

REPEATED GAMES WITH  
DIFFERENTIAL TIME PREFERENCES\* \*\*

Ehud Lehrer<sup>1</sup> and Ady Pauzner<sup>2</sup>

First version: May, 95

This Version: February, 98

**Abstract**

When players have identical time preferences, the set of feasible repeated game payoffs coincides with the convex hull of the underlying stage-game payoffs. Moreover, all feasible and individually rational payoffs can be sustained by equilibria if the players are sufficiently patient. Neither of these facts generalizes to the case of different time preferences. First, players can mutually benefit from trading payoffs across time. Hence, the set of feasible repeated game payoffs is typically larger than the convex hull of the underlying stage-game payoffs. Second, it is not usually the case that every trade plan that guarantees individually rational payoffs can be sustained by an equilibrium, no matter how patient the players are. This paper provides a simple characterization of the sets of Nash and of subgame perfect equilibrium payoffs in two-player repeated games.

---

\* We wish to thank Jonny Shalev, Jeroen Swinkels and three anonymous referees for helpful comments. We especially thank Itzhak Gilboa for his valuable help. An earlier version of the paper was circulated as: "Breaking the Barriers of the Feasible Set: On Repeated Games with Different Time Preferences".

\*\* A number of (less central) proofs were moved, because of space limitations, from this paper to Lehrer and Pauzner (1997). Word 6.0 and pdf versions of that paper can be downloaded from \_\_\_\_\_.

<sup>1</sup> School of Mathematical Sciences, Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 69978 and Department of Managerial Economics and Decision Sciences, J.L. Kellogg Graduate School of Management, Northwestern University. e-mail: lehrer@math.tau.ac.il

<sup>2</sup> Eitan Berglas School of Economics, Faculty of Social Sciences, Tel Aviv University, Tel Aviv 69978, Israel. e-mail: ady@econ.tau.ac.il

## 1. Introduction

Repeated games in which all players have identical time preferences have been extensively studied. For such games, the set of feasible payoffs of the repeated game coincides with that of the stage-game. Moreover, folk theorems assert that, as players become very patient, the set of equilibrium payoffs of the repeated game approaches the set of its feasible and individually rational payoffs.<sup>1</sup> In contrast, when players have different discount factors both statements are typically false. First, the set of feasible payoffs of the repeated game is generally larger than that of the stage-game.<sup>2</sup> Second, even when players are very patient, not all the feasible and individually rational payoffs of the repeated game can be supported by equilibria.

The first fact arises from the possibility of “trading” payoffs over time. Trade is made possible by differences in time preferences. An impatient player cares more than a patient one about payoffs received in early stages, while a patient player cares relatively more about later periods. Thus, both players may benefit from playing actions that the impatient player prefers in early stages and actions that the patient player prefers in later stages. The gains from this trade can push the players’ overall utility outside the feasible set of the stage-game. Therefore, the set of all feasible payoffs of the repeated game is typically larger than that of the stage-game.

The second fact, that not every feasible payoff can be sustained by an equilibrium, is due to individual rationality considerations. Intertemporal trade requires trust. The patient player is willing to forego early payoffs only if she can trust the impatient player to reciprocate later on. And this requires that the impatient player’s individually rational payoff level be low enough that he can be punished should he deviate. In other words, the benefits of intertemporal trade can be reaped only if the impatient player is vulnerable enough to be trusted.

In some cases, most notably zero-sum games, no mutually beneficial trade plan is enforceable. That is, all equilibrium payoffs belong to the feasible set of the stage-game. In other cases, every feasible and individually rational *repeated-game* payoff is sustainable in equilibrium. What characterizes the set of equilibrium payoffs? Which factors determine whether there are equilibrium payoffs outside the stage-game’s feasible set? When are there feasible and individually rational repeated-game payoffs that cannot be supported by equilibrium? These are the issues we address. Specifically, our main result is a characterization of the Nash and subgame perfect equilibrium payoffs in two-player games.

Discounting of future payoffs reflects the players' tastes. Since people often differ in their time preferences, it is natural to consider the case of different discount factors. However, in cases where payoffs are monetary, one may argue that differential time preferences do not matter. Is it not the case that players can smooth out their payoffs at a common interest rate determined by the market? Indeed, if they can, that interest rate is the relevant discount factor and the classical folk theorems apply. But in many situations intertemporal markets may not exist or may not be accessible to all agents. For example, consider the interaction between an employee and her employer. The employer may be able to borrow money at an interest rate lower than that accessible to the employee. Whereas wage negotiations are often thought of as zero-sum, intertemporal trade is one way in which workers and employers can devise Pareto-improving contracts.

Differential time preferences have appeared in a number of applications. Rubinstein (1982) discusses the alternating offers model of bargaining between two players having different discount factors. Fudenberg and Levine (1989), Aoyagi (1993) and Celentani, Fudenberg, Levine, and Pesendorfer (1995) study repeated games in which a relatively patient player establishes her reputation in early stages of the game. While the bargaining and reputation models focus on how differences in patience affect the balance of power between the players, we focus instead on how this difference creates new cooperative possibilities, and on how such possibilities can be exploited in equilibrium.

The remainder of the paper is organized as follows. In Section 2 we analyze a few motivating examples. Section 3 contains the formal model and a characterization of the set of feasible payoffs of the repeated game. Section 4 is devoted to our main result: a characterization of the set of equilibrium (Nash and subgame perfect) payoffs in the two-player case. Section 5 concludes with a discussion of related issues and directions for future research. In particular, we discuss some difficulties in extending the results to games with incomplete information and to the general case of more than two players.

## 2. Illustrative examples

Consider the (two-player zero-sum) stage-game “matching pennies”:

1,-1	-1,1
-1,1	1,-1

At each stage  $k$ , players choose mixed actions. The impatient player receives the stage-payoff  $X(k) \in [-1,1]$  and the patient one receives  $Y(k) = -X(k)$ . By mixing evenly between the two actions, each player can guarantee her individually rational (henceforth,

IR) level of 0. Assume that the players evaluate their infinite streams of stage-payoffs using discount factors  $1 > \delta_p \geq \delta_i > 0$  (P and I stand for patient and impatient, respectively). The repeated-game payoffs are then,

$$U_I = (1 - \delta_i) \sum \delta_i^k X(k),$$

$$U_P = (1 - \delta_p) \sum \delta_p^k Y(k) = -(1 - \delta_p) \sum \delta_p^k X(k).$$

If the discount factors are identical,  $U_I + U_P \equiv 0$ . That is, the repeated game is also zero-sum. Since both IR levels are 0, the only feasible and IR payoff pair is  $(U_I, U_P) = (0, 0)$ . This is, therefore, the only repeated-game equilibrium payoff.

In the case where  $\delta_i < \delta_p$ , there exist feasible payoffs of the repeated game that are Pareto superior to  $(0, 0)$ . For example, the players may agree on receiving the payoff pair  $(X(k), Y(k)) = (1, -1)$  up to a certain period, and  $(X(k), Y(k)) = (-1, 1)$  thereafter. In other words, the patient player lends payoffs to the impatient one, and is refunded afterwards. As a result, both receive a positive payoff. However, this plan is not an equilibrium of the repeated game, because the impatient player will refuse to repay the debt. We later show that  $(0, 0)$  is, indeed, the unique equilibrium payoff.

Consider now the following modification of the stage-game:

1,-1	-1,1	-2,-2
-1,1	1,-1	-2,-2
-2,-2	-2,-2	-2,-2

The additional “threat” actions reduce the IR levels from 0 to -2, but do not affect the (zero-sum) Pareto-frontier. Now, if  $\delta_i$  is close enough to 1, the borrowing plan suggested above is enforceable: if the impatient player defaults on his loan he may be punished, and his payoff may be driven down to his IR level. Thus, a strictly positive payoff for both players is sustainable at equilibrium.

In the first example the set of equilibrium payoffs of the repeated game coincides with the set of feasible and individually rational payoffs of the stage-game. In contrast, in the second example a payoff outside the feasible set of the stage-game is supported by an equilibrium. In fact, all the feasible payoffs of this repeated game can be supported by an equilibrium if both players are patient enough. These examples illustrate a general phenomenon. For a fixed Pareto frontier of the stage-game, when the IR levels are reduced, the repeated-game equilibrium set is “pushed out.” That is, there may be new equilibrium payoffs that Pareto-dominate formerly efficient equilibrium points. More vulnerable players can trust each other more, and thereby achieve a higher degree of cooperation.

An intermediate case, where the set of equilibrium payoffs contains points outside the stage-game feasible set, while some (repeated-game) feasible payoffs cannot be supported by equilibrium, is the repeated prisoners' dilemma:

	L	R
U	2,2	0,3
D	3,0	1,1

Assume, for simplicity, that while both players are very patient on an absolute scale, the “patient” player is much more patient than the “impatient” one.<sup>3</sup> Consequently, the impatient player cares almost exclusively about a (long) initial period, while the patient one cares primarily about the long run. By playing the point  $(7/3, 4/3)$  at the early stages of the game and switching to the point  $(4/3, 7/3)$  for the tail of the game,<sup>4</sup> the players receive a total payoff close to  $(7/3, 7/3)$ , which is not feasible in the stage-game. This plan can be implemented as an equilibrium: a deviation of the impatient player will provoke a punishment to his IR level, 1, which is lower than his tail payoff,  $4/3$ .

An alternative plan could have generated even higher payoffs. Playing the point  $(3, 0)$  first and switching to  $(0, 3)$  later generates a total payoff close to  $(3, 3)$ . However, this plan cannot be supported by an equilibrium because the impatient player cannot be forced to accept a tail payoff of 0, which is lower than 1, his IR level.

### 3. The repeated game and its feasible set

#### 3.1 The stage-game

We consider a stage-game with two players, I (impatient) and P (patient), and finite pure action sets,  $A_I$  and  $A_P$ .  $\mathbf{A}_I$  and  $\mathbf{A}_P$  are the players mixed action sets and  $\mathbf{A}$  is the set of correlated actions, i.e., probability distributions over  $A_I \times A_P$ . The (expected) payoff functions are  $(g_I, g_P): \mathbf{A} \rightarrow \mathfrak{R}$ . Let  $V$  be the set of feasible stage-game payoffs,

$$V = \{(g_I(a), g_P(a)): a \in \mathbf{A}\}.$$

(I.e.,  $V$  is the convex hull of the pure-action payoffs.) Let  $ir_I$  and  $ir_P$  denote the IR levels,

$$ir_I = \min_{a_P \in A_P} \max_{a_I \in A_I} g_I(a_I, a_P), \quad ir_P = \min_{a_I \in A_I} \max_{a_P \in A_P} g_P(a_I, a_P).$$

$IR_I$  and  $IR_P$  denote the half planes of individually rational payoffs,

$$IR_I = \{(x, y): x \geq ir_I\}, \quad IR_P = \{(x, y): y \geq ir_P\}.$$

For a given  $\varepsilon > 0$ ,  $IR_I^\varepsilon$  and  $IR_P^\varepsilon$  are the sets of “ $\varepsilon$ -strong” individually rational payoffs,

$$IR_I^\varepsilon = \{(x, y): x \geq ir_I + \varepsilon\}, \quad IR_P^\varepsilon = \{(x, y): y \geq ir_P + \varepsilon\}.$$

Finally, let  $IR$  and  $IR^\varepsilon$  be the intersections of the corresponding sets across players:

$$IR = IR_I \cap IR_P, \quad IR^\varepsilon = IR_I^\varepsilon \cap IR_P^\varepsilon.$$

### 3.2 The repeated game

The stage-game is repeated infinitely. We assume perfect monitoring: each player can condition her action at stage  $k$  on the past realized actions. We also permit public randomization: in each stage the players observe the realization of a continuous, exogenous random variable and can condition their action on its outcome. Accordingly, the players can play any correlated action in  $\mathbf{A}$  and receive any (expected) stage-payoffs pair  $(X(k), Y(k))$  in  $V$ .<sup>5</sup>

The players discount future payoffs according to discount factors  $1 > \delta_p > \delta_I > 0$ . These discount factors are the subjective present values of one payoff unit, received after a delay of one time unit. Suppose that the interval between two consecutive repetitions of the stage-game is  $\Delta$  time units. Then, a unit of payoff received at the  $k$ -th stage is worth to the impatient (patient) player  $\delta_I^{k\Delta}$  ( $\delta_p^{k\Delta}$ ) units of payoff at the outset.<sup>6</sup> Properly normalized, the present values of the payoff streams are,

$$U_I^\Delta = (1 - \delta_I^\Delta) \sum_{k=0}^{\infty} \delta_I^{k\Delta} X(k), \quad U_p^\Delta = (1 - \delta_p^\Delta) \sum_{k=0}^{\infty} \delta_p^{k\Delta} Y(k).$$

The stage- $k$  continuation payoff is the present value of future payoffs, evaluated at stage  $k$ :

$$U_I^\Delta(k) = (1 - \delta_I^\Delta) \sum_{i=0}^{\infty} \delta_I^{i\Delta} X(k+i), \quad U_p^\Delta(k) = (1 - \delta_p^\Delta) \sum_{i=0}^{\infty} \delta_p^{i\Delta} Y(k+i).$$

It is useful to consider the case where players receive (integrable) continuous-time payoff streams  $X(t)$  and  $Y(t)$ . The evaluations of these streams at time 0 are<sup>7</sup>

$$U_I^0 = (-\log \delta_I) \int_0^{\infty} \delta_I^t X(t) dt, \quad U_p^0 = (-\log \delta_p) \int_0^{\infty} \delta_p^t Y(t) dt.$$

Let  $F^\Delta(V)$  be the feasible set of the repeated game with stage length  $\Delta$ , and let  $F^0(V)$  be its continuous time counterpart. I.e.,  $F^\Delta(V)$  and  $F^0(V)$  are the ranges of  $U^\Delta = (U_I^\Delta, U_p^\Delta)$  and  $U^0 = (U_I^0, U_p^0)$ , respectively. When no confusion is likely to arise, we will omit  $V$  and denote  $F^\Delta(V)$  and  $F^0(V)$  by  $F^\Delta$  and  $F^0$ . Clearly,  $F^\Delta$  and  $F^0$  are supersets of  $V$ . Finally, note that  $F^\Delta$  and  $F^0$  are closed and convex sets.

### 3.3 The feasible set

The examples in Section 2 show that  $F^\Delta$  may be a strict superset of  $V$ . We now turn to characterize the boundary of  $F^\Delta$ . Since  $F^\Delta$  is convex, every point on its boundary is a maximizer of a certain weighted sum of the players' payoffs. Therefore, one way to fully characterize  $F^\Delta$  is to consider all possible weight pairs  $(\alpha_I, \alpha_p)$  (including negative ones), and, for each pair, to identify all points  $(x, y)$  that solve,

$$(1) \quad \text{Max } \alpha_I x + \alpha_P y \quad \text{s.t. } (x, y) \in F^\Delta.$$

Let there be given, then,  $\alpha = (\alpha_I, \alpha_P)$ . The explicit formulation of (1) is:

$$(2) \quad \text{Max}_{\{X(k), Y(k)\}_{k=0}^{\infty}} \alpha_I U_I^\Delta + \alpha_P U_P^\Delta \quad \text{s.t. } \forall k, (X(k), Y(k)) \in V.$$

We can decompose the problem of maximizing the discounted sum of stage-payoffs (2) into separately maximizing, for each stage  $k$ , the weighted sum of the players' payoffs,<sup>8</sup>

$$(3) \quad \text{Max}_{X(k), Y(k)} \alpha_I (1 - \delta_I^\Delta) \delta_I^{k\Delta} X(k) + \alpha_P (1 - \delta_P^\Delta) \delta_P^{k\Delta} Y(k) \quad \text{s.t. } (X(k), Y(k)) \in V.$$

For any integer  $k$ , let  $h_\alpha(k) = (\alpha_I (1 - \delta_I^\Delta) \delta_I^{k\Delta}, \alpha_P (1 - \delta_P^\Delta) \delta_P^{k\Delta})$ .  $h_\alpha(k)$  is a vector in  $\mathfrak{R}^2$ , representing a direction in the plane. (3) can be written as the linear program:

$$(4) \quad \text{Max}_v h_\alpha(k) \cdot v \quad \text{s.t. } v \in V.^9$$

The feasible polygon  $V$  is the same for every  $k$ , while the ascent direction of the objective function,  $h_\alpha(k)$ , changes with  $k$ . The direction corresponding to  $k = 0$  is  $h_\alpha(0)$ . Then, for each successive  $k$ , the objective function is multiplied coordinate-wise by  $(\delta_I^\Delta, \delta_P^\Delta)$ . Thus, as  $k$  increases,  $h_\alpha(k)$  tilts gradually (clockwise or counterclockwise depending on  $\alpha$ ). Finally, for very large  $k$ ,  $h_\alpha(k)$  is almost vertical. This is so since the ratio  $\alpha_P (1 - \delta_P^\Delta) \delta_P^{k\Delta} / \alpha_I (1 - \delta_I^\Delta) \delta_I^{k\Delta}$  tends to infinity as  $k$  tends to infinity.

Consequently, the optimal solutions to (4) form a path of vertices that moves along the frontier of  $V$ . This path starts at a certain vertex corresponding to the direction  $h_\alpha(0)$ , and ends at the vertex where the patient player achieves her highest or lowest payoff (depending on the sign of  $\alpha_P$ ). Note that if  $\Delta$  is small enough, the change in  $h_\alpha(k)$  between consecutive stages is small. In this case, the optimal path goes through all the vertices between the first and last one. (No vertex is ever skipped.)

For generic values of  $\alpha$ , the maximum of  $h_\alpha(k) \cdot v$  over the polygon  $V$  is attained at one vertex for any  $k$ . However, for some values of  $\alpha$ , there are periods  $k$  at which the maximum is attained over a whole facet of  $V$ . By choosing different points on this facet, payoff can be transferred between the two players at a fixed ratio, without violating optimality. In this case,  $F^\Delta$  itself contains an entire facet, perpendicular to the direction  $\alpha$ .<sup>10</sup>

To gain some intuition into the geometric structure of  $F^\Delta$ , consider again the game “matching pennies.” We construct the Pareto frontier of  $F^\Delta$  as follows. The point on the Pareto frontier that corresponds to  $\alpha = (\alpha_I, \alpha_P) = (0, 1)$  (north) is  $(-1, 1)$ . It is generated by the constant path,  $(X(k), Y(k)) = (-1, 1)$  for every  $k$ . This is also the optimal path for directions  $\alpha$  close enough to  $(0, 1)$ . As  $\alpha$  tilts eastward, a direction, say,  $\alpha_1$ , is reached

where there are two pure action paths that maximize the problem in (2). The first consists of playing constantly  $(-1,1)$ , and the second consists of playing  $(1,-1)$  in the first period and  $(-1,1)$  thereafter. The direction  $\alpha_1$  is perpendicular to the first facet of  $F^\Delta$ . This facet corresponds to playing any mixture of  $(-1,1)$  and  $(1,-1)$  in the first period, and playing  $(-1,1)$  in all subsequent periods. Again, slightly shifting  $\alpha$  eastward from  $\alpha_1$  does not alter the optimal path (playing  $(1,-1)$  in the first period and  $(-1,1)$  thereafter). These values of  $\alpha$  correspond to the next vertex of the Pareto frontier. As  $\alpha$  shifts further, the next facet is reached, which is perpendicular to the direction  $\alpha_2$ . This facet is generated by the paths consisting of  $(1,-1)$  in the first stage, any split in the second stage, and  $(-1,1)$  ever after. This facet is shorter than the first, since the weight of the second period in each player's discounted payoffs' sum is smaller than that of the first. Continuing to move clockwise along the Pareto frontier, we encounter an infinite sequence of facets, corresponding to longer prefixes of  $(1,-1)$  and shorter tails of  $(-1,1)$ . The facets become unboundedly small, and converge to the east-most point of the Pareto frontier  $(1,-1)$ .

The feasible set of the continuous time case,  $F^0$ , is found in a similar way. Given a direction  $\alpha$ , we solve a continuum of problems; for any  $t \in [0, \infty)$ ,

$$(5) \quad \text{Max}_{X(t), Y(t)} h_\alpha^0(t) \cdot (X(t), Y(t)) \quad \text{s.t.} \quad (X(t), Y(t)) \in V,$$

where  $h_\alpha^0(t) = (\alpha_I (-\log \delta_I) \delta_I', \alpha_P (-\log \delta_P) \delta_P')$ . The time axis divides into a finite number of intervals  $[0, t_1), [t_1, t_2), \dots, [t_z, \infty)$  such that the solution to (5) is constant over each interval; a path of adjacent vertices of  $V$  is followed. While  $F^\Delta$  is a polygon with an infinite number of facets,  $F^0$  has a smooth frontier. This is illustrated in Figure 1: The innermost polygon is the feasible set of the stage-game,  $V$ . The intermediate polygon is the feasible set of the repeated game,  $F^\Delta$  (the parameters are  $\delta_P = 0.5$ ,  $\delta_I = 0.05$  and  $\Delta = 1$ ). The outermost boundary is that of  $F^0$ .<sup>11</sup>

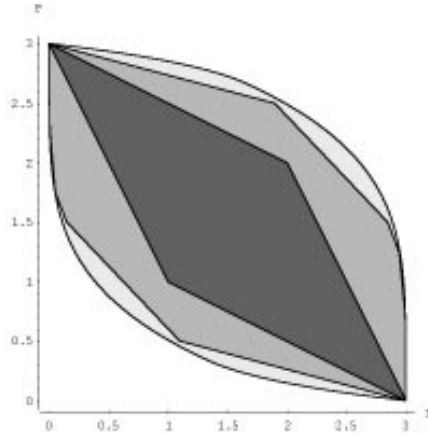


Figure 1: The feasible sets  $F^\Delta$  and  $F^0$  in the prisoner's dilemma.



$F^\Delta(\cdot)$  may be viewed as an operator that transforms any given convex polygon  $B$  to a set  $F^\Delta(B)$ , of all feasible repeated-game payoffs where the set of stage payoffs is  $B$ . Proposition 1 states that  $F^\Delta(B)$  converges uniformly, from inside, to  $F^0(B)$  as  $\Delta \rightarrow 0$ . Moreover, it is increasing and uniformly continuous in  $B$ .

**Proposition 1:**

Let  $B$  be a convex polygon of feasible stage-payoffs.

- a. For any  $\Delta > 0$ ,  $F^\Delta(B) \subseteq F^0(B)$ ; and
- b. for any  $\varepsilon > 0$  there exists  $\underline{\Delta} > 0$  such that for any  $f^0 \in F^0(B)$  and any  $\Delta < \underline{\Delta}$ , there exists  $f^\Delta \in F^\Delta(B)$  satisfying  $\|f^0 - f^\Delta\| < \varepsilon$ .

Moreover, given  $\Delta \geq 0$  and convex polygons  $B$  and  $B'$ ,

- c. for any  $\varepsilon > 0$ , there exists  $d > 0$  such that if  $\max_{b \in B} \min_{b' \in B'} \|b - b'\| < d$ , then  $\max_{f \in F^\Delta(B)} \min_{f' \in F^\Delta(B')} \|f - f'\| < \varepsilon$ ; and
- d. if  $B' \subseteq B$ , then  $F^\Delta(B') \subseteq F^\Delta(B)$ .

**Proof:** See appendix.

By Proposition 1, the feasible set of the limit, continuous-time “game”,  $F^0 = F^0(V)$ , is an upper bound, and (when  $\Delta$  is small) a good approximation of  $F^\Delta = F^\Delta(V)$ . In particular, the Pareto frontier of  $F^0$  delineates the boundary of all possible cooperative outcomes. An explicit formula of this frontier is presented in Proposition 2. (This formula can easily be modified to the three other parts of the boundary of  $F^0$ .)

**Proposition 2:**

Let  $B$  be a convex polygon of feasible stage-payoffs and denote the vertices on the Pareto frontier of  $B$  as  $(x_0, y_0), (x_1, y_1), \dots, (x_l, y_l)$ , with  $x_0 > \dots > x_l$  (and thus  $y_0 < \dots < y_l$ ). The Pareto frontier of  $F^0(B)$  is the graph of the function  $U_p(U_l)$ :

$$U_p = y_m + (x_m - U_l)^{\frac{1}{r}} (S_m)^{\frac{r-1}{r}} \text{ whenever } \mu_m \geq U_l \geq \mu_{m+1}, m = 0 \dots l-1,$$

where

$$(i) \quad r = \frac{\log(\delta_l)}{\log(\delta_p)} (> 1)$$

$$(ii) \quad S_m = \sum_{i=m}^{l-1} (x_i - x_{i+1}) \left( \frac{y_{i+1} - y_i}{x_i - x_{i+1}} \right)^{\frac{r}{r-1}}$$

$$(iii) \quad \mu_0 = x_0, \text{ and } \mu_{m+1} = x_m - S_m \left( \frac{y_{m+1} - y_m}{x_m - x_{m+1}} \right)^{\frac{r}{1-r}}, m = 0 \dots l-1.^{12}$$

**Proof:** See Lehrer and Pauzner (1997).

To illustrate Proposition 2, we compute the Pareto frontier of the feasible set of the prisoners' dilemma game presented in Section 1. The vertices of the Pareto frontier of  $V$  are  $(x_0, y_0) = (3, 0)$ ,  $(x_1, y_1) = (2, 2)$  and  $(x_2, y_2) = (0, 3)$ . Assume, for example, that  $\delta_I = e^{-.02}$  and  $\delta_P = e^{-.01}$ . Hence,  $r = 2$ ,  $S_1 = 2 \cdot \frac{1}{2}^2 = \frac{1}{2}$ , and  $S_0 = 2^2 + \frac{1}{2} = \frac{9}{2}$ . Thus,

$$U_P = \begin{cases} 2 + \sqrt{\frac{1}{2}} \cdot \sqrt{2 - U_I} & 0 \leq U_I \leq \frac{15}{8} \\ \sqrt{\frac{9}{2}} \cdot \sqrt{3 - U_I} & \frac{15}{8} \leq U_I \leq 3 \end{cases}$$

One implication of Proposition 2 is the following. For any fixed payoff of the impatient player,  $U_I$ , the patient player's payoff,  $U_P$ , approaches her highest possible one,  $y_I$ , as the patience ratio,  $r$ , increases.<sup>13</sup> In the limit, the players can jointly attain their maximal payoffs,  $x_0$  and  $y_I$ . In the repeated prisoners dilemma, for instance, they can attain payoffs close to (3,3).

#### 4. The Sets of Equilibrium Payoffs

In this section we characterize the sets of Nash and subgame-perfect equilibrium payoffs. These equilibrium sets are closed and convex<sup>14</sup>. Hence, they can be characterized using the technique developed in the previous section. To find the points on the boundary of the equilibrium set corresponding to a given direction  $\alpha$ , we solve a maximization problem similar to that used in the characterization of the feasible set. However, additional constraints must be added. A play path can be sustained by an equilibrium only if, at any stage, each player's continuation payoff is individually rational. This condition is clearly necessary. We later show that it is also sufficient (for both Nash and subgame perfect equilibrium) if the time  $\Delta$  between any two consecutive stages is small enough.

The additional constraints imposed on all continuation payoffs might render the maximization problem too complicated. However, all play paths that solve the maximization problem share one property, which greatly simplifies our task: along an optimal path, each player's stream of payoffs is monotone. For example, along a Pareto-optimal path (i.e., a path corresponding to  $\alpha = (\alpha_I, \alpha_P) \gg 0$ ), the stage-payoffs of the patient player are increasing, while those of the impatient player are decreasing. Consequently, the large set of constraints reduces to much simpler restrictions.

Along a Pareto-optimal path, the impatient player should never receive a stage-payoff below his IR level. This is so because his stage-payoffs are decreasing, and therefore, if one stage-payoff is below his IR level, so are all subsequent payoffs and, as a result, also the corresponding continuation payoff. This means that only stage-payoffs in

$V \cap IR_I$  can be used along a Pareto-optimal path. Thus, the equilibrium set cannot exceed the Pareto frontier of  $F^\Delta(V \cap IR_I)$ .

As for the patient player, all the constraints on her continuation payoffs reduce to one: that her overall repeated-game payoff be individually rational. Since her stage-payoffs are increasing along a Pareto-optimal path, her continuation payoffs are also increasing. Thus, if the initial one,  $U_p^\Delta$ , is individually rational, so are all the other continuation payoffs. Geometrically, this means that the Pareto frontier of  $F^\Delta(V \cap IR_I)$ , intersected with  $IR_p$ , forms the Pareto frontier of the equilibrium payoffs set.

Figure 2 illustrates the construction of the Pareto frontier of the equilibrium set, for  $\Delta$  close to 0, in three steps. By Proposition 1, when  $\Delta$  is small enough  $F^\Delta$  can be replaced by its approximation,  $F^0$ .

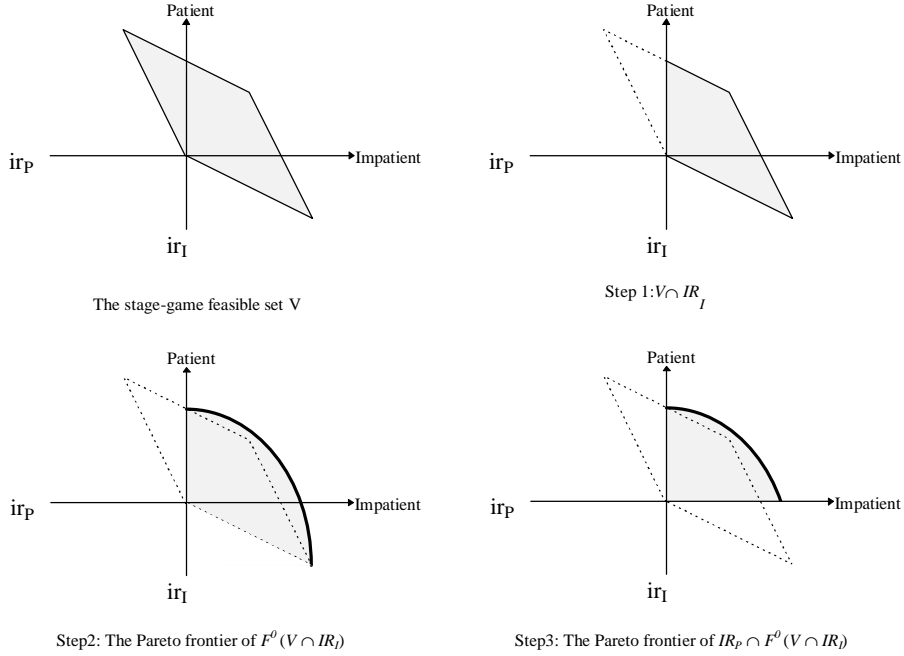


Figure 2: Constructing the Pareto frontier of the equilibrium set (Nash or SP)

The Pareto frontier of the equilibrium set corresponds to the directions  $\alpha = (\alpha_I, \alpha_p) \gg 0$  (northeast). There are three other cases:  $\alpha_I > 0, \alpha_p < 0$  (southeast),  $\alpha_I < 0, \alpha_p > 0$  (northwest) and  $\alpha_I < 0, \alpha_p < 0$  (southwest). In all cases, the payoff streams to each player are monotone. However, whether they are increasing or decreasing depends on the case. Consequently, the IR constraints have to be treated differently in each case. We first characterize the Pareto frontier of the equilibrium payoffs sets (Theorem 1). This is the most interesting part of the frontier from an economic point of view. Next, we explain the difference between this construction and those pertaining to the three other

cases. We conclude with a full characterization (Theorem 2). We start with some notation.

Given discount factors  $1 > \delta_p \geq \delta_l > 0$  and the time  $\Delta$  between any two consecutive repetitions of the stage-game, denote:

$E^\Delta$ : The set of Nash equilibrium payoffs of the repeated game.

$SPE^\Delta$ : The set of subgame-perfect equilibrium payoffs of the repeated game.

Recall that  $F^\Delta(B)$  is the operator producing the set of feasible payoffs in the repeated game when  $B$  is a polygon of available stage-payoffs. Finally, for two sets in  $\mathfrak{R}^2$ ,  $B$  and  $B'$ , we say that  $B \leq B'$  if for every  $b \in B$  there exists  $b' \in B'$  such that  $b'$  weakly Pareto-dominates  $b$  or equals  $b$ .

**Theorem 1.**

For any  $\varepsilon > 0$ , there exists  $\Delta > 0$ , such that for any  $\Delta < \Delta$ ,

$$IR_p^\varepsilon \cap F^\Delta(V \cap IR_l^\varepsilon) \leq SPE^\Delta \leq E^\Delta \leq IR_p \cap F^\Delta(V \cap IR_l)$$

Theorem 1 says that the Pareto frontier of the equilibrium set  $E^\Delta$ , as well as its subgame-perfect counterpart  $SPE^\Delta$ , are bounded between the Pareto frontiers of two sets that are close to each other. By Proposition 1, an approximation to the Pareto frontier is obtained: as  $\Delta$  becomes close to 0, the Pareto frontier of the equilibrium set uniformly approaches, from inside, that of  $IR_p \cap F^0(V \cap IR_l)$ .<sup>15</sup> In particular, Theorem 1 implies that if some point on the Pareto frontier of  $V$  is outside  $IR_l$ , then the repeated game's equilibrium set,  $E^\Delta$ , does not converge to the set of feasible and individually rational payoffs of the repeated game,  $F^\Delta \cap IR$ .<sup>16</sup> That is, not any feasible and individually rational payoff of the repeated game can be sustained by an equilibrium. This is impossible when both players have the same time preferences.

**Proof of Theorem 1:**

The key to the left-side inequality is Lemma 1, which states that the players' payoffs along an optimal path are monotone.

**Lemma 1:**

Let  $B$  be a convex and compact set in  $\mathfrak{R}^2$  and assume that  $\{(X(k), Y(k))\}_{k=0}^\infty$  maximizes:

$$\sum_{k=0}^\infty \alpha_l (1 - \delta_l^\Delta) \delta_l^{k\Delta} X(k) + \alpha_p (1 - \delta_p^\Delta) \delta_p^{k\Delta} Y(k) \quad s.t. \forall k, (X(k), Y(k)) \in B.$$

Then,  $\alpha_p Y(k)$  is (weakly) increasing and  $\alpha_l X(k)$  is (weakly) decreasing.

**Proof:** See Appendix.

Let  $f$  be a Pareto-optimal point in  $IR_p^\varepsilon \cap F^\Delta(V \cap IR_l^\varepsilon)$ . The path generating  $f$  solves a maximization problem as in Lemma 1, with  $(\alpha_l, \alpha_p) \gg 0$  and  $B = V \cap IR_l^\varepsilon$ . By

the lemma, the patient player's payoffs are increasing along the path. Thus, for any stage  $k$ , each player's continuation payoff is strongly individually rational:  $U_I^\Delta(k) > ir_I + \varepsilon$  and  $U_P^\Delta(k) > ir_P + \varepsilon$ .

For  $\Delta$  small enough, this path can be easily extended to a Nash equilibrium (for example, punishing deviations using trigger strategies). The extension to a subgame perfect equilibrium is essentially the same as in the case of identical discount factors; this is proved in Lemma 2.

**Lemma 2:**

For any  $\varepsilon > 0$ , there exists  $\underline{\Delta} > 0$  such that any payoff path, along which all continuation payoffs are in  $IR^\varepsilon$ , can be extended to a subgame perfect equilibrium when  $\Delta < \underline{\Delta}$ .

**Proof:** See Lehrer and Pauzner (1997).

By lemma 2,  $f \in SPE^\Delta$ . This proves the left-side inequality in the theorem. The middle inequality is trivial, since  $SPE^\Delta \subseteq E^\Delta$ . To show the right-side inequality, we have to take an equilibrium point  $f \in E^\Delta$  and construct a point in  $IR_P \cap F^\Delta(V \cap IR_I)$  that Pareto dominates it. Recall that, along an equilibrium path, all continuation payoffs are individually rational. For the patient player, this is a stronger constraint than that pertaining to paths supporting points in  $IR_P \cap F^\Delta(V \cap IR_I)$ , since the latter only requires that her stage-0 continuation payoff be individually rational. However, for the impatient player, the constraints associated with an equilibrium path are weaker than the requirement that all stage-payoffs be (impatient player) individually rational. Lemma 3 helps us overcome this obstacle.

**Lemma 3:**

Assume that  $\{(X(k), Y(k))\}_{k=0}^\infty \in V^\infty$  maximizes  $\alpha_I U_I^\Delta(0) + \alpha_P U_P^\Delta(0)$  subject to:  $\forall k, U_I^\Delta(k) \geq ir_I, U_P^\Delta(k) \geq ir_P$ . Suppose that  $\alpha_I > 0$ . Then  $\forall k, X(k) \geq ir_I$ .

**Proof:** See Appendix.

Now, let  $f$  be a Pareto-optimal point in  $E^\Delta$ . There exists a point  $\hat{f}$  that Pareto-dominates (or is equal to)  $f$ , such that the path  $\{(\hat{X}(k), \hat{Y}(k))\}_{k=0}^\infty$  generating it solves a maximization problem as in Lemma 3, with  $(\alpha_I, \alpha_P) \gg 0$ . By the lemma,  $\forall k, \hat{X}(k) \geq ir_I$ . Thus,  $\hat{f} \in IR_P \cap F^\Delta(V \cap IR_I)$ . This concludes the proof of Theorem 1. ■

We now apply theorem 1, together with the formula given in Proposition 2, to find the Pareto frontier of the equilibrium set in the prisoner's dilemma, presented earlier. The vertices on the Pareto frontier of  $V \cap IR_I$  are:  $(x_0, y_0) = (3, 0)$ ,  $(x_1, y_1) = (2, 2)$  and

$(x_2, y_2) = (1, 2.5)$ . Let  $r = 2$  (as in Section 3). The Pareto frontiers of the equilibrium sets approach that of  $IR_P \cap F^0(V \cap IR_I)$ , which is the graph of the function:

$$U_P = \begin{cases} 2 + \sqrt{1/4} \sqrt{2 - U_I} & 1 \leq U_I \leq 17/16 \\ \sqrt{17/4} \sqrt{3 - U_I} & 17/16 \leq U_I \leq 3 \end{cases}$$

The following figure illustrates the Pareto-frontiers of the feasible set and of the limit equilibrium set in the prisoner's dilemma, for three different patience ratios:

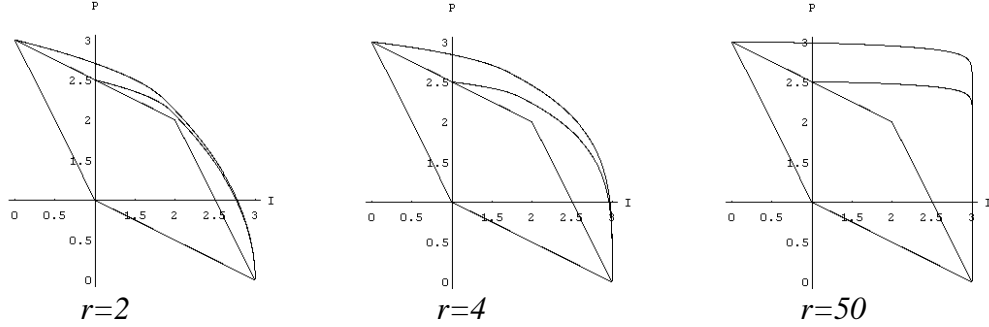


Figure 3: Feasible sets and equilibrium sets (Nash or SP) in the “prisoner's dilemma.”

We now extend Theorem 1 to the other three directions. Recall that, along a Pareto-optimal path corresponding to  $(\alpha_I, \alpha_P) \gg 0$ , the payoffs of the patient player increase, whereas those of the impatient player decrease. This property was key to the characterization of the Pareto frontier. Similar regularities characterize the paths of payoffs that generate points on the other three frontiers, as summarized in the following table:

Direction	$\alpha_I$	$\alpha_P$	Impatient Player's Payoff	Patient Player's Payoff
NE	+	+	decreasing	increasing
NW	-	+	increasing	increasing
SE	+	-	decreasing	decreasing
SW	-	-	increasing	decreasing

The construction of the frontier for the other directions is generalized in the following way. First, we intersect  $V$  with the IR half-planes of the players whose payoffs are decreasing along optimal paths in the given direction<sup>17</sup>. This is done because, for such players, if one stage-payoff is not individually rational, neither is the continuation payoff at that stage. Next, we apply the operator  $F^0(\cdot)$  to construct the feasible frontier for the repeated game. Finally, we intersect the resulting set with the IR half-planes of the players whose payoffs are increasing. This can be done because, for such players, if the initial present value is individually rational, so are all continuation payoffs.

Let  $F_D^\Delta(B)$  denote the frontier of  $F^\Delta(B)$  corresponding to the direction  $D$ . I.e.,  $F_{NE}^\Delta(B)$  is the Pareto-frontier,  $F_{NW}^\Delta(B)$  is the northwest frontier, etc. The above argument implies that the frontiers of the equilibrium sets converge, as  $\Delta \rightarrow 0$ , to the following curves:  $IR_p \cap F_{NE}^0(V \cap IR_I)$  (Pareto frontier),  $IR \cap F_{NW}^0(V)$  (northwest frontier),  $F_{SE}^0(V \cap IR)$  (southeast frontier) and  $IR_I \cap F_{SW}^0(V \cap IR_p)$  (southwest frontier)<sup>18</sup>.

We denote the limit equilibrium sets, i.e., the sets of payoffs that can be sustained as equilibrium outcomes when the time between stages is short enough, by  $E = \cup_{\Delta > 0} E^\Delta$  and  $SPE = \cup_{\Delta > 0} SPE^\Delta$ . Intuitively, these sets are the convex hulls of their four frontiers. More precisely, Theorem 2 states that the interiors of  $E$  and  $SPE$  coincide, and equal the interior of  $W$ , which is the convex hull of three sets: the northeast frontier, the northwest frontier, and the (convex) set  $F^0(V \cap IR)$  which includes the other two frontiers<sup>19</sup>. The proof of Theorem 2 is tedious and has no further insight beyond that of Theorem 1. It is therefore omitted.

### Theorem 2

$$\text{interior}(W) \subseteq SPE \subseteq E \subseteq W$$

where  $W = \text{convex hull}(F^0(V \cap IR), IR_p \cap F_{NE}^0(V \cap IR_I), IR \cap F_{NW}^0(V))$

**Proof:** See Lehrer and Pauzner (1997).

## 5. Concluding Remarks

### 5.1 Convergence of discount factors to 1

In order to establish a folk theorem, one needs to have discount factors close to 1. There are many converging paths of the two dimensional vector of discount factors to the vector (1,1). Which path is the appropriate one?

Our interest is in players with differential time preferences. Therefore, we need to retain the difference between the players while the discount factors converge to 1. To do so, we consider specific players, with fixed time preferences, and shorten the time between any two consecutive stages. The discount factors  $\delta_j$ , representing the present value of payoff delayed by 1 time unit, are fixed throughout. The stage discount factors, i.e., the factors that represent the difference in the valuation of payoff received at two consecutive stages, are  $\delta_j^\Delta$ . When  $\Delta$  approaches 0, both stage discount factors converge to 1.

This approach is formally equivalent to taking a path of discount factors that converges to 1 while keeping the patience ratio,  $r = \log \delta_I / \log \delta_p$ , constant. This ratio measures the relative patience of the players: the impatient player values one dollar

received after one time unit as much as the patient player values one dollar received after  $r$  time units. When the two players are very patient and have a patience ratio  $r$ , the feasible set of the game is close to the set  $F^0$  corresponding to that  $r$ .

## 5.2 The stage length and the Pareto frontier

Consider a game in which the highest payoff of the patient in  $V$  is not in  $IR_I$ . Recall that, along a path generating a Pareto-optimal equilibrium payoff, the stage-payoffs of the impatient player are decreasing, whereas those of the patient player are increasing. Most importantly, at the tail of the path optimality dictates that the patient player receive her highest possible payoff, subject to the (binding) constraint that the payoff to the impatient player is at least  $\varepsilon$  above his IR level.

The minimal increment  $\varepsilon$  is needed in order to make deviations nonprofitable for the impatient player. It depends on the stage length  $\Delta$ , since  $\Delta$  determines how much the player can gain from a one-stage deviation. Hence,  $\Delta$  always imposes an active constraint on equilibrium payoffs; as  $\Delta$  shrinks, more stage-payoffs can be used and, as a result, the whole Pareto frontier of the equilibrium set (Nash or SP) is pushed further out. Such a tension between efficiency and incentive compatibility does not exist in the case of identical discount factors, where, once  $\Delta$  is below some threshold, the only equilibrium payoffs that are added when  $\Delta$  is reduced further are those close to the IR levels. The reason is that, when players have identical discount factors, a given Pareto-optimal equilibrium payoff can be generated by playing that point at every stage. That is, there is no need to use payoffs that are close to the IR level.

## 5.3 Games with incomplete information

Our analysis is confined to games with complete information, where both players know the stage-game played. The case of incomplete information is significantly different. The following example shows that even when the stage-game is zero-sum, the repeated game may have equilibria in which the sum of payoffs is not zero.

Nature chooses, with equal probabilities, one of the following games:

1,-1	0,0
0,0	0,0

0,0	0,0
0,0	1,-1

The stage-game chosen is repeatedly played by two players, patient and impatient, with discount factors close to 1 and 0, respectively. The impatient player is informed about the game chosen, while the patient player knows only the probabilities. The following



strategies form an equilibrium. At the first stage, the impatient player plays top or bottom according to the game played, and the patient player mixes between left and right with equal probabilities. The players receive expected stage-payoffs of .5 and -.5. At this point the impatient player's information is revealed, and the patient player secures a continuation payoff of 0. Since the discount factors are nearly 0 and 1, this equilibrium generates repeated-game payoffs close to .5 and 0, the sum of which is not zero.

The reason why this phenomenon may occur is that when a player acts upon her information, this information is partially revealed and the remaining game is no longer the same as the original one. For more details, see Lehrer and Yariv (1995).

#### 5.4 n-player games

Section 4 provides a characterization of the equilibrium payoffs set only for two-player games. Our analysis relies on the fact that, for any given  $\alpha \in \mathfrak{R}^2$ , the path generating an extreme point follows a simple one-dimensional curve (e.g., the Pareto frontier of  $V$ ). Along the path, the players' payoff streams are monotone. In contrast, when there are more than two players, the sequence of extreme points of  $V$  corresponding to an  $\alpha$ -optimal path ( $\alpha \in \mathfrak{R}^n$ ) does not have any monotonicity property. Consider, for example, a three-player game with only two possible (pure) payoff combinations:  $a=(1,-1,1)$  and  $b=(-1,1,-1)$  (players are ordered by their degree of patience). Some Pareto-optimal paths consist of playing "a" during an initial period, switching to "b" for an intermediate time, and coordinating again on "a" for the rest of the game. This generates payoff streams that are not monotone. Therefore, our method fails in the n-player case. We leave open the problem of characterizing the equilibrium payoffs in the general case.<sup>20</sup>

## Appendix: proofs

### Proof of Proposition 1:

**Part a:** Since  $(-\log \delta) \int_{k\Delta}^{(k+1)\Delta} \delta^t dt = (1 - \delta^\Delta) \delta^{k\Delta}$ , every payoff in  $F^\Delta(B)$  can be achieved in (4) by setting  $(X(t), Y(t))$  to be constant over intervals of the form  $[k\Delta, (k+1)\Delta)$ : Let  $((\tilde{X}(1), \tilde{Y}(1)), (\tilde{X}(2), \tilde{Y}(2)), \dots)$  be the discrete path that generates some point in  $F^\Delta(B)$  when evaluated by  $U^\Delta$ . The path:  $((X(t), Y(t)) = ((\tilde{X}(k), \tilde{Y}(k))$  whenever  $k\Delta \leq t < (k+1)\Delta$ , sustains the same point when evaluated by  $U^0$ .

**Part b:** Since  $F^\Delta(B)$  is convex it is sufficient to show that the frontier of  $F^0(B)$  can be uniformly approximated by points in  $F^\Delta(B)$ . Thus, let  $f$  be a payoff vector on the frontier of  $F^0(B)$ . As explained in Section 3.2, there is a path  $((X(t), Y(t))$ , which is constant over the time intervals  $[0, t_1], [t_1, t_2], \dots, [t_\ell, \infty)$ , that generates  $f$ . ( $\ell$  is, at most, the number of vertices on the corresponding frontier of  $V$ .)

For a given  $\Delta$  define the discrete path  $((\tilde{X}(k), \tilde{Y}(k)) = ((X(k\Delta), Y(k\Delta)))$ . As in part (a), we extend the discrete path into a continuous-time path,  $(\hat{X}(t), \hat{Y}(t))$ , by setting it constant over intervals of the form  $[k\Delta, (k+1)\Delta)$ . As before,  $((\tilde{X}(k), \tilde{Y}(k))$ , evaluated by  $U^\Delta$  and  $(\hat{X}(t), \hat{Y}(t))$ , evaluated by  $U^0$ , yield the same payoff.

The paths  $\hat{X}(\cdot)$  and  $X(\cdot)$  differ from each other on at most  $\ell$  intervals  $[k\Delta, (k+1)\Delta)$ , because there are  $\ell$  times when  $X(\cdot)$  changes its value. Player  $j$  assigns a weight of  $(1 - \delta_j^\Delta) \delta_j^{k\Delta}$  to each interval. Therefore, the difference between player  $j$ 's payoffs from the two paths is bounded by  $M\ell(1 - \delta_j^\Delta)$  (where  $M$  is the maximal difference between stage payoffs). This bound tends to 0 as  $\Delta$  goes to 0. Thus, any point on the frontier of  $F^0(B)$  can be approximated by points in  $F^\Delta(B)$  when  $\Delta$  is sufficiently small.

We now show that the approximation is uniform. Suppose to the contrary that there is some  $\varepsilon > 0$ , a sequence  $\Delta_n \rightarrow 0$  and a sequence  $f_n \in F^0(B)$  such that for every  $n$  and  $f \in F^{\Delta_n}(B)$ , the distance between  $f$  and  $f_n$  is greater than  $\varepsilon$ . Due to compactness of  $F^0(B)$  we may assume (taking a converging subsequence if needed) that  $f_n$  converges to, say,  $f_0$ . We have shown that when  $\Delta_n$  is sufficiently small, there exists a point  $f$  in  $F^{\Delta_n}(B)$  whose distance to  $f_0$  is less than  $\varepsilon/2$ . As  $f_n$  converges to  $f_0$ , the distance between  $f$  and  $f_n$  is less than  $\varepsilon$  when  $n$  is large. This is a contradiction.

**Part c:** Let  $d = \varepsilon/2$  and let  $\{(X(k), Y(k))\}_{k=0}^\infty$  be the path that generates  $(X, Y) = f \in F^\Delta(B)$ . Since  $B'$  is close to  $B$ , there is a path  $\{(X'(k), Y'(k))\}_{k=0}^\infty$  of payoffs in  $B'$  such that for all  $k$ ,  $|X'(k) - X(k)| < d$  and  $|Y'(k) - Y(k)| < d$ . Since the repeated-game payoff to a player from a path of stage-payoffs is a weighted average of the stage-payoffs,

the repeated-game payoffs  $(X', Y')$  generated by the new path satisfy  $|X' - X| < d$  and  $|Y' - Y| < d$ . This means the point  $(X', Y') = f' \in f^\Delta(B')$  satisfies  $\|f' - f\| < 2d = \varepsilon$ .

**Part d:** Trivial. ■

**Remark:** Notice that although the sets  $F^\Delta(B)$  converge to the larger set  $F^0(B)$  as  $\Delta \rightarrow 0$ , this convergence is not necessarily monotone: for fixed  $\Delta_1 < \Delta_2$ , unless  $\Delta_1$  is an integer fraction of  $\Delta_2$ ,  $F^{\Delta_2}(B)$  is not always a subset of  $F^{\Delta_1}(B)$ .

**Proof of Lemma 1:**

Assume that  $k_2 > k_1$  and consider the following modification of the path:

$$(\hat{X}(k), \hat{Y}(k)) = (X(k), Y(k)) \text{ for } k \neq k_1, k_2$$

$$(\hat{X}(k_2), \hat{Y}(k_2)) = (X(k_1), Y(k_1))$$

$$(\hat{X}(k_1), \hat{Y}(k_1)) = (1 - \delta_I^{(k_2 - k_1)\Delta})(X(k_1), Y(k_1)) + \delta_I^{(k_2 - k_1)\Delta}(X(k_2), Y(k_2))$$

The impatient player's valuation of the path is unchanged, while the patient's one changes by  $d = \delta_P^{k\Delta}(\delta_P^{(k_2 - k_1)\Delta} - \delta_I^{(k_2 - k_1)\Delta})(Y(k_1) - Y(k_2))$ . If the path is already a solution to the maximization problem, the proposed modification must not increase the optimal value. In particular,  $\alpha_P d$  has to be non-positive. Since  $\delta_P^{k\Delta}(\delta_P^{(k_2 - k_1)\Delta} - \delta_I^{(k_2 - k_1)\Delta}) > 0$ , we must have  $\alpha_P(Y(k_1) - Y(k_2)) \leq 0$ .

A slightly different modification of the path yields  $\alpha_I(X(k_1) - X(k_2)) \geq 0$ . ■

**Proof of Lemma 3:**

Assume to the contrary that there exists  $\hat{k}$  such that  $X(\hat{k}) < ir_I$ . Let  $k_0$  be the first  $k_0 \geq \hat{k}$  such that  $X(k_0 + 1) > X(k_0)$  (such  $k_0$  must exist since  $U_I^\Delta(\hat{k}) \geq ir_I$ , and  $U_I^\Delta(\hat{k})$  is a weighted average of  $X(\hat{k}), X(\hat{k} + 1), \dots$ ). Since  $X(\hat{k}) < ir_I$ , also  $X(k_0) < ir_I$ . However, since  $U_I^\Delta(k_0) \geq ir_I$ , we must have  $U_I^\Delta(k_0 + 1) > ir_I$  (recall that  $U_I^\Delta(k_0)$  is the weighted average  $(1 - \delta_I^\Delta)X(k_0) + \delta_I^\Delta U_I^\Delta(k_0 + 1)$ ). Choose  $\varepsilon > 0$  s.t.  $U_I^\Delta(k_0 + 1) > ir_I + \varepsilon M$ , where  $M$  is the maximal difference between the impatient player's payoff in  $V$ .

Consider now the following modification of the path,

$$(\hat{X}(k), \hat{Y}(k)) = (X(k), Y(k)) \text{ for } k \neq k_0, k_0 + 1$$

$$(\hat{X}(k_0 + 1), \hat{Y}(k_0 + 1)) = (1 - \varepsilon)(X(k_0 + 1), Y(k_0 + 1)) + \varepsilon(X(k_0), Y(k_0))$$

$$(\hat{X}(k_0), \hat{Y}(k_0)) = (1 - \varepsilon)(X(k_0), Y(k_0)) + \varepsilon[(1 - \delta_P^\Delta)(X(k_0), Y(k_0)) + \delta_P^\Delta(X(k_0 + 1), Y(k_0 + 1))]$$

In the new path,  $U_P^\Delta(0)$  is unchanged, while  $U_I^\Delta(0)$  is increased by  $d = \varepsilon \delta_I^{k_0\Delta}(\delta_P^\Delta - \delta_I^\Delta)(X(k_0 + 1) - X(k_0))$ . Since  $X(k_0 + 1) > X(k_0)$  we have  $d > 0$ , and thus  $\alpha_I d > 0$ . To show that this contradicts the assumption that the original path is  $\alpha$ -optimal, we only need to show that all continuation payoffs along the path  $\{(\hat{X}(k), \hat{Y}(k))\}_{k=0}^\infty$  are individually rational.

The impatient player's constraints are satisfied since for  $k \geq k_0 + 2$  his payoffs are unchanged, for  $k \leq k_0$   $U_I^\Delta(k)$  are increased, and  $U_I^\Delta(k_0 + 1)$  is at least  $ir_I$  (because it has been reduced by no more than  $\varepsilon \cdot M$ ). As for the patient player, her tail payoffs at stages  $k \neq k_0 + 1$  are unchanged. Verifying that  $U_P^\Delta(k_0 + 1) \geq ir_P$  is trivial in the case where  $Y(k_0 + 1) \leq Y(k_0)$ . To prove that this is also the case when  $Y(k_0 + 1) > Y(k_0)$ , we show that  $U_P^\Delta(k_0 + 1)$  of the original path is *strictly* greater than  $ir_P$ , and choose small enough  $\varepsilon$  to guarantee that both  $U_P^\Delta(k_0 + 1)$  and  $U_I^\Delta(k_0 + 1)$  remain individually rational after the modification.

Assume to the contrary that  $U_P^\Delta(k_0 + 1) = ir_P$ . Since  $U_P^\Delta(k_0 + 2) \geq ir_P$ , we have  $Y(k_0 + 1) \leq ir_P$  (recall that  $U_P^\Delta(k_0 + 1)$  is a weighted average of the two). Since  $Y(k_0 + 1) > Y(k_0)$  we obtain  $Y(k_0) < ir_P$ , which implies that  $U_P^\Delta(k_0) < ir_P$ . This violates the constraints and therefore  $U_P^\Delta(k_0 + 1) > ir_P$ . ■

## References

- Abreu, D., P. Dutta and J. Smith (1994), "The folk theorem for Repeated Games: A NEU Condition", *Econometrica*, 62, 939-948
- Aoyagi, M. (1993), "Reputation and Dynamic Stackelberg Leadership with Random Public Signals", mimeo.
- Aumann, R. (1981), "Survey of Repeated Games", in *Essays in Game Theory and Mathematical Economics in Honor of Oscar Morgenstern*, Mannheim Bibliographisches Institut, 11-42.
- Aumann, R. and L. Shapley (1976), "Long Term Competition -- A Game Theoretic Analysis", mimeo.
- Celentani, M., D. Fudenberg, D. Levine and W. Pesendorfer (1995), "Maintaining a Reputation Against a Long Lived Opponent", *Econometrica*, 64, 691-704.
- Fudenberg, D. and D. Levine (1989), "Reputation and Equilibrium Selection in Games with a Patient Player", *Econometrica*, 57, 579-578
- Fudenberg, D. and E. Maskin (1986), "The Folk Theorem in Repeated Games with Discounting or without Incomplete Information", *Econometrica*, 54, 533-554.
- Fudenberg, D. and E. Maskin (1991), "On the dispensability of Public Randomization in Discounted Repeated games", *Journal of Economic theory*, 53, 428-438.
- Lehrer, E. and A. Pauzner (1997), "Repeated Games with Differential Time Preferences: Appendix for Referees (Proofs of Proposition 1 and Theorem 2)", Mimeo.
- Lehrer, E. and A. Pauzner (1996), "Repeated Games with Differential Time Preferences: The Case of More than Two Players", Mimeo.
- Lehrer, E. and L. Yariv (1995), "Repeated Games with Incomplete Information on One Side: The Case of Different Discounting Factors", Mimeo.
- Osborne, M. and A. Rubinstein (1994), *A Course in Game Theory*, MIT Press.
- Rubinstein, A. (1979), "Equilibrium in Supergames with the Overtaking Criterion", *Journal of Economic Theory*, 39, 83-96.
- Rubinstein, A. (1982), "Perfect Equilibrium in a Bargaining Model", *Econometrica*, 50, 97-109.

## Footnotes:

---

<sup>1</sup> See, for instance, Aumann and Shapley (1976), Rubinstein (1976), Aumann (1981), Fudenberg and Maskin (1986), Abreu, Dutta, and Smith (1994).

<sup>2</sup> This observation has apparently been made by several authors. See, for instance, Rubinstein and Osborne (1994), Exercise 139.1.

<sup>3</sup> See Section 5.1 for a formal definition.

<sup>4</sup> The stage-payoffs pair  $(7/3, 4/3)$  is generated by mixing  $2/3$ - $1/3$  between UL and DL (i.e., the patient player plays L and the impatient mixes between U and D). Similarly, mixing between UL and UR generates the payoffs pair  $(4/3, 7/3)$ .

<sup>5</sup> The assumption that the players can use a public randomization device is almost without loss of generality. Fudenberg and Maskin (1991) show explicitly that any correlated mixed action can be approximated by alternating between pure actions with the appropriate frequency if the players are very patient.

<sup>6</sup> Notice that as  $\Delta$  approaches zero,  $\delta^\Delta$  approaches 1. For details, see Section 5.1.

<sup>7</sup> These continuous time payoffs are obtained by taking the limit, as  $\Delta \rightarrow 0$ , of the first order element in the Taylor expansion of  $1 - \delta^\Delta$ . This yields  $-\log \delta$ .

<sup>8</sup> The boundary of  $F^\Delta$  can be characterized in other ways as well. For instance, one may maximize  $U_p$  subject to  $U_l \geq u$  for every  $u$ . However, this maximization problem would not readily decompose across periods.

<sup>9</sup> A centered dot,  $\cdot$ , denotes the inner product of two vectors.

<sup>10</sup> More formally, for a given facet of  $V$  (whose vertices are  $(x, y)$  and  $(x', y')$ ), let  $\alpha(k)$  be the value of  $\alpha$  such that  $h_{\alpha(k)}(k)$  is perpendicular to that facet. That is,  $h_{\alpha(k)}(k) \cdot (x, y) = h_{\alpha(k)}(k) \cdot (x', y')$ . When solving for the optimal path corresponding to  $\alpha(k)$ , we may let  $(x(k), y(k))$  be any convex combination of  $(x, y)$  and  $(x', y')$ . If we let  $(U_l, U_p)$  and  $(U'_l, U'_p)$  denote the pairs of discounted payoffs corresponding to  $(x, y)$  and  $(x', y')$ , then  $U_l - U'_l = (1 - \delta_l^\Delta) \delta_l^{k\Delta} (x - x')$  and  $U_p - U'_p = (1 - \delta_p^\Delta) \delta_p^{k\Delta} (y - y')$ . This shows that  $\alpha_l(k)(U_l - U'_l) + \alpha_p(k)(U_p - U'_p) = h_{\alpha(k)}(k) \cdot (x - x', y - y') = 0$ , and that the length of the line segment connecting  $(U_l, U_p)$  and  $(U'_l, U'_p)$  decreases with  $k$ .

<sup>11</sup> Typically,  $F^\Delta$  is strictly inside  $F^0$ . This is so because points on the boundary of both,  $F^\Delta$  and  $F^0$ , are generated by optimal divisions of the time axis between playing different vertices of  $V$ . In the case of  $F^\Delta$ , these divisions are constrained to integer multiples of  $\Delta$ . Since this constraint is typically binding,  $F^0$  typically exceeds  $F^\Delta$  (even the vertices of  $F^\Delta$

do not touch  $F^0$ ). An exception is the case where the (Pareto) frontier of  $V$  has only two vertices. In this case, all the vertices on the (Pareto) frontier of  $F^\Delta$  are on  $F^0$ .

<sup>12</sup> Notice that  $\mu_l = x_l$ , i.e., the formula covers the entire range  $U_l \in [x_l, x_0]$ .

<sup>13</sup> As  $r$  goes to infinity,  $r/(1-r)$  goes to 1 and  $1/r$  goes to 0. For any fixed  $U_l$ ,  $S_m$  tends to  $y_l - y_m$ . Thus,  $U_p \approx y_m + 1 \cdot (y_l - y_m) = y_l$ .

<sup>14</sup> Convexity follows from the assumption that the players can use a continuous correlating device. Closedness relies on the fact that the correlating device can be replaced by one that generates only a finite number of signals. For the proofs, see Lehrer and Pauzner (1997).

<sup>15</sup> Notice that the lower bound,  $IR_p^\varepsilon \cap F^\Delta(V \cap IR_l^\varepsilon)$ , may be empty for every  $\varepsilon > 0$ . In this case, the upper bound  $IR_p \cap F^\Delta(V \cap IR_l)$  must consist of points on the IR level of one of the players. For instance, in zero-sum games, the upper bound is the singleton set  $\{(i_r, i_p)\}$ . Since the equilibrium set is not empty, it contains exactly that point.

<sup>15</sup> To see why, recall that any Pareto-optimal point (in any direction  $\alpha$ ) is generated by a path that ends at the vertex that gives the patient player her highest stage payoff (see page 6). Eliminating this vertex strictly reduces the optimum in any direction. If some point on the Pareto frontier of  $V$  is outside  $IR_l$ , the highest vertex is eliminated when  $V$  is intersected with  $IR_l$ . Therefore, the Pareto frontier of  $F^\Delta(V \cap IR_l)$  is strictly dominated by that of  $F^\Delta(V)$ .

<sup>17</sup> There may be 0, 1 or 2 such players. If the number is 0, nothing is done at this step.

<sup>18</sup> The southwest frontier of the equilibrium set is simply that of  $V \cap IR$ . The reason is that the southwest frontier of  $V$  is never in the interior of  $IR$ . This is because  $V$  must have at least one point (weakly) Pareto-dominated by  $(i_r, i_p)$ . Such a point is obtained, for instance, when both players are minimaxing each other.

<sup>19</sup> To see why  $IR_l \cap F_{sw}^0(V \cap IR_p) \subset F^0(V \cap IR)$ , refer to footnote 18.

<sup>20</sup> In Lehrer and Pauzner (1996), some partial results concerning the n-player case are obtained. Briefly, it is shown that there always exist equilibrium payoffs that Pareto-dominate payoffs on the Pareto frontier of  $V$ , unless (1) there is a mutual minmax action profile and (2) the corresponding payoff is on the Pareto frontier of  $V$ . Moreover, in the case where all the Pareto optimal points in  $V$  are strongly individually rational, any Pareto optimal point in  $F^\Delta$  is an equilibrium payoff when  $\Delta$  is small enough.