be indifferent at both points is if  $-(\ell' - \ell)a + (z - z')b \leq 0$ , or if  $(z - z')/(\ell' - \ell) \leq a/b = c$ . This shows that  $Z_0$  is Lipschitz with constant  $c$ .

Now suppose that  $Z_{n-1}$  is Lipschitz with constant  $c$ . We prove that  $Z_n$  has the same property. Otherwise there would be two points  $(\ell, z)$  and  $(\ell', z')$  on  $Z_n$  with  $\ell' > \ell$  and  $z' < z$ , satisfying  $(z' - z)/(\ell' - \ell) > c$ . We again compare payoffs at the two points path by path. The key is noticing that the difference in future values of  $L^X$  is still no greater than  $\ell' - \ell$ . Why? The difference could grow only if there were a time at which the state  $(L_v^X, z_v)$  on the path that started at  $(\ell, z)$  was to the left of  $Z_{n-1}$  while the state  $(L_v^{X'}, z'_v)$  on the other path was to the right. But this cannot be: up until the first such time  $v$  at which this were to happen, the difference in  $L^X$  could only shrink while the difference in the payoff parameter would remain constant. Hence, the ratio  $(z_v - z'_v)/(L_v^{X'} - L_v^X)$  would have to be greater than  $c$  while the slope of  $Z_{n-1}$  is less than  $c$ . Knowing that the difference in  $L^X$  can only shrink, we can apply the same calculation as in the case of  $Z_0$ . Q.E.D.

**Proof of Theorem 2:** To show this, we perform the iterative procedure from the right using translations of the curve  $Z$  defined in the text. Let  $Z_\infty$  be the limit. As in the proof of Theorem 1, there must be a point on  $Z_\infty$  at which an agent is indifferent between X and C if she expects all other agents to pick X to the right and C to the left.

Now let us consider an agent who chooses sectors at the indifference point  $(L^X, Z_\infty(L^X))$ . She expects the dynamics shown in Figure VI. These dynamics are unstable, since the movement in  $L^X$  always pulls away from  $Z_\infty$ . (With a bit of algebra, one can verify that  $Z$  and hence  $Z_\infty$  is strictly downward sloping if  $\theta > 0$ .) When the trend and the variance

in  $z$  are small, the movement in  $L^X$  is fast relative to the movement in  $z$ , so these dynamics very soon pull the system ‘far away’ from the curve. In other words, the system very quickly *bifurcates*, either upwards (sending all agents to the X sector) or downwards (sending them all to C).

By Theorem 2 and Corollary 1 in BFP (1997b), as the variance and trend of  $z$  shrink to zero, the amount of time that passes before a bifurcation occurs goes to zero. Moreover, the chance of bifurcating up (to X) converges to  $P_X(L^X) = \delta(1 - L^X) / (\delta(1 - L^X) + \delta L^X)$ , while the chance of bifurcating down (to C) goes to  $P_C(L^X) = 1 - P_X(L^X)$ . Hence, the agent's relative payoff from choosing X is approximately

$$(1 - L^X) \cdot \int_{t=0}^{\infty} e^{-(\theta+\delta)t} (\pi(L_t^\uparrow, Z_\infty(L^X)) - 1) dt + L^X \cdot \int_{t=0}^{\infty} e^{-(\theta+\delta)t} (\pi(L_t^\downarrow, Z_\infty(L^X)) - 1) dt$$

where  $L_t^\uparrow = 1 - (1 - L^X)e^{-\delta t}$  and  $L_t^\downarrow = L^X e^{-\delta t}$ . This must equal zero since the agent is indifferent. By performing the changes of variables  $\ell = L_t^\uparrow$  and  $\ell = L_t^\downarrow$ , one can verify that  $Z_\infty(L^X) \approx Z(L^X)$ . Since the two curves have the same shape, in the limit X must be chosen to the right of  $Z$ . An analogous argument shows that C must be selected to the left of  $Z$ . Q.E.D.

**Proof of Theorem 3:** This is proved by a simple rescaling of time that permits us to apply Theorem 2. The new time unit is  $\tilde{t} = t / \delta$ . In the new time units, the parameters are  $\tilde{\delta} = 1$ ,  $\tilde{\theta} = \theta / \delta$ ,  $\tilde{\sigma}^2 = \sigma^2 / \delta$ ,  $\tilde{\mu}(\tilde{t}, z) = \mu(t, z)$ , and  $\tilde{\lambda} = 1 / \delta$ . By Theorem 2, in the limit as  $\delta \rightarrow \infty$ , agents choose X whenever  $z > Z(L^X)$  and C whenever  $z < Z(L^X)$ . Moreover, since  $\tilde{\theta} = \theta / \delta \rightarrow 0$ , the weights  $w_t$  converge to 1, which implies that  $Z(L^X) \rightarrow z^*$  for all  $L^X$ .

Q.E.D.

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