ON THE TIME AND DIRECTION OF STOCHASTIC BIFURCATION

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1. Introduction and main results.

This paper is a mathematical companion to an article introducing a new economics model, by Burdzy, Frankel and Pauzner (1997). The motivation of this paper is applied, but the results may have some mathematical interest in their own right. Our model, i.e., equation (1.1) below, does not seem to be known in literature. Despite its simplicity, it generated some interesting and non-trivial mathematical questions.

In this paper, we limit ourselves to mathematical results; those interested in their economic motivation should consult Burdzy, Frankel and Pauzner (1997). To make this easier, the two papers have been written using comparable notation. A related paper by Bass and Burdzy (1997) will analyze a simplified version of our model and derive a number of new results of a purely mathematical nature.

We will first prove existence and uniqueness for differential equations of the form (1.1) below. These equations involve Brownian motion but they do not fall into the category of classical "stochastic differential equations" as they do not involve the Itô theory of integration. Typical solutions of these equations are Lipschitz functions rather than semi-martingales. It turns out that the excursion theory for Markov processes is the appropriate probabilistic tool for treatment of this family of equations.

We also establish several properties of the "bifurcation time," to be defined below. We prove that the bifurcation time for (1.1) goes to 0 as the random noise becomes smaller and smaller. More importantly, we determine the asymptotic values for probabilities of upward and downward bifurcations.

The simplicity of equation (1.1) is misleading. If the process B_t in (1.1) is not a Brownian motion but a fractional Brownian motion, none of the results in Theorems 1 and 2 can be proved using the same methods, except the existence part of Theorem 1. In fact, we currently do not know any method of proving results analogous to Theorems 1 and 2 for fractional Brownian motion B_t.

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We say that Brownian motion B_t has drift μ and variance σ^2 if $E(B_t - B_0) = \mu t$ and $E(B_t - B_0 - \mu t)^2 = \sigma^2 t$. Recall that a function g is called Lipschitz with constant c if $|g(t) - g(s)| \leq c|t-s|$ for all t and s.

Theorem 1. *Let B_t be a Brownian motion with drift μ and variance σ^2. Suppose that $k > 0$, $x_0 \in (0,1)$ and b_0 are fixed real numbers, and $B_0 = b_0$, a.s. Assume that f is a decreasing Lipschitz function with constant c_1. Consider the following differential equation:*

$$dX/dt = \begin{cases} k(1-X_t) & \text{if } B_t > f(X_t), \\ -kX_t & \text{if } B_t < f(X_t). \end{cases} \tag{1.1}$$

For almost every path B_t there exists a unique Lipschitz solution $\{X_t, t \geq 0\}$ of (1.1) which starts from x_0.

Note that dX/dt is not specified by (1.1) for t such that $B_t = f(X_t)$. A typical solution X_t does not have a derivative for such t. If f^{-1} exists and is a Lipschitz function then the function $X_t = f^{-1}(B_t)$ is a continuous solution of (1.1) but it is not Lipschitz. Hence, uniqueness holds only for Lipschitz solutions. We note that k is a Lipschitz constant for the Lipschitz solution of (1.1).

It will be shown in Bass and Burdzy (1997) that existence and uniqueness for solutions to another, related differential equation can be proved using the results of Engelbert and Schmidt (see Karatzas and Shreve (1988) Section 5.5). We have not been able to apply that theory to (1.1). However, our proof of Theorem 1 seems to be more intuitive than an application of general theorems from stochastic calculus.

The following simple results are quite useful.

Lemma 1. *Suppose that X_t^1 and X_t^2 are Lipschitz solutions to equation (1.1) corresponding to non-increasing Lipshitz functions f_1 and f_2 which satisfy $f_1(x) \leq f_2(x)$ for all x. We assume that the solutions X_t^1 and X_t^2 are defined relative to the same Brownian motion B_t. We also assume that $X_0^1 \geq X_0^2$. Then $X_t^1 \geq X_t^2$ for all $t \geq 0$ a.s.*

Lemma 2. *Suppose that X_t is the solution of (1.1). Let $\widetilde{X}_t^{b,x}$ be the solution to (1.1) starting from $\widetilde{X}_0^{b,x} = x_0 + x$ and corresponding to $\widetilde{B}_t = B_t + b$ (f and k remain the same in parts (i) and (ii) of the lemma).*

(i) If $b, x > 0$ then $\widetilde{X}_t^{b,0} \geq X_t$ for all $t \geq 0$ a.s. and $\widetilde{X}_t^{0,x} \geq X_t$ for all $t \geq 0$ a.s.

(ii) As b and x go to 0, the processes $\widetilde{X}_t^{b,x}$ converge a.s. to X_t. The convergence is uniform on every fixed interval $[0,T]$.

(iii) Suppose that $\{f_n : n = 1, 2, \dots\}$ are Lipschitz functions with the same Lipschitz constant. Fix some x_0, b_0 and k. For each n, let $\widehat{X}_t^n$ be the solution to (1.1) corresponding to the function f_n in place of f. If the functions f_n converge to (necessarily Lipschitz) f on every bounded interval then the solutions $\widehat{X}_t^n$ converge to X_t, the solution of (1.1) corresponding to f.

The statement of the next theorem is rather complicated in order to be directly applicable in Burdzy, Frankel and Pauzner (1997). We precede it with a simplified version of the result, to help the reader grasp its meaning.

Fix some $x_0 \in (0,1)$ and suppose that f is a decreasing Lipschitz function with $f'(x_0) \neq 0$. Fix some μ and σ^2. Let B_t^k be a Brownian motion with drift μ/k, variance σ^2/k, and starting from $B_0^k = f(x_0)$ for every k. Let X_t^k be the solution of

$$dX^k/dt = \begin{cases} (1 - X_t^k) & \text{if } B_t^k > f(X_t^k), \\ -X_t^k & \text{if } B_t^k < f(X_t^k). \end{cases}$$

Fix arbitrarily small $c_0 \in (0, \min(x_0, 1 - x_0))$, and let $T_1^k = \inf\{t > 0 : X_t^k \notin (c_0, 1 - c_0)\}$ and $T_0^k = \sup\{t < T_1^k : B_t^k = f(X_t^k)\}$. Then as $k \to \infty$ (as the variance and drift of $B_\cdot^k$ go to 0), the random bifurcation times T_0^k go to 0 in distribution. Moreover, the probability of a "positive" bifurcation (defined as $dX^k/dt > 0$ for all $t \in (T_0^k, T_1^k)$) converges to $1 - x_0$ as $k \to \infty$. The probability of a "negative" bifurcation goes to x_0. Theorem 2 proves a result that is even stronger, since the function f and other parameters can vary with k.

Theorem 2. *For each $k > 0$, let B_t^k be a Brownian motion with drift μ_k and variance σ_k^2, where $\lim_{k\to\infty} \mu_k = \lim_{k\to\infty} \sigma_k^2 = 0$. Assume that for each k, we have a continuously differentiable decreasing function f_k. Suppose that $x_0^k \in [0,1]$ are numbers which converge to some fixed $x_0 \in (0,1)$ as $k \to \infty$. Assume that $\lim_{k\to\infty} f_k'(x_0) \neq 0$, and the derivatives are asymptotically uniformly continuous at x_0, i.e., for every $\varepsilon > 0$ there exist $k_0 < \infty$ and $\delta > 0$ such that $|f_k'(x) - f_k'(x_0)| < \varepsilon$ for all $x \in [x_0 - \delta, x_0 + \delta]$ and all $k > k_0$. Let X_t^k be the solution to the following differential equation, with $B_0^k = f_k(x_0^k)$, a.s.,*

$$dX_t^k/dt = \begin{cases} \lambda_k(1 - X_t^k) & \text{if } B_t^k > f_k(X_t^k), \\ -\widehat{\lambda}_k X_t^k & \text{if } B_t^k < f_k(X_t^k), \end{cases} \tag{1.2}$$

where $\lim_{k\to\infty} \lambda_k = \lambda \in (0,\infty)$ and $\lim_{k\to\infty} \widehat{\lambda}_k = \widehat{\lambda} \in (0,\infty)$.

Fix arbitrarily small $c_0 \in (0, \min(x_0, 1 - x_0))$, and let $T_1^k = \inf\{t > 0 : X_t^k \notin (c_0, 1 - c_0)\}$ and $T_0^k = \sup\{t < T_1^k : B_t^k = f_k(X_t^k)\}$.

(i) The random times T_0^k converge to 0 in distribution as $k \to \infty$.

(ii) The probability that $dX^k/dt > 0$ for all $t \in (T_0^k, T_1^k)$ converges to $\frac{\lambda(1-x_0)}{\lambda(1-x_0)+\widehat{\lambda}x_0}$ as $k \to \infty$. Consequently, the probability that the derivative of X^k is negative on the same interval converges to $\frac{\widehat{\lambda}x_0}{\lambda(1-x_0)+\widehat{\lambda}x_0}$.

For brevity, Theorem 2 is stated only for the case $x_0 \in (0,1)$. It also holds when $x_0 \in \{0,1\}$, with a slight change in the definition of T_1^k. Fix any arbitrarily small $c_0 > 0$. If $x_0 = 0$, let $T_1^k = \inf\{t > 0 : X_t^k \geq 1 - c_0\}$. If $x_0 = 1$, let $T_1^k = \inf\{t > 0 : X_t^k \leq c_0\}$. (The definition of T_0^k is unchanged.)

In Theorem 1, we prove the existence and uniqueness of the solutions to (1.1). The analogous results hold for the solutions of (1.2), without assuming that $\lambda_k = \widehat{\lambda}_k$. We omit the proof, which is analogous to that of Theorem 1. Note that Lemmas 1 and 2 hold as well if (1.2) is substituted for (1.1).

We note that Theorem 2 remains true if one (but only one) of the constants λ or $\widehat{\lambda}$ is equal to zero. The proof of such modified theorem does not require any conceptual changes.

Corollary 1 below shows that the results of Lemmas 1 and 2 and Theorems 1 and 2 continue to hold if the trend in the Brownian motion is a more general function of t and B_t.

Corollary 1. *Let the functions $\eta(b)$ and $\nu(t,b)$ be Lipschitz in all arguments, and assume that $|\nu(t,b)| < \eta(b)$ for all b and t. Let the diffusion process $\widehat{B}$ be defined by $d\widehat{B}_t = d\theta_t + \mu\nu(t,\widehat{B}_t)dt$, where θ is a Brownian motion with variance σ^2 and zero drift. Then Theorem 1 and Lemmas 1 and 2 hold for $\widehat{B}$ in place of B. Theorem 2 remains true if we replace B^k with the diffusion process $\widehat{B}^k$, which has variance σ_k^2 and drift $\mu_k\nu(t,\widehat{B}_t^k)$ and where σ_k^2 and μ_k satisfy the same properties as in the statement of Theorem 2.*

Consider a right-continuous process $(A_t)_{t\geq 0}$. We will say that $(A_t)_{t\geq 0}$ *has i.i.d. jumps* if: (1) for some random times $\{t_i\}_{i\geq 0}$, $t_0 = 0$, the process A_t is constant on every interval $[t_{i-1}, t_i)$; and (2) the random vectors $(t_i - t_{i-1}, A_{t_i} - A_{t_i-})$ are independent and identically distributed.

Consider a process A_t with i.i.d. jumps and $A_0 = b_0$ and the following differential equation,

$$dX/dt = \begin{cases} k(1-X_t) & \text{if } A_t > f(X_t), \\ -kX_t & \text{if } A_t < f(X_t). \end{cases} \tag{1.3}$$

Since A_t is constant almost everywhere, there need not be a unique Lipschitz solution to (1.3). Let $\overline{X}_t$ and $\underline{X}_t$ be the maximal and the minimal Lipschitz solutions; their existence can be shown using the approach given in the proof of Theorem 1.

Fix a constant $c > 0$ and let $g(b, x)$ be any Lipschitz function. Let A be either a Brownian motion or a process with i.i.d. jumps and

$$\overline{\Phi}(x, b; A, f) = E\left[\int_{t=0}^{\infty} e^{-ct} g(A_t, \overline{X}_t) dt \;\mid\; (A_0, \overline{X}_0) = (b, x)\right],$$
$$\underline{\Phi}(x, b; A, f) = E\left[\int_{t=0}^{\infty} e^{-ct} g(A_t, \underline{X}_t) dt \;\mid\; (A_0, \underline{X}_0) = (b, x)\right].$$

Suppose that A_t^j is a sequence of processes with i.i.d. jumps such that the distribution of $(j \cdot (t_i - t_{i-1}), \sqrt{j} \cdot (A_{t_i} - A_{t_i -}))$ is the same for all i and j, with the mean $(1, 0)$ and the variance of the second component equal to 1. Then it is standard to show (see Billingsley (1968)) that the processes A_t^j converge in distribution to the Brownian motion with mean 0 and variance 1.

Proposition 1. *Let $\{A_t^i\}_{i \geq 1}$ be a sequence of processes with i.i.d. jumps that converges in distribution to a Brownian motion B as $i \to \infty$. Let f^i be a sequence of strictly decreasing Lipschitz functions that converges to f as $i \to \infty$. Suppose that (x^i, b^i) converges to (x, b). Then $\overline{\Phi}(x^i, b^i; A^i, f^i)$ and $\underline{\Phi}(x^i, b^i; A^i, f^i)$ both converge to $\overline{\Phi}(x, b; B, f) = \underline{\Phi}(x, b; B, f)$.*

2. Proofs. We will first prove Theorem 1. The proof of existence is quite elementary and perhaps it is an easy corollary of known results. We provide it here for completeness. We will give a quite intuitive proof of uniqueness instead of trying to derive uniqueness from general results on stochastic differential equations.

Proof of Theorem 1. We will give the proof for the case $\sigma^2 = 1$ and $\mu = 0$, i.e., the standard Brownian motion. The case of arbitrary σ^2 needs only minor adjustments. For arbitrary fixed μ, the distribution of Brownian motion without drift and the distribution of Brownian motion with drift μ are mutually absolutely continuous on any finite interval $[0, t]$. Hence, the existence and uniqueness of solutions to (1.1) follows for the case of arbitrary constant drift from the case with no drift.

The first step is to prove the existence of a solution. Consider a $\delta \in (0, k/2)$ and define a δ-approximate solution X_t^δ as follows. Suppose that $b_0 \neq f(x_0)$. It will be obvious from the proof how to deal with the case when $b_0 = f(x_0)$. Recall that almost all Brownian paths are continuous. This easily implies that the condition $X_0^\delta = x_0$ and the equation (1.1) define a continuous function X_t^δ in a unique way until the first time $t = t_1$ when $B_{t_1} = f(X_{t_1}^\delta)$. We then let $X_t^\delta = X_{t_1}^\delta + (t - t_1)k(1 - X_{t_1}^\delta)$ for all $t \in [t_1, t_1 + \delta]$. With probability 1, $B_{t_1+\delta} \neq B_{t_1} = f(X_{t_1+\delta}^\delta)$. Since $\delta < k/2$, the total increment of X_t^δ over the interval $[t_1, t_1 + \delta]$ is bounded by $(1 - X_{t_1}^\delta)/2$ and so we must have $X_t^\delta \in (0, 1)$ for all $t \leq t_1 + \delta$. We extend the function X_t^δ in a unique way so that it is continuous at time $t_1 + \delta$ and it satisfies (1.1) for all $t > t_1 + \delta$ until the first time $t_2 > t_1 + \delta$ such that $B_{t_2} = f(X_{t_2}^\delta)$. We let $X_t^\delta = X_{t_2}^\delta + (t - t_2)k(1 - X_{t_2}^\delta)$ for all $t \in [t_2, t_2 + \delta]$. If we continue in this way, we will define a δ-approximate solution for all $t \geq 0$ since after every time t_j when the functions B_t and $f(X_t^\delta)$ are equal, we extend the solution for δ units of time. We note that the sequence of times t_j will be infinite a.s. but we do not need this property in our proof. The function X_t^δ takes values in $(0, 1)$ for all t, by the same arguement that showed this for $t \leq t_1 + \delta$. Note that the δ-approximate solution satisfies (1.1) for all $t \in (t_j + \delta, t_{j+1})$, for all j, and it is continuous for all t. Hence, the derivative of X_t^δ is defined almost everywhere and its absolute value is bounded by k, in view of (1.1). It follows that the δ-approximate solution is a Lipschitz function with the Lipschitz constant k. Next, let $\delta = 1/m$, and for every integer $m \geq 1$, consider a $1/m$-approximate solution. Let X_t be the essential supremum of $X_t^{1/m}$, i.e.,

$$X_t = \lim_{n\to\infty} \sup_{m>n} X_t^{1/m}.$$

The supremum of an arbitrary family of Lipshitz functions with constant k is a Lipshitz function with the same constant, and the same remark applies to the limit of a sequence of such functions. Hence, X_t is a Lipschitz function with constant k.

For a fixed t, we can find a subsequence of $X_t^{1/m}$ converging to X_t. Using the diagonal method and the Lipschitz property of $X_t^{1/m}$'s, we see that there exists a subsequence of $X_t^{1/m}$ which converges to X_t uniformly on compact sets. Without loss of generality we will assume that $X_t^{1/m}$ itself converges to X_t uniformly on compacts.

We will show that X_t is a solution to (1.1). The set of t such that $B_t = f(X_t)$ is closed because both functions B_t and $f(X_t)$ are continuous. Consider any interval (s_1, s_2) such that $B_t \neq f(X_t)$ for all $t \in (s_1, s_2)$. Choose arbitrary small $\delta_1 > 0$. Let $\delta_2 = \inf_{t\in(s_1+\delta_1, s_2-\delta_1)} |B_t - f(X_t)|$. Since the $1/m$-approximate solutions converge uniformly to X_t on $[s_1, s_2]$ and f is Lipschitz, we have $|f(X_t^{1/m}) - f(X_t)| < \delta_2/2$ for all $t \in (s_1 + \delta_1, s_2 - \delta_1)$ and sufficiently large m. It follows that for such t and m, we have $f(X_t^{1/m}) \neq B_t$. Let us assume that $f(X_t^{1/m}) > B_t$ for $t \in (s_1 + \delta_1, s_2 - \delta_1)$ and large m, the other case being analogous. When m is large enough so that $1/m < \delta_1$, it follows from our construction of δ-approximate solutions and from (1.1) that $X_t^{1/m}/dt = -kX_t^{1/m}$ for $t \in (s_1+2\delta_1, s_2-\delta_1)$. Let $s_0 = s_1 + 2\delta_1$. We obtain $X_t^{1/m} = X_{s_0}^{1/m} e^{-k(t-s_0)}$ for $t \in (s_1+2\delta_1, s_2-\delta_1)$. Since $X_{s_0}^{1/m} \to X_{s_0}$, we see that $X_t = X_{s_0} e^{-k(t-s_0)}$ for $t \in (s_1+2\delta_1, s_2-\delta_1)$. Thus X_t satisfies (1.1) on $(s_1+2\delta_1, s_2-\delta_1)$ and in view of arbitrary nature of δ_1, the same claim extends to the whole interval (s_1, s_2). The argument applies to all intervals (s_1, s_2) such that $B_t \neq f(X_t)$ for all $t \in (s_1, s_2)$. This implies that X_t is a Lipschitz solution to (1.1). The proof of existence of a Lipschitz solution is complete.

Since the functions $X_t^{1/m}$ are adapted to the Brownian filtration $\mathcal{F}_t^B = \sigma(B_v, v \leq t)$, so is their essential supremum, X_t. Moreover, the process $\{(B_t, X_t), t \geq 0\}$ is strong Markov with respect to the filtration $\{\mathcal{F}_t^B, t \geq 0\}$.

We will show that X_t is the largest of all Lipschitz solutions to (1.1), i.e., if X_t^* is another Lipschitz solution then $X_t \geq X_t^*$ for all t. Consider any Lipschitz solution X_t^* to (1.1) and suppose that $X_t^* > X_t$ for some t. Then there must exist $\delta = 1/m$ such that $X_t^* > X_t^\delta$ for some t. Fix such δ and let S be the infimum of t such that $X_t^* > X_t^\delta$. If $S \in [t_j + \delta, t_{j+1})$ for some j, then $f(X_S^*) = f(X_S^\delta) \neq B_S$ a.s., and, by continuity, the same relationship extends to some non-degenerate interval to the right of S. On this interval, both X_t^* and X_t^δ satisfy one of the conditions in (1.1), so they must agree, and this contradicts the definition of S. Next suppose that $S \in [t_j, t_j + \delta)$ for some j. On this interval, the derivative of X_t^δ is equal to $k(1 - X_{t_j}^\delta)$. It is is easy to see that no Lipshitz solution to (1.1) can grow faster than that on this interval, and so $S \geq t_j + \delta$, a contradiction which completes the proof of our claim.

The solution X_t is consistent in the following sense. Consider a fixed path $\{B_t, t \geq 0\}$ and the solution X_t. Now choose any $s > 0$ and suppose that $X_s = z$. Let $\{X_u^*, u \geq s\}$ be the largest Lipschitz solution with constant k for the equation (1.1) on the interval $[s, \infty)$ with $X_s^* = z$ and the path $\{B_t, t \geq 0\}$ truncated to $\{B_t, t \geq s\}$. Then it is easy to see that $X_u^* = X_u$ for all $u \geq s$. It follows that the portion $\{X_t, t \in [s, u]\}$ of the solution to (1.1) may be defined only in terms of X_s and $\{B_t, t \in [s, u]\}$.

Let $D = \{(b, x) \in R^2 : b = f(x)\}$. We will apply the results of Maisonneuve (1975) to construct an exit system $(H^{b,x}, dL)$ for the process of excursions of (B_t, X_t) from the set D. We will briefly describe the elements of the exit system. See Blumenthal (1992), Burdzy (1987), Maisonneuve (1975), or Sharpe (1989) for various versions of excursion theory. The first element of an exit system, $H^{b,x}$'s, are excursion laws, i.e., $H^{b,x}$ is an infinite σ-finite measure defined on the space C^* of functions (e_t^B, e_t^X) defined on $(0, \infty)$

(note that 0 is excluded) which take values in $\mathbf{R}^2 \cup \{\Delta\}$. Here Δ is the coffin state. Let ν be the lifetime of an excursion, i.e., $\inf\{t > 0 : (e_t^B, e_t^X) = \Delta\}$. Then under $H^{b,x}$, we have $(e_t^B, e_t^X) \in \mathbf{R}^2$ for $t \in (0, \nu)$ and $(e_t^B, e_t^X) = \Delta$ for $t \in [\nu, \infty)$, except for the set of excursions of $H^{b,x}$-measure zero. The measure $H^{b,x}$ is strong Markov with respect to the transition probabilities of the process $\{(B_t, X_t), t \geq 0\}$ killed at the hitting time of D. Moreover, the $H^{b,x}$-measure of the set of paths for which $\lim_{t \downarrow 0}(e_t^B, e_t^X) \neq (b, x)$ is equal to 0. The second element of the exit system, dL, denotes the measure defined by a non-decreasing process L_t. The process L_t is a continuous additive functional which will be referred to as the *local time* for (B_t, X_t) on D. The process L_t does not increase on any interval (s, u) such that $(B_v, X_v) \notin D$ for $v \in (s, u)$; in other words, $L_s = L_u$ for such intervals.

Next we will study the excursion laws $H^{b,x}$. Note that we need to consider only $H^{b,x}$ with $b = f(x)$. It is clear that $e_t^B \neq f(e_t^X)$ for $t \in (0, \nu)$. Hence, X_t is governed by one and only one of the formulae given in (1.1), on the whole interval $(0, \nu)$. In either case, e_t^X is an exponential function or a linear transformation of an exponential function for the duration of an excursion. Consider the process $\{(\xi_t, e_t^X)\} := \{(e_t^B - f(e_t^X), e_t^X)\}$ under the measure $H^{b,x}$. Let $\widehat{H}^{0,x}$ be its distribution (b will be suppressed in the notation). Since $H^{b,x}$ is strong Markov with respect to the transition probabilities of the process $\{(B_t, X_t), t \geq 0\}$ killed at the hitting time of D, it follows that the distribution of ξ_t is an excursion law from 0 whose transition probabilities are those for Brownian motion with drift $-f(X_t)$ killed upon returning to 0. Let $\widetilde{H}^{0,x}$ be the excursion law on the paths $\{(\xi_t, e_t^X)\}$ such that $e_0^X = x$, the function e_t^X is governed by the same deterministic equation as in the case of $\widehat{H}^{0,x}$, and the distribution of ξ_t is an excursion law from 0 whose transition probabilities are those for Brownian motion with no drift killed upon returning to 0. The only difference between $\widetilde{H}^{0,x}$ and $\widehat{H}^{0,x}$ is the absence of the drift in the former one.

Let $S(\lambda)$ be the amount of time spent by ξ_t within $(0, \lambda)$, for small $\lambda > 0$. Since the drift $-f(X_t)$ under $\widehat{H}^{0,x}$ is bounded, it is not hard to see that the local properties of excursions under $\widehat{H}^{0,x}$ are similar to those of $\widetilde{H}^{0,x}$. In particular, we have for some $\beta_1 \in (0, \infty)$,

$$\lim_{\lambda \to 0} \widehat{H}^{0,x}(S(\lambda))/\lambda = \beta_1,$$

since a similar result is true for $\widetilde{H}^{0,x}$.

Note that $\widehat{H}^{0,x}$ is the sum of two measures; the first one is supported on paths for which ξ_t stays above 0 and the other one is supported on the paths below 0. Let $S^-(\lambda)$

be the amount of time spent by ξ_t within $(-\lambda, 0)$. By analogy,

$$\lim_{\lambda\to 0} \widehat{H}^{0,x}(S^-(\lambda))/\lambda = \beta_2,$$

where β_2 is another normalizing constant. We will show that $\beta_2 = \beta_1$; this claim requires a proof since the normalizations for the "positive" and "negative" parts of $\widehat{H}^{0,x}$, relative to $\widetilde{H}^{0,x}$, might not agree.

Suppose that T^* is any stopping time for the filtration generated by L_t, for example, we may take $\ell > 0$ and let T^* be the stopping time $\inf\{t > 0 : L_t \geq \ell\}$. Since L_t does not increase when B_t is away from $f(X_t)$, we must have $B_{T^*} = f(X_{T^*})$. In view of the Lipschitz property of f, we have $|f(X_{T^*+t}) - f(X_{T^*})| \leq c_1 kt$, and so for small t we have $|f(X_{T^*+t}) - f(X_{T^*})| \leq t^{5/6}$. Let $A_1 = A_1(s)$ be the event that the first excursion of (B_t, X_t) from the set D after the time T^* with $|e^B_t - f(e^X_t)| > t^{3/4}$ for some $t < \nu$, is such that $e^B_t > f(e^X_t)$ for $t < \nu$. Let A_2 be the analogous event with $e^B_t < f(e^X_t)$ for $t < \nu$. Let $T(a)$ be the hitting time of a for B_t and let θ be the usual Markovian shift. We have for small s,

$$\{T(B_{T^*} + s^{3/4} + s^{5/6}) \circ \theta_{T^*} < T(B_{T^*} - s^{3/4} + s^{5/6}) \circ \theta_{T^*} < T^* + s\} \subset A_1,$$

and

$$\{T(B_{T^*} - s^{3/4} - s^{5/6}) \circ \theta_{T^*} < T(B_{T^*} + s^{3/4} - s^{5/6}) \circ \theta_{T^*} < T^* + s\} \subset A_2.$$

It is easy to see that

$$\begin{aligned}
&\lim_{s\to 0} P(T(B_{T^*} + s^{3/4} + s^{5/6}) \circ \theta_{T^*} < T(B_{T^*} - s^{3/4} + s^{5/6}) \circ \theta_{T^*} < T^* + s) \\
&\quad = \lim_{s\to 0} P(T(B_{T^*} - s^{3/4} - s^{5/6}) \circ \theta_{T^*} < T(B_{T^*} + s^{3/4} - s^{5/6}) \circ \theta_{T^*} < T^* + s) = 1/2.
\end{aligned}$$

It follows that

$$\lim_{s\to 0} P(A_1(s)) = \lim_{s\to 0} P(A_2(s)) = 1/2.$$

Hence, as s goes to 0, the probability that the first excursion of ξ after T^* which hits $s^{3/4}$ will come before the excursion which hits $-s^{3/4}$ converges to $1/2$. Since T^* is an arbitrary stopping time for L_t, the normalizations for the "positive" and "negative" parts of $\widehat{H}^{0,x}$, relative to $\widetilde{H}^{0,x}$, must agree for almost all x (strictly speaking, they must agree only on the set that may be charged by dL_t but we can make them always equal without loss of generality). This completes the proof that $\beta_1 = \beta_2$.

Even though we already know that both parts of $\widehat{H}^{0,x}$ have identical normalizations relative to $\widetilde{H}^{0,x}$, the value of β_1 is not yet determined. At this point we choose the normalization so that we have

$$\lim_{\lambda\to 0} \widehat{H}^{0,x}(S(\lambda))/\lambda = \lim_{\lambda\to 0} \widehat{H}^{0,x}(S^-(\lambda))/\lambda = 1.$$

We choose the normalization of the local time to match that of the excursion laws so that we can apply the excursion system theory. The last formula implies, just like for the standard local time of Brownian motion at 0, that

$$\lim_{\lambda\downarrow 0} \frac{1}{\lambda} \int_0^t \mathbf{1}_{\{B_s - f(X_s)\in(0,\lambda)\}} ds = L_t, \tag{2.1}$$

with probability 1, for every $t \geq 0$.

Now we apply the results from excursion theory proved above to show the uniqueness of X_t. Suppose that X_t^* is another Lipschitz solution starting from the same point $X_0^* = x_0 = X_0$. First we prove that the set Q of times t such that $B_t = f(X_t^*)$ has zero Lebesgue measure. Fix arbitrarily small $a > 0$. Consider $\eta > 0$, to be specified later. Note that for any integer j,

$$|f(X^*_{j\eta+s}) - f(X^*_{j\eta})| \leq c_1 ks.$$

By conditioning on the values of $B_{j\eta}$ and $X^*_{j\eta}$, and by using Brownian scaling, we obtain for all integer $j \geq 0$,

$$P(B_{j\eta+s} \in [f(X^*_{j\eta+s}) - c_1 ks, f(X^*_{j\eta+s}) + c_1 ks]) < a,$$

provided $s < \eta$ and η is sufficiently small. Hence, the expected value of $\int_0^t \mathbf{1}_{\{B_s = f(X_s^*)\}} ds$ is less than at. The estimate holds for arbitrarily small a and so the expected value of the integral is 0. By the Fubini theorem, the set Q has zero Lebesgue measure, with probability 1. Let $\widehat{Q}$ be the set of all t such that $B_t = f(X_t^*)$ or $B_t = f(X_t)$. Clearly, the same argument shows that $\widehat{Q}$ has zero measure.

For every $s \in [0,t] \setminus \widehat{Q}$, the derivatives $\frac{\partial}{\partial s} X_s^*$ and $\frac{\partial}{\partial s} X_s$ exist and are defined by (1.1). Recall that X_t is the maximal Lipschitz solution and so $X_t \geq X_t^*$ for all t. It follows directly from (1.1) that if $B_t < f(X_t) \leq f(X_t^*)$ or $f(X_t) \leq f(X_t^*) < B_t$ then $\frac{\partial}{\partial t} X_t^* > \frac{\partial}{\partial t} X_t$. On the other hand, the condition $f(X_t) < B_t < f(X_t^*)$ implies that $\frac{\partial}{\partial t} X_t^* < \frac{\partial}{\partial t} X_t$. However, we always have $\frac{\partial}{\partial t} X_t - \frac{\partial}{\partial t} X_t^* \leq 2k$. In view of the fact that $\widehat{Q}$ has zero measure, we do not need to analyze other cases for the relative position of B_t, $f(X_t)$ and $f(X_t^*)$. We obtain

$$(X_t - X_t^*) - (X_s - X_s^*) \leq \int_s^t \mathbf{1}_{\{f(X_u) < B_u < f(X_u^*)\}} 2k du.$$

Let $U(s) = \inf\{t > 0 : L_t \geq s\}$. In view of (2.1) we can find small $\lambda > 0$ such that

$$\frac{1}{2\lambda}\int_0^{U(1/16c_1k)} \mathbf{1}_{\{B_s - f(X_s)\in(0,2\lambda)\}} ds \leq 2L_{U(1/16c_1k)} = 1/(8c_1k).$$

Let $V = U(1/(16c_1k)) \wedge \inf\{t > 0 : |f(X_t) - f(X_t^*)| \geq 2\lambda\}$. Then

$$\begin{aligned}(X_V - X_V^*) - (X_0 - X_0^*) &\leq \int_0^V \mathbf{1}_{\{f(X_u)<B_u<f(X_u^*)\}} 2k\,du \\ &\leq \int_0^V \mathbf{1}_{\{B_u - f(X_u)\in(0,2\lambda)\}} 2k\,du \\ &\leq \int_0^{U_{1/16k}} \mathbf{1}_{\{B_u - f(X_u)\in(0,2\lambda)\}} 2k\,du \\ &\leq 4\lambda k/(8c_1k) = \lambda/(2c_1).\end{aligned}$$

Since $X_V - X_V^* \leq \lambda/(2c_1)$, we must have $|f(X_t) - f(X_t^*)| \leq \lambda/2$, and so $V = U(1/16c_1k)$. Since this is true for arbitrarily small $\lambda > 0$, we conclude that $X_t = X_t^*$ for $t \leq U(1/16c_1k)$. An induction argument based on the strong Markov property applied at the stopping times $U(j/16c_1k)$, $j = 1, 2, \ldots$, shows that $X_t = X_t^*$ for $t \leq U((j+1)/16c_1k)$ and every $j \geq 1$. This implies that $X_t = X_t^*$ for all t. $\square$

Proof of Lemma 1. Assume first that $X_0^1 = x_1 > x_2 = X_0^2$. Let T be the first time t when $X_t^1 = X_t^2$. We will argue that $T = \infty$. On the interval $[0, T)$ we have $X_t^1 > X_t^2$. Hence, for any $t \in [0, T)$, except a set of measure zero, we have either

(i) $B_t < f_1(X_t^1) \leq f_1(X_t^2) \leq f_2(X_t^2)$; or
(ii) $f_1(X_t^1) \leq f_1(X_t^2) \leq f_2(X_t^2) < B_t$; or
(iii) $f_1(X_t^1) < B_t < f_2(X_t^2)$.

In cases (i) and (ii), we have

$$\frac{d(X^1 - X^2)}{dt} = -k(X_t^1 - X_t^2),$$

while in case (iii),

$$\frac{d(X^1 - X^2)}{dt} = k(1 - X_t^1) + kX_t^2 > -k(X_t^1 - X_t^2).$$

It follows that $X_t^1 - X_t^2 \geq (x_1 - x_2)e^{-kt}$ for $t < T$, and so $T = \infty$.

Now consider the case when $X_0^1 = X_0^2 = x_0$. Let X_t^n be the solution of (1.1) defined relative to f_1, the same Brownian motion B_t, and such that $X_0^n = x_0 + 1/n$. By the first part of the proof, $X_t^n \geq X_t^2$ for all t a.s. Now let n go to infinity. Let X_t^* be the limit of a subsequence of X_t^n. The limit exists for a subsequence because all functions X_t^n are Lipschitz with constant k. One can prove that X_t^* is a solution to (1.1) starting from x_0 just like in the proof of Theorem 1. By uniqueness, $X_t^* = X_t^1$. Since all the functions X_t^n are greater than or equal to X_t^2, we must have $X_t^1 \geq X_t^2$ a.s. □

Proof of Lemma 2. We will deduce part (i) from Lemma 1. The condition $\widetilde{B}_t > f(X_t)$ is equivalent to $B_t > f(X_t) - b$, and this may be rewritten as $B_t > f_1(X_t)$, where $f_1(x) = f(x) - b$. Since $f_1(x) \leq f(x)$, Lemma 1 implies that $\widetilde{X}_t^{b,0} \geq X_t$. The assertion $\widetilde{X}_t^{0,x} \geq X_t$ follows directly from Lemma 1.

For part (ii), take any sequence $\{(b_n, x_n)\}$ such that $b_n \to 0$ and $x_n \to 0$ as n goes to infinity. For a fixed t, there exists a subsequence $\{(b_{n_j}, x_{n_j})\}$ such that $\widetilde{X}_t^{b_{n_j}, x_{n_j}}$ converges. By extracting further subsequences and then using the diagonal method we can obtain a subsequence $\{(b'_n, x'_n)\}$ of the original sequence $\{(b_n, x_n)\}$ such that $\widetilde{X}_s^{b'_n, x'_n}$ converges to a limit X_s^* for every rational $s > 0$. The convergence is uniform on compact sets because all functions $\widetilde{X}_s^{b'_n, x'_n}$ are Lipschitz with constant k. We see that X_s^* must be a solution to (1.1) by the same argument as in the proof of Theorem 1. By uniqueness, $X_s^* = X_s$ for all s. Since the same is true for any initial sequence $\{(b_n, x_n)\}$, we conclude that $\widetilde{X}_t^{b,x}$ converge to X_t a.s., uniformly on compacts.

The proof of part (iii) is completely analogous to that for part (ii). One can show that for every subsequence of $\widehat{X}_t^n$, there is a further subsequence which converges and, moreover, it converges to a solution of (1.1). The argument is finished by invoking the uniqueness of the solution. □

Lemma 3. *Let excursion laws $H^{b,x}$ be defined as in the proof of Theorem 1 but relative to the solution of (1.2) in Theorem 2. Fix arbitrarily small $\varepsilon > 0$ and $x \in (0, 1-\varepsilon)$. Let A be the event that for the excursion (e_t^B, e_t^X) with lifetime ν under $H^{b,x}$, we have $\lim_{t \to \nu-} e_t^X \geq 1 - \varepsilon$. Assume that the derivatives f'_k of the functions in (1.2) are uniformly continuous, with the modulus of continuity independent of k. Moreover, we assume that $|f'_k(x)| > c_0$, where $c_0 > 0$ is independent of x and k. Suppose $b_k = f_k(x)$. There exists an absolute constant $\alpha \in (0, \infty)$ such that*

$$\lim_{k \to \infty} \frac{H^{b_k, x}(A)}{\alpha |f'_k(x)| \lambda_k (1-x)} = 1.$$

The convergence is uniform in x on every interval $(0, 1-\varepsilon_1) \subset (0, 1-\varepsilon)$.

Proof. To simplify the proof, we will consider only the case when $\lambda = \lim_{k\to\infty} \lambda_k = 1$.

Let $g_t^{x,k} = f_k(1-(1-x)\exp(-\lambda_k t))$. The function $g_t^{x,k}$ may be represented as $f_k(X_t^k)$ where X_t^k is the solution to (1.2) starting from $X_0^k = x$, and assuming that the first condition in (1.2) is always satisfied, i.e., $B_t^k > f_k(X_t^k)$ for all $t \geq 0$. The derivative of $g_t^{x,k}$ is a continuous function of t and its value at $t=0$ is $f_k'(x)\lambda_k(1-x)$. Let $T_{b,y}$ be the first time $t \geq 0$ with $B_t^k = f_k(X_t^k)$, assuming $B_0^k = b$ and $X_0^k = y$. Let τ_k be the first time t when $X_t^k \geq 1-\varepsilon$.

Fix an arbitrarily small $\xi > 0$. We will show that there exists $\eta > 0$ such that when $|y-x| \leq \eta$, $k > 1/\eta$, and $\sigma_k^6 < b - f_k(y) < \sigma_k^4$, then

$$(1-\xi)(b-f_k(y))\beta \leq P(T_{b,y} > \tau_k) \leq (1+\xi)(b-f_k(y))\beta, \tag{2.2}$$

where

$$\beta = \beta(k,x,\sigma^2) = 2|f_k'(x)|(1-x)/\sigma_k^2.$$

Consider a small $\zeta > 0$ whose value will be chosen later in the proof. Assume that $|y-x|$ is sufficiently small so that one can find small $u > 0$ (not depending on k) such that for any $t_1 \in (0,u)$,

$$\begin{aligned}\left|\frac{\partial}{\partial t} g_t^{y,k}|_{t=t_1}\right| &< \left|(1+\zeta/2)\frac{\partial}{\partial t} g_t^{y,k}|_{t=0}\right| < \left|(1+\zeta)\frac{\partial}{\partial t} g_t^{x,k}|_{t=0}\right| \\ &= (1+\zeta)|f_k'(x)|\lambda_k(1-x) \leq (1+\zeta)|f_k'(x)|(1-x).\end{aligned}$$

Let K be the line passing through $(0, f_k(y))$ with the slope $(1+\zeta)f_k'(x)(1-x)$. Let A_1 denote the event that the process $t \to (t, B_t^k)$ intersects K for some $t > 0$, and let A_2 be the event that the process (t, B_t^k) intersects K at some time t greater than u. In view of our assumptions on the derivatives of f_k's, we must have $\tau_k > u$, if u is sufficiently small. This and the fact that K lies below the graph of $g_t^{y,k}$ for $t \in (0,u)$ imply that the event $\{T_{b,y} \leq \tau_k\}$ contains $A_1 \setminus A_2$.

The probability that Brownian motion B_t with drift $\mu > 0$, variance $\sigma^2 > 0$, starting from $B_t = b > 0$, will ever hit 0 is equal to

$$\exp(-2b\mu/\sigma^2), \tag{2.3}$$

by a formula from page 362, Section 7.5, of Karlin and Taylor (1975). In particular, the probability of this event is strictly between 0 and 1. The formula (2.3) applies also to lines with a constant slope, with the drift of Brownian motion being increased by the slope of the line.

Assume that k is so large that $|\mu_k| < (1+\zeta)|f_k'(x)|(1-x)/4$. An application of (2.3) gives

$$P(A_1) = \exp(-2(b - f_k(y))[|(1+\zeta)f_k'(x)(1-x)| + \mu_k]/\sigma_k^2).$$

In order to estimate the probability of A_2, we will apply the Markov property at time u. Either the Brownian motion decreases by more than $|u(1+\zeta)f_k'(x)(1-x)/2|$ units over the interval $(0,u)$ or its distance from K at time u is greater than this quantity. A standard estimate shows that for $v > 1$, the probability that the normal distribution deviates by more than v standard units from its mean is bounded above by $\exp(-v^2/2)$. We have assumed that k is so large that the absolute value $|\mu_k|$ of the drift of B_t^k is less than $|(1+\zeta)f_k'(x)(1-x)/4|$. Hence if the Brownian motion B_t^k decreases by more than $|u(1+\zeta)f_k'(x)(1-x)/2|$ units over the interval $(0,u)$ then its value at time u is more than

$$\frac{|u(1+\zeta)f_k'(x)(1-x)/4|}{\sqrt{u}\sigma_k}$$

standard units away from its center. The probability of this event is bounded by

$$\exp(-u[(1+\zeta)f_k'(x)(1-x)]^2/(32\sigma_k^2)). \tag{2.4}$$

If B_u^k is more than $u(1+\zeta)|f_k'(x)|(1-x)/2$ units above K then the probability that it will ever hit K after time u is bounded by

$$\exp(-2(u(1+\zeta)|f_k'(x)|(1-x)/2)[|(1+\zeta)f_k'(x)(1-x)| + \mu_k]/\sigma_k^2), \tag{2.5}$$

by (2.3). The probability of A_2 is bounded by the sum of (2.4) and (2.5). Since $\sigma_k^6 < b - f_k(y) < \sigma_k^4$, it is elementary to check that for $x < 1-\varepsilon$ and large k, the sum of (2.4) and (2.5) is less than $\zeta(1 - P(A_1))$. It follows that

$$P(T_{b,y} > \tau_k) \le 1 - (P(A_1) - P(A_2)) \le (1+\zeta)(1 - P(A_1)).$$

Thus

$$P(T_{b,y} > \tau_k) \le (1+\zeta)(1 - \exp(-2(b - f_k(y))[|(1+\zeta)f_k'(x)(1-x)| + \mu_k]/\sigma_k^2)).$$

In view of $\sigma_k^6 < b - f_k(y) < \sigma_k^4$, this gives for large k,

$$P(T_{b,y} > \tau_k) \le (1+\zeta)^2(2(b - f_k(y))(1+2\zeta)|f_k'(x)|(1-x)/\sigma_k^2).$$

Since ζ can be arbitrarily small, we obtain for any $\xi > 0$ and large k,

$$P(T_{b,y} > \tau_k) \le (1+\xi)2(b - f_k(y))|f_k'(x)|(1-x)/\sigma_k^2.$$

This proves the upper bound in (2.2).

The proof of the lower bound in (2.2) proceeds along similar lines. We consider a small $\zeta > 0$. Suppose that k is sufficiently large so that $\lambda_k > (1-\zeta)/(1-\zeta/2)$. Assume that $|y-x|$ is sufficiently small so that one can find small $u > 0$ (not depending on k) such that for $t_1 \in (0, u)$,

$$\begin{aligned}\left|\frac{\partial}{\partial t} g_t^{y,k}|_{t=t_1}\right| &> \left|(1-\zeta/4)\frac{\partial}{\partial t} g_t^{y,k}|_{t=0}\right| > \left|(1-\zeta/2)\frac{\partial}{\partial t} g_t^{x,k}|_{t=0}\right| \\ &= |(1-\zeta/2) f_k'(x)\lambda_k(1-x)| > (1-\zeta)|f_k'(x)|(1-x).\end{aligned}$$

Let K_1 be the line passing through $(0, f_k(y))$ with the slope $(1-\zeta)f_k'(x)(1-x)$ and let K_2 be the horizontal line passing through the point $(u, g_u^{y,k})$. Let A_1 denote the event that the process (t, B_t^k) intersects K_1 for some $t > 0$, and let A_2 be the event that (t, B_t^k) intersects K_2 at some $t \in [u, \tau_k]$. Since K_1 lies above the graph of $g_t^{y,k}$ for $t \in (0, u)$ and K_2 lies above the graph of $g_t^{y,k}$ for $t \in (u, \tau_k)$, the event $\{T_{b,y} \le \tau_k\}$ is contained in $A_1 \cup A_2$. We have, by (2.3),

$$P(A_1) = \exp(-2(b - f_k(y))[(1-\zeta)|f_k'(x)|(1-x) + \mu_k]/\sigma_k^2).$$

In order to estimate the probability of A_2, we will apply the Markov property at time u. Either the Brownian motion decreases by more than $u(1-\zeta)|f_k'(x)|(1-x)/2$ units over the interval $(0, u)$ or its distance from K_2 at time u is greater than this quantity. Suppose that k is so large that the absolute value $|\mu_k|$ of the drift of B_t^k is less than $(1-\zeta)|f_k'(x)|(1-x)/4$. Then if the Brownian motion decreases by more than $u(1-\zeta)|f_k'(x)|(1-x)/2$ units over the interval $(0, u)$ then its value at time u is more than

$$\frac{u(1-\zeta)|f_k'(x)|(1-x)/4}{\sqrt{u}\sigma_k}$$

standard units away from its center. The probability of this event is bounded by

$$\exp(-u[(1-\zeta)f_k'(x)(1-x)]^2/(32\sigma_k^2)). \tag{2.6}$$

Note one can find a constant $t_1 < \infty$ which depends on ϵ but does not depend on k or x and such that if $X_t^k = g_t^{y,k}$ for all $t < \tau_k$ then $\tau_k < t_1$ a.s. Let K_3 be the horizontal line which is $u(1-\zeta)|f_k'(x)|(1-x)/8$ units above K_2. Assume that k is so large that the absolute value $|\mu_k|$ of the drift of B_t^k is less than $u(1-\zeta)|f_k'(x)|(1-x)/(8t_1)$. Then the absolute value of the integral of the drift μ_k over the interval (u,τ_k) is bounded by $u(1-\zeta)|f_k'(x)|(1-x)/8$. It follows that the probability that the Brownian motion with drift μ_k and starting from $B_0=b$, will hit K_2 after time u but before time τ_k is bounded by the probability that Brownian motion with no drift starting from the level B_u^k will hit the line K_3 before time t_1. This probability is in turn bounded by two times the probability that the Brownian motion with no drift starting from the level B_u^k at time u will be below K_3 at time $u+t_1$. If B_u^k is more than $u(1-\zeta)|f_k'(x)|(1-x)/2$ units above K_2 then it is at least $u(1-\zeta)|f_k'(x)|(1-x)/4$ units above K_3. If this condition is fulfilled, the probability that the Brownian motion with no drift is below K_3 at time $u+t_1$ is bounded above by

$$\exp(-(1/2)[u(1-\zeta)f_k'(x)(1-x)/4]^2/(t_1\sigma_k^2)). \tag{2.7}$$

In view of our previous remarks, a bound for the probability of A_2 may be obtained by multiplying (2.7) by 2 and adding it to (2.6). If $b-f_k(y)\in[\sigma_k^6,\sigma_k^4]$ and k is large then the sum of (2.6) and two times (2.7), and so the probability of A_2, is less than $\zeta(1-P(A_1))$. It follows that

$$P(T_{b,y}>\tau_k)\geq 1-(P(A_1)+P(A_2))\geq 1-P(A_1)-\zeta(1-P(A_1))\geq(1-\zeta)(1-P(A_1)).$$

Thus

$$P(T_{b,y}>\tau_k)\geq(1-\zeta)(1-\exp(-2(b-f_k(y))[(1-\zeta)|f_k'(x)|(1-x)+\mu_k]/\sigma_k^2)).$$

For $b-f_k(y)<\sigma_k^4$ and large k, this gives

$$P(T_{b,y}>\tau_k)\geq(1-\zeta)^2(2(b-f_k(y))(1-2\zeta)|f_k'(x)|(1-x)/\sigma_k^2).$$

Since ζ can be arbitrarily small, we obtain for any $\xi>0$ and large k,

$$P(T_{b,y}>\tau_k)\geq(1-\xi)2(b-f_k(y))|f_k'(x)|(1-x)/\sigma_k^2.$$

This proves the lower bound in (2.2) and so the proof of (2.2) is complete.

We will use (2.2) to estimate the $H^{b,x}$-measure of excursions whose second component exits the interval $[0,1-\epsilon]$ before the lifetime of the excursion. By the abuse of the notation, we will refer only to the first component and ignore the second component of the excursion (e_t^B,e_t^X) under the excursion laws. We recall from the proof of Theorem 1 that locally near the starting point, the excursion laws $\widehat{H}^{0,x}$ may be approximated by the excursion laws $\widetilde{H}^{0,x}$ of Brownian motion with no drift. The renormalized $\widetilde{H}^{0,x}$-distribution of the excursion at time $t > 0$, truncated to excursions with lifetimes exceeding t, is the same as the distribution of Brownian motion starting from 0 and conditioned not to return to 0 before time t. By scaling, this distribution is the same for any t, up to the usual Brownian scaling factor, and so it has a density $q(z/\sigma_k\sqrt{t})/(\sigma_k\sqrt{t})$. The $\widetilde{H}^{0,x}$-mass of excursions which have lifetimes greater than $t > 0$ is a constant times $\sigma_k/\sqrt{t}$, assuming that for any σ_k we normalize the local time to be the density of the occupation measure. It follows that, for small t, we can approximate the density of $\widetilde{H}^{0,x}$-excursions and also of $\widehat{H}^{0,x}$-excursions by $q(z/\sigma_k\sqrt{t})/t$. We will apply the Markov property at $t = \sigma_k^{10}$ because a typical excursion position at the time $t = \sigma_k^{10}$ is σ_k^5 away from the starting point, and so we can apply formula (2.2). By applying the Markov property at time $t = \sigma_k^{10}$ we see that the ratio of $H^{b,x}(A)$ and

$$\int_0^\infty [2z|f_k'(x)|(1-x)/\sigma_k^2] \cdot [q(z/\sigma_k^6)/\sigma_k^{10}]dz$$

converges to 1 as $k \to \infty$. The last quantity is equal to

$$|f_k'(x)|(1-x)\int_0^\infty (2z/\sigma_k^2) \cdot [q(z/\sigma_k^6)/\sigma_k^{10}]dz = \alpha|f_k'(x)|(1-x).$$

Recall that we considered only the case when $\lambda_k \to \lambda = 1$ to see that this completes the proof. $\square$

Proof of Theorem 2. (i) In order to simplify the notation, we will assume in part (i) that $\lambda = \lim_{k\to\infty}\lambda_k = \widehat{\lambda} = \lim_{k\to\infty}\widehat{\lambda}_k = 1$.

Fix some arbitrarily small $p_1 > 0$ and $t_0 > 0$. We will show that for large k the random time T_0^k is less than t_0 with probability greater than $1 - p_1$. Suppose that $M > 1$ is a large integer whose value will be specified later. Recall the meaning of c_0 from the statement of the theorem and let $t_1 = t_0/(2M)$. If $B_s^k \neq f_k(X_s^k)$ for all s in some interval (s_1, s_2) then $s \to X_s^k$ is monotone on this interval and it is an exponential function or a linear transformation of an exponential function. It is easy to see that for any given t_1 we can find $k_0 < \infty$ such that for $k > k_0$, if $B_s^k \neq f_k(X_s^k)$ for all $s \in (s_1, s_1 + t_1)$ then we must have $T_1^k < s_1 + t_1$, no matter what the value of $X_{s_1}^k$ is. From now on we will assume that $k > k_0(t_1)$.

Find $c_1 > 0$ and $c_2 \in (0, \min(x_0 - c_0, 1 - c_0 - x_0))$, such that $|f_k'(x)| \in (c_1, c_1^{-1})$ for $x \in [x_0 - c_2, x_0 + c_2]$ and k greater than some k_0. We will assume without loss of generality that $t_1 < c_2/2$. Recalling that $x_0^k \to x_0$ we see that the endpoints of the interval $[f_k(x_0 + c_2/2), f_k(x_0 - c_2/2)]$ are at least $c_1c_2/4$ units away from $f_k(x_0^k)$, for large k.

Let $S_0 = 0$, and for $j \geq 1$, let S_j be the smallest $t \in [S_{j-1} + \sigma_k^2, S_{j-1} + \sigma_k^2 + t_1]$ with $B_t^k = f_k(X_t^k)$. If there is no such t, we let $S_j = S_{j-1} + \sigma_k^2 + t_1$. Let $A_j = \{T_1^k \geq S_j\}$ and $C_j = \{B_{S_j}^k \in [f_k(x_0 + c_2/2), f_k(x_0 - c_2/2)]\}$. We have

$$\begin{aligned} P(A_j) &= P(A_j \cap A_{j-1} \cap \ldots \cap A_0) \qquad (2.8) \\ &\leq P(A_j \cap C_j \cap A_{j-1} \cap C_{j-1} \cap \ldots \cap A_0 \cap C_0) + P([C_j \cap C_{j-1} \cap \ldots \cap C_0]^c) \\ &= \prod_{m=1}^{j} P(A_m \cap C_m \mid A_{m-1} \cap C_{m-1} \cap \ldots \cap A_0 \cap C_0) + P([C_j \cap C_{j-1} \cap \ldots \cap C_0]^c) \\ &\leq \prod_{m=1}^{j} P(A_m \mid A_{m-1} \cap C_{m-1} \cap \ldots \cap A_0 \cap C_0) + P([C_j \cap C_{j-1} \cap \ldots \cap C_0]^c). \end{aligned}$$

Recall that $B_0^k = f_k(x_0^k)$, the points x_0^k converge to x_0, and

$$[f_k(x_0^k) - c_1c_2/4, f_k(x_0^k) + c_1c_2/4] \subset [f_k(x_0 + c_2/2), f_k(x_0 - c_2/2)],$$

for large k. As k goes to infinity, the variance and drift of B_t^k go to zero. Hence we may and will assume that k is so large that the probability that B_s^k is outside the interval $[f_k(x_0 + c_2/2), f_k(x_0 - c_2/2)]$ for some $s \in (0, t_0)$ is less than $p_1/2$. This implies that

$$P([C_j \cap C_{j-1} \cap \ldots \cap C_0]^c) \leq p_1/2. \qquad (2.9)$$

Next we will estimate $P(A_m \mid A_{m-1} \cap C_{m-1} \cap \ldots \cap A_0 \cap C_0)$. Let us assume that the event $A_{m-1} \cap C_{m-1} \cap \ldots \cap A_0 \cap C_0$ holds. We will further condition on the value of $B_{S_{m-1}}^k$ and $X_{S_{m-1}}^k$. Since we are assuming that C_{m-1} holds, we must have $B_{S_{m-1}}^k \in [f_k(x_0 + c_2/2), f_k(x_0 - c_2/2)]$. We will assume without loss of generality that $B_{S_{m-1}}^k \geq f_k(X_{S_{m-1}}^k)$; the opposite case may be treated in an analogous way. Recall that $s \to X_s^k$ is a Lipschitz function with constant 1, and the Lipschitz constant for f_k is c_1^{-1}, on the interval $[x_0 - c_2, x_0 + c_2]$. Thus,

$$f_k(X_{S_{m-1}+s}^k) \leq f_k(X_{S_{m-1}}^k) + c_1^{-1}s,$$

for $s \leq c_2/2$. If k is large enough so that $\sigma_k^2 < c_2/2$ then

$$f_k(X_{S_{m-1}+\sigma_k^2}^k) \leq f_k(X_{S_{m-1}}^k) + c_1^{-1}\sigma_k^2. \qquad (2.10)$$

Suppose that k is sufficiently large so that and $|\mu_k| < 1$. The variance of B^k_t is equal to $t\sigma_k^2$, so

$$P(B^k_{S_{m-1}+\sigma_k^2} - B^k_{S_{m-1}} > 2c_1^{-1}\sigma_k^2) > p_2 > 0,$$

provided k is sufficiently large. This and (2.10) yield

$$P(B^k_{S_{m-1}+\sigma_k^2} - f_k(X^k_{S_{m-1}+\sigma_k^2}) > c_1^{-1}\sigma_k^2) > p_2. \tag{2.11}$$

For sufficiently large k we obtain from (2.10),

$$\begin{aligned} f_k(X^k_{S_{m-1}+\sigma_k^2}) &\leq f_k(X^k_{S_{m-1}}) + c_1^{-1}\sigma_k^2 \leq B^k_{S_{m-1}} + c_1^{-1}\sigma_k^2 \\ &\leq f_k(x_0 - c_2/2) + c_1^{-1}\sigma_k^2 \leq f_k(x_0 - c_2), \end{aligned}$$

and so $X^k_{S_{m-1}+\sigma_k^2} \geq x_0 - c_2$. Note that for large k we have

$$\left| \frac{\partial}{\partial t} f_k(X^k_t) \right| > (c_1/2)\min(x_0, 1 - x_0) = c_3,$$

provided $X^k_t \in [x_0 - c_2, x_0 + c_2]$ and $B^k_t \neq f_k(X^k_t)$. Let K be the straight line passing through the point $(S_{m-1} + \sigma_k^2, f_k(X^k_{S_{m-1}+\sigma_k^2}))$ with the slope $c_4 = -c_3/2$. Recall that we have assumed that $t_1 < c_2$, that $s \to X^k_s$ is Lipschitz with constant 1, and that the absolute value of the derivative of f_k on $[x_0 - c_2, x_0 + c_2]$ is greater than c_1. All these facts imply that if $X^k_{S_{m-1}+\sigma_k^2} \in [x_0 - c_2, x_0 + c_2/2]$ and if $B^k_{S_{m-1}+\sigma_k^2+s} > f_k(X^k_{S_{m-1}+\sigma_k^2+s})$ for all $s \in (0, t_1)$ then $X^k_{S_{m-1}+\sigma_k^2+s} \in [x_0 - c_2, x_0 + c_2]$ for $s \in (0, t_1)$ and so the graph of the function $f_k(X^k_s)$ stays below the line K for $s \in (S_{m-1} + \sigma_k^2, S_{m-1} + \sigma_k^2 + t_1)$. By monotonicity, the same is true if we relax the assumption $X^k_{S_{m-1}+\sigma_k^2} \in [x_0 - c_2, x_0 + c_2/2]$ and suppose instead that $X^k_{S_{m-1}+\sigma_k^2} \geq x_0 - c_2$. It follows that if the Brownian motion $B^k_{S_{m-1}+\sigma_k^2+s}$ stays above the line K for all $s > 0$ then it cannot cross the graph of $f_k(X^k_s)$ before the time $S_{m-1} + \sigma_k^2 + t_1$. This would imply that $T^k_1 < T_m$. Hence, the (conditional) probability of A_m is less than the probability of hitting K at any time greater than $S_{m-1} + \sigma_k^2$. We estimate the last probability using (2.3). Let us assume that

$$B^k_{S_{m-1}+\sigma_k^2} - f_k(X^k_{S_{m-1}+\sigma_k^2}) > c_1^{-1}\sigma_k^2,$$

as in (2.11). For large k, the absolute value of drift of B^k is bounded by $|c_4|$, the same quantity as the slope of K. The probability of hitting K is therefore less or equal to

$$\exp\Big(- \frac{2(c_1^{-1}\sigma_k^2)|2c_4|}{\sigma_k^2} \Big) = p_3 < 1.$$

This and (2.11) give,

$$P(A_m \mid A_{m-1} \cap C_{m-1} \cap \ldots \cap A_0 \cap C_0) \le (1 - p_2) + p_2 p_3 = p_4 < 1.$$

Combining this with (2.8) and (2.9) yields

$$P(A_M) \le p_4^M + p_1/2.$$

We now choose M so that $p_4^M < p_1/2$ and then we choose large k so that $M\sigma_k^2 < t_0/2$ to obtain

$$P(T_1^k \ge t_0) \le P(T_1^k \ge M(\sigma^2/k + t_1)) \le P(T_1^k \ge S_M) = P(A_M) \le p_1.$$

This completes the proof of part (i) of Theorem 2.

(ii) We have just proved that the bifurcation time T_0^k converges to 0 in distribution, as k goes to infinity. By the continuity of Brownian paths, the position of the Brownian motion at the bifurcation time converges to x_0, in distribution. Lemma 3 shows that the ratio of the mass that the excursion laws give to excursions bifurcating upward to the mass given to excursions going downward and starting from the same point x has the same limit as

$$\frac{\alpha |f_k'(x_0)| \lambda_k (1 - x_0)}{\alpha |f_k'(x_0)| \widehat{\lambda}_k x_0}$$

which is

$$\frac{\lambda(1 - x_0)}{\widehat{\lambda} x_0}, \tag{2.12}$$

assuming that $x \to x_0$ and $k \to \infty$. The processes of upward excursions and downward excursions are independent Poisson processes with random intensities. The ratio of the intensities converges to the quantity in (2.12), in distribution, on the whole interval $[0, T_0^k]$. This implies that the ratio of the probability of having an upward bifurcation to the probability of having a downward bifurcation converges to $\lambda(1 - x_0)/\widehat{\lambda} x_0$. $\square$

Proof of Corollary 1. By the Girsanov theorem, the distribution of $\widehat{B}_t$ on any finite interval $[0, t_0]$ is mutually absolutely continuous with the distribution of B_t (see Karatzas and Shreve (1988)). The properties of solutions to (1.1) proved in Theorem 1 and Lemmas 1 and 2 hold with probability 1, so they hold when we replace B_t with $\widehat{B}_t$.

The argument used to prove the original Theorem 2 can be used to prove its new version, when we replace B^k by $\widehat{B}^k$. We will limit ourselves to the observation that the drift of $\widehat{B}^k$ is a continuous function of the value of the process and is effectively bounded on a bounded time interval as the values for the drift when $\widehat{B}^k$ is outside a finite interval are irrelevant to the estimates. With this fact in hand it is not hard to modify the original proof but the details are left to the reader. $\square$

Proof of Proposition 1. We will only sketch the proof as it is quite standard. See Billingsley (1968) for the methods applied below. We will only deal with the maximal solution $\overline{X}_t^i$ to (1.3) with A and f replaced by A^i and f^i, and $A_0^i = b^i$, $\overline{X}_t^i = x^i$. Let $\mathcal{P}^i$ be the joint distribution of the pair of processes $(A_t^i, \overline{X}_t^i)$. It is easy to prove the tightness of the sequence of measures $\mathcal{P}^i$ as A_t^i converge to Brownian motion by assumption and $\overline{X}_t^i$ are Lipschitz. Let $\mathcal{P}$ be the limit of some convergent subsequence of $\mathcal{P}^i$. The first component of the process under $\mathcal{P}$ is a Brownian motion and the second one is a process with Lipschitz trajectories. We can prove that they satisfy the equation (1.1) by using the same argument as was used for sequences of approximate solutions in the proof of Theorem 1. By uniqueness of the solutions to (1.1) (see Theorem 1), the whole sequence $\mathcal{P}^i$ converges to $\mathcal{P}$.

In order to finish the proof, it remains to check the uniform integrability of $\overline{\Phi}(x^i, b^i; A^i, f^i)$. This is a standard excercise in view of the Gaussian estimates for the tails of the one-dimensional distributions of Brownian motion and the Lipshitz character of both $\overline{X}_t^i$ and g. $\square$

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