Optimal Bilateral Trade of Multiple Objects*

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Abstract

We study a private-values buyer-seller problem with multiple objects. Valuations are binary and i.i.d. We construct mechanisms that span the set of all Pareto-efficient outcomes. The induced trading rules for objects are linked in a simple way.

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1 Introduction

Bilateral trade with private values is a fundamental problem in mechanism design. Myerson and Satterthwaite (1983) have shown that in any non-trivial buyer-seller problem with continuous valuations, full efficiency cannot be attained. Also with discrete (finite) valuations, for many parameter values the first-best outcome cannot be reached.

Jackson and Sonnenschein (2007) show that with many objects (and valuations that are independent across objects) the problem disappears in the limit: the first-best outcome can be approached as the number of objects tends to infinity. While their mechanism approaches the first best in the limit, for a finite number of objects it is not the optimal one (not second best).[1] Cohn (2010) constructs a more elaborated mechanism which, although also not optimal, achieves exponential convergence to the first best, rather than polynomial as in Jackson and Sonnenschein (2007).[2]

This paper looks for second-best mechanisms, focusing on the case of binary valuations. We construct a set of mechanisms that spans the entire Pareto frontier of the buyer's and seller's ex ante gains from trade. These mechanisms share a simple structure. Objects which both agents are eager to trade (low value for the seller and high for the buyer) are traded. Objects which both agents are reluctant to trade (high value for the seller and low for the buyer) are not traded. Objects with a mixed desire to trade are traded only if the reluctant agent announces that the number of (other) objects she is eager to trade is above some threshold. This threshold depends on the designer's objective function: the stronger the preference for an agent, the lower the threshold.

In the extreme case in which the designer's objective identifies with the seller's, our mechanism becomes the monopolist's profit-maximizing one. Our results imply that the monopoly sells the first M goods at a price equal to the buyer's high valuation, while additional goods are discounted and sold at the buyer's low valuation (provided that the monopoly wishes to sell at that price).

The benefits of such "mixed bundling" – offering a bundle of goods at a price lower than the sum of individual prices – have been studied in the literature on auctions and on monopolistic screening. McAfee, McMillan and Winston (1989) show that a monopoly selling two goods can benefit from bundling, at least in the case that the buyer's valuations are independent. Armstrong (2000) studies revenue-maximizing auctions for two goods. In the part of his analysis that deals (like this paper)

[1] Jackson and Sonnenschein treat a more general collective decision problem; in this context their mechanism works even without monetary transfers between agents.

[2] Athey and Miller (2007) consider a related problem in a context of sequential trade. They identify conditions under which a first-best outcome is attainable under ex post incentive compatibility.

with the i.i.d. and binary valuations case, he shows that a bidder announcing a high value for one object receives priority in the bidding for the other object. Avery and Hendershott (2000) study revenue-maximizing auctions for two objects, with one buyer interested in both and many buyers interested in only one object. They show that, for the multi-object buyer, the probability of receiving one object and the price paid for it may depend on the announced valuation for the other, leading sometimes to an inefficient allocation. While all these papers deal with the case of two objects, our paper deals with an arbitrary number of objects. (Our problem remains tractable by the assumption that objects' valuations are i.i.d.) Our paper also differs in that the uncertainty is double sided (except for Avery and Hendershott 2000, where the opportunity cost generated by the one-object bidders makes the uncertainty in the interaction between the seller and the multi-object buyer double-sided). Finally, we characterize the whole efficient frontier of the trading problem, while the above papers focus only on the seller's preferred mechanism.[3]

In a closely related paper, Fang and Norman (2008) show that bundling can also be beneficial in the context of provisioning multiple public goods (when exclusion is allowed). They study a model with many potential users, whose willingness to pay is private information. After they report their types the government decides whether to produce each good, whether to exclude certain agents from using it, and how much to charge each agent. For the case of binary valuations, they characterize the optimal mechanism (with equal weights on all consumers). They show that as the number of agents grows without bounds: (1) either none or all of the public goods are provided, and (2) a user with a low valuation is excluded iff she announced low valuations too many times. There are strong mathematical connections between our paper and Fang and Norman's. However, their main result, regarding the exclusion rule, is not parallel to our main result. This is because the public-goods analogue to trade in the buyer-seller problem is provision, rather than exclusion.

In their survey of the vast literature on multidimensional screening, Rochet and Stole (2003) explain that such environments tend to be difficult to solve when they lack an exogenous type-ordering, i.e., when the set of binding IC constraints depends on the set of trading probabilities of the different objects (or quantities, in their general formulation). A related difficulty, they add, is that the IC conditions are frequently binding not only among adjacent types. In our model, the assumptions of symmetry across objects and of binary valuations give rise to a natural ordering of

[3]Note that our problem is different from the standard monopolistic bundling model in that the monopoly's type is private information (in other words, the monopoly here is an "informed principal" – see Myerson 1983 and Maskin and Tirole 1990). Moreover, and to its further advantage, the monopoly devises its optimal mechanism at the ex ante stage, and then both the buyer and the monopoly submit their reports to the mechanism.

types – according to the number of eager valuations. This ordering is not "exogenous" in Rochet and Stole's sense: whether the IC constraints between adjacent types are the only binding ones depends on the trading probabilities. Nonetheless, we solve for the optimal mechanism in a relaxed problem, in which IC is required to hold only between adjacent types relative to this ordering, and show that the optimal trading probabilities in the relaxed problem are monotone. This, in turn, implies that these constraints are the only binding ones in the original problem. To what extent can this methodology be generalized to problems with more than two valuations per object remains an open question for future research. The main challenge is, presumably, to identify an appropriate ordering of types. Unlike the binary-valuations case, however, this ordering may depend on parameters of the model. With three valuations, for example, type $(reluctant, eager)$ may either be "above" or "below" type $(intermediate, intermediate)$, depending on the specific problem's parameters.

2 The Model

Consider a private values buyer-seller problem with N objects, labeled $i \in I = \{1, ..., N\}$. Each agent assigns one of two possible valuations to each object. To treat the two agents in a symmetric way, we denote the valuations of the buyer (b) by $v_e^b > v_r^b > 0$ ("e" stands for "eager" to trade the object, and "r" stands for "reluctant"); for the seller (s) the valuations are $v_r^s > v_e^s \geq 0$. We also denote the difference between the valuations of agent $j \in \{b, s\}$ by $\Delta^j = \left| v_e^j - v_r^j \right|$. For non-triviality of the problem, we assume that $v_e^b > v_r^s > v_r^b > v_e^s$.

A type, for agent $j \in \{b, s\}$, is a vector of object valuations $w^j = \left(w_1^j, ..., w_N^j\right) \in W^j$. Valuations are independent across objects and agents. Specifically, each w_i^j takes the values v_e^j and v_r^j with probabilities $q^j \in [0, 1]$ and $\left(1 - q^j\right)$, respectively. We assume that agents are risk-neutral, and their utilities from possessing multiple objects are additive.

By the revelation principle (Myerson 1979), we can restrict attention to incentive-compatible and direct mechanisms. Mechanism $\Gamma = \left\langle p, t^s, t^b \right\rangle$ specifies, for each pair of announcements $\left(w^s, w^b\right)$, a probability $p\left(i; w^s, w^b\right)$ of trade for each object i and a monetary transfer $t^j\left(w^s, w^b\right)$ to agent j.

The utility of agent j of type w^j who announces $\hat{w}^j$, when the rival $-j$ announces $\hat{w}^{-j}$, is:

$$\begin{aligned} u^b(w^b, \hat{w}^b; \hat{w}^s) &= \sum_{i=1}^N p\left(i; \hat{w}^s, \hat{w}^b\right) w_i^b + t^b\left(\hat{w}^b, \hat{w}^s\right) \\ u^s(w^s, \hat{w}^s; \hat{w}^b) &= -\sum_{i=1}^N p\left(i; \hat{w}^s, \hat{w}^b\right) w_i^s + t^s\left(\hat{w}^b, \hat{w}^s\right). \end{aligned}$$

Under truth telling of $-j$, the interim expected utility of agent j of type w^j who announces $\hat{w}^j$ is:

$$U^j(w^j, \hat{w}^j) = E_{w^{-j}}\left[u^j(w^j, \hat{w}^j; w^{-j})\right],$$

where $E_{w^{-j}}$ denotes the expectation over all types of $-j$. Agent j's ex ante utility (under truth telling) is:

$$EU^j = E_{w^j}\left[U^j(w^j, w^j)\right],$$

where E_{w^j} is the expectation over all of j's types. Finally, the revenue of the mechanism is:

$$R\left(w^s, w^b\right) = -\left(t^s(w^s, w^b) + t^b(w^s, w^b)\right). \tag{1}$$

3 Optimal Mechanisms

For any pair of weights $\alpha = \left(\alpha^s, \alpha^b\right)$ (non-negative and summing to 1), an α-*optimal* mechanism is one that maximizes the α-weighted sum of the agents' ex ante utilities:

$$Max_{\ p,\ t^s,\ t^b} \ \ \alpha^s EU^s + \alpha^b EU^b$$

subject to the constraints:

$$\begin{array}{ll} \text{(IC)} & U^j(w^j, w^j) \geq U^j\left(w^j, \hat{w}^j\right) \ \text{ for any } w^j, \hat{w}^j \in W^j \\ \text{(IR)} & U^j\left(w^j, w^j\right) \geq 0 \ \text{ for any } w^j \in W^j \\ \text{(BB)} & R\left(w^s, w^b\right) \geq 0 \ \text{ for any } w^s \in W^s,\ w^b \in W^b. \end{array}$$

A utility pair $\left(EU^s, EU^b\right)$ induced by an α-*optimal* mechanism is called an α-*optimal outcome*. An α-*facet* of the Pareto frontier is the set of all α-optimal outcomes.

Our main result characterizes a set of simple mechanisms that spans the entire Pareto frontier:

Theorem 1 *Any α-optimal outcome can be achieved by a direct mechanism in which:*

- *An object which both agents are eager to trade is traded with probability 1.*
- *An object which both agents are reluctant to trade is traded with probability 0.*
- *An object which agent $-j$ is eager to trade and agent j is reluctant to trade is traded with:*
  - *probability* 1*, if the number of objects that j is eager to trade exceeds threshold M^j,*
  - *probability* 0*, if it is below M^j,*
  - *probability σ^j, if it is exactly M^j,*

where $M^j \in \{0, ..., N\}$ and $0 < \sigma^j \leq 1$. Moreover, denoting by $M^j(\alpha)$ the set of all thresholds M^j in mechanisms that generate an α-facet of the Pareto frontier, the correspondence $M^j(\alpha)$ is weakly decreasing in α^j.

A constructive proof of the theorem is given in Section 4. The proof also shows how to compute M^j, σ^j and the payments to agents. We proceed by explaining the intuition of the result, and conclude with an application to bundling and consumer surplus in the special case of monopoly.

Intuition for Theorem 1

In a first-best solution, each object is traded whenever the buyer's valuation is higher than the seller's, i.e., as long as at least one agent is eager to trade. This outcome, however, is generally infeasible: for many parameter values, the information rents that are required to induce the agents to reveal their types and agree to participate are higher than the total surplus generated by trade, and thus budget balance is violated. To restore budget balancedness, the mechanism must restrict trade between some of the types, thereby reducing rent payments. Second-best optimality, then, requires to identify the most economical trade restrictions.

It is convenient to think of the mechanism as if it collects all the surplus from trades (temporarily leaving agents with zero net utilities) and uses it to pay the agents the required information rents. These rents are then the agents' final payoffs. The mechanism sets trade probabilities for each pair of types and each object. Increasing each of these probabilities induces (1) a change in the total surplus generated by the mechanism and (2) a change in the sum of required rents to other types of both agents. The difference (2)-(1) is the *budgetary cost* of increasing that probability. The *direct benefit* (change in the objective function) is the α-weighted sum of the same induced rents. A second-best mechanism prioritizes trades with lower ratios of budgetary cost to direct benefit.

A priory, agents have an incentive to lie *downwards*: pretend to be eager to trade fewer objects, in order to obtain better trading prices. Rents are thus paid per object that the agent admits being *eager* to trade. The rent equals the expected gain from lying, which is proportional to the probability that the agent is allowed to trade after a "*reluctant*" report. There are thus two obvious cases with no trade-off between the cost and the benefit of trades: Whenever both agents are eager to trade an object there is no reason to restrict trade, since trade increases surplus without generating rent payments for other types. And, whenever both agents are reluctant, they should not trade – such trade decreases the surplus and only increases the incentive of other types to lie.

A tradeoff exists when one agent is eager to trade an object and the other is reluctant. Call the set of all the types of an agent, who have exactly m eager valuations, "level m". Allowing (all) level m types of one agent to trade objects they are reluctant to (when the other agent is eager to trade) necessitates rent payments to all the types at levels above m. These trades also generate a surplus, which is proportional to the number of types in level m times the number of reluctant valuations per type. Our key (and non-trivial) result is that the induced budgetary cost to direct benefit ratio is decreasing in m. We thus obtain, for each agent j, a threshold M^j such that levels above M^j trade their reluctant objects and those below do not.

The thresholds for the two agents are the lowest integers for which the total rents do not exceed the surplus generated by the mechanism.[4] Since decreasing threshold M^j increases j's rent, there is a trade-off between the two thresholds. Recall that the objective function is the α-weighted sum of the rents. Thus, threshold M^j is (weakly) decreasing in α^j.

If α^j is sufficiently high, M^j equals 0. Agent j is always allowed to trade, and trade restrictions apply to agent $-j$ only. As we further increase α^j, the α-optimal mechanism picks an even higher M^{-j}, and pays the economized rent (minus the lost surplus) as a lump sum payment to j.[5]

Monopoly and consumer surplus:

Consider the extreme case of $\alpha^s = 1$ and $\alpha^b = 0$. Here, the seller can be viewed as a monopolist who sets the mechanism in his best interest. If N is small, the monopolist sets $M^b = N$, which implies that only objects which the buyer is eager to trade (high valuation) are traded, at the highest price he is willing to pay, v_e^b. Since there is no trade when the buyer is reluctant to trade, the buyer receives no rents at all. For large N, the monopolist sets $M^b < N$, i.e., agrees to sell some of the objects at a price equal to the buyer's low valuation v_r^b (provided the monopoly has low cost, v_e^s, for these objects). In these cases, the buyer is paid a positive expected rent, i.e., ends up with a positive (expected) consumer surplus. This is formalized in the following proposition (proved in the appendix), which applies also to the opposite, monopsony case:

Proposition 1 *Assume that $\alpha^j = 0$. Then, in any α-optimal mechanism, agent j has a strictly positive ex ante utility, or equivalently $M^j < N$, if and only if $N \geq \frac{q^j}{1-q^j} \frac{\Delta^j}{v_e^b - v_e^s - \Delta^j}$.*

[4] The probability of trade at the threshold may be between 0 and 1 so that total surplus and rents are equated.

[5] For some parameter values, a first-best solution (no trade restrictions) is feasible. Here, the total surplus exceeds the required rents, and both agents may receive lump sum payments. Note, however, that for a first-best solution, α must be in a neighborhood of $(0.5, 0.5)$. At extreme values of α, the optimal solution must involve restricting trade (and reducing rents) for the agent with low α^j, in order to to increase the lump sum payment to the preferred agent.

4 Proof of Theorem 1

We start with a convenient decomposition of the transfers t^j, into two parts:

$$\begin{aligned} t^b(w^s, w^b) &= \sum_{i=1}^{N} -p\left(i; w^s, w^b\right) w_i^b + \tau^b\left(w^s, w^b\right) \\ t^s(w^s, w^b) &= \sum_{i=1}^{N} p\left(i; w^s, w^b\right) w_i^s + \tau^s\left(w^s, w^b\right). \end{aligned} \tag{2}$$

This decomposition can be interpreted as follows. First, each agent gets paid (or pays) the left summand; as long as his announcement is truthful, this payment exactly compensates for the expected loss (or gain) from giving away (or receiving) the objects. Then, the agent is also paid a *rent*, τ^j, implicitly defined by (2). Under truth-telling, this rent is in fact his net utility: denoting $\bar{\tau}^j\left(w^j\right) = E_{w^{-j}}\left[\tau^j\left(w^j, w^{-j}\right)\right]$, we observe that $U^j(w^j, w^j) = \bar{\tau}^j\left(w^j\right)$.

Since the agents are risk neutral and their types are independent, for any mechanism satisfying ex ante budget balance there exists a mechanism satisfying ex post budget balance with the same trading rule and same interim utilities to both agents (Proposition 2 in Borgers and Norman 2008). We can thus replace the ex post budget-balance condition (BB) by its weaker ex ante counterpart:

$$\text{(BB')} \quad E_{w^b, w^s}\left[R\left(w^s, w^b\right)\right] \geq 0.$$

For any type w^j, we say that a report $\hat{w}^j$ is a *a local downward deviation* if there is exactly one object $i \in N$ such that $w_i^j = v_e^j$, $\hat{w}_i^j = v_r^j$ and $w_{i'}^j = \hat{w}_{i'}^j$ for all $i' \neq i$. We say that a mechanism is LDIC (Local Downward Incentive Compatible) if all types prefer truth telling to a local downward deviation:

Definition 1 *A mechanism is LDIC if $U^j(w^j, w^j) \geq U^j\left(w^j, \hat{w}^j\right)$ for any $w^j \in W^j$ and any local downward deviation $\hat{w}^j$ of w^j.*

We proceed by looking for a LDIC-optimal mechanism: one that maximizes $\alpha^s EU^s + \alpha^b EU^b$ subject to LDIC, IR and BB' (the final step of the proof shows that this mechanism satisfies, in fact, the full set of IC constraints).[6] We start with the following lemma:

Lemma 1 *In any LDIC-optimal mechanism, the probability of trade is* 1 *for objects which both agents are eager to trade, and* 0 *for objects which both agents are reluctant to trade.*[7]

[6] For a closely related discussion, in the case of unidimensional trade, see Bolton & Dewatripont (2005), pp. 78-80.

[7] In fact, the proof can be easily adapted to show that the lemma also holds if "LDIC-optimal" is replaced by "X-optimal" where X is any subset of the IC conditions.

Since our problem is symmetric across objects (each agent's valuations are i.i.d. across objects), we can restrict the search to the domain of *symmetric* mechanisms, which are agnostic to changing the names of the objects. Let $\pi : I \rightarrow I$ denote a permutation mapping, and let M_π denote the corresponding permutation operator on vectors, so that $M_\pi w$ is a vector in which the i^{th} element is the $\pi(i)^{th}$ element in w (formally, M_π is the matrix defined by $[M_\pi]_{(i,j)} = [I_{N\times N}]_{(\pi^{-1}(i),j)}$). Then:

Definition 2 *Mechanism $\Gamma = \langle p, \tau^s, \tau^b \rangle$ is symmetric if for any (w^s, w^b) and any permutation π, (1) $p(i; w^s, w^b) = p(\pi(i); M_\pi w^s, M_\pi w^b)$ for all $i \in I$, and (2) $\tau^j(w^s, w^b) = \tau^j(M_\pi w^s, M_\pi w^b)$.*

Lemma 2 *For any LDIC mechanism $\Gamma = (p, \tau^s, \tau^b)$ there exists a symmetric LDIC mechanism $\hat{\Gamma} = (\hat{p}, \hat{\tau}^s, \hat{\tau}^b)$ with the same ex ante utilities for the agents and revenue for the mechanism.*[8]

Let $g(w^j)$ denote the number of objects that type w^j is eager to trade. In a symmetric mechanism Γ, the rents τ^s and τ^b depend only on $g(w^s)$ and $g(w^b)$; the probability of trade in object i, $p(i, w^s, w^b)$, depends only on the valuations of the specific object, w_i^s and w_i^b, and on $g(w^s)$ and $g(w^b)$. We thus partition the set W^j of all types of agent j to $N+1$ disjoint sets $\{G_m^j\}_{m=0}^N$ such that all types w^j in G_m^j satisfy $g(w^j) = m$. We refer to types in G_m^j as types in "level m", and denote an arbitrary type in G_m^j by $w^{j,m}$. Denote the probability that type $w^{j,m}$ trades an object he is reluctant to, conditional on $-j$ being eager to trade that object, by μ_m^j.

A *tight* mechanism is one in which each type is *exactly indifferent* to a local downward deviation:

Definition 3 *A mechanism is tight if $U^j(w^j, w^j) = U^j(w^j, \hat{w}^j)$ for any $w^j \in W^j$ and any local downward deviation $\hat{w}^j$ of w^j.*

The search for optimal LDIC mechanisms can be further restricted to the smaller and more structured domain of tight and symmetric ones (TSLDIC):

Lemma 3 *For any symmetric LDIC mechanism $\Gamma = (p, \tau^s, \tau^b)$ there exists a TSLDIC mechanism $\hat{\Gamma} = (p, \hat{\tau}^s, \hat{\tau}^b)$ with the same ex ante utilities for the agents and revenue for the mechanism.*[9]

In a TSLDIC-optimal mechanism, the expected rent $\bar{\tau}^j$ of type $w^{j,m}$ is uniquely determined by the probabilities $\{\mu_m^j\}_{m=0}^{N-1}$ and by c^j, the rent of the lowest type $w^{j,0}$. Since $w^{j,m+1}$ is exactly

[8] We omit the explicit proof since this is a standard technique in the literature – see, e.g., Fang and Norman (2008). Note that like the IC property, LDIC is also preserved after any permutation of the objects.

[9] Note that this lemma would not necessarily hold if the LDIC condition were replaced by full IC. This is because, in general, the addition of "tightness" might violate some of the IC constraints.

indifferent to a local downward deviation, $\bar{\tau}\left(w^{j,m+1}\right) = \bar{\tau}\left(w^{j,m}\right) + \left(q^{-j} \cdot \mu_m^j + (1-q^{-j}) \cdot 0\right) \Delta^j$ (recall that, by Lemma 1, if $-j$ is reluctant to trade object i, the probability of trade is 0). Thus:

$$\bar{\tau}\left(w^{j,m}\right) = c^j + q^{-j}\Delta^j \sum\nolimits_{k=0}^{m-1} \mu_k^j. \tag{3}$$

for any $m \in \{0, ..., N\}$, $j \in \{b, s\}$. The *ex ante* utility of agent j is then:

$$T^j\left(c^j, \mu_0^j ... \mu_{N-1}^j\right) = c^j + q^{-j}\Delta^j \sum\nolimits_{m=0}^{N} \left[f_{BD}(m, N, q^j) \sum\nolimits_{k=0}^{m-1} \mu_k^j\right]$$

where f_{BD} is the p.d.f. of the binomial distribution. Let S denote the ex ante surplus generated by trade. The surplus comes from eager-eager and eager-reluctant encounters, and is given by:

$$\begin{aligned} S\left(c^s, \mu_0^s ... \mu_{N-1}^s, c^b, \mu_0^b ... \mu_{N-1}^b\right) \quad = \quad & N q^s q^b (v_e^b - v_e^s) \\ & + q^s (v_r^b - v_e^s) \sum\nolimits_{m=0}^{N-1} \left[(N-m) f_{BD}(m, N, q^b) \mu_m^b\right] \\ & + q^b (v_e^b - v_r^s) \sum\nolimits_{m=0}^{N-1} \left[(N-m) f_{BD}(m, N, q^s) \mu_m^s\right]. \end{aligned}$$

By (3), the IR constraint reduces to $c^j \geq 0$. Ex ante budget balance requires $S(\cdot) \geq T^s(\cdot) + T^b(\cdot)$. In an optimal mechanism this condition must hold with equality (otherwise, a constant could be added to the payments of one agent, increasing his utility without violating IC).

For brevity of notation denote $\mu_{-1}^j = c^j$ and $\mu^j = \{\mu_m^j\}_{m=-1}^{N-1}$. A TSLDIC mechanism that satisfies the conditions of Lemma 1 can then be characterized by two vectors μ^b, μ^s. Given weights $\left(\alpha^s, \alpha^b\right)$, we look for $\left\{\mu^s, \mu^b\right\}$ that maximize the α-weighted sum of ex ante utilities:

$$\begin{array}{llll} & MAX_{\mu^s,\mu^b} & \alpha^s T^s(\mu^s) + \alpha^b T^b\left(\mu^b\right) & \\ \textit{subject to:} & \text{(BB')} & T^s(\mu^s) + T^b\left(\mu^b\right) - S\left(\mu^s, \mu^b\right) = 0 & \\ & \text{(IR)} & \mu_{-1}^j \geq 0, & j \in \{s, b\} \\ & \text{(PROB)} & 1 \geq \mu_m^j \geq 0, & m \in \{0, ..., N-1\},\ j \in \{s, b\} \end{array} \tag{4}$$

Let r_m^j denote the ratio of the derivatives of the BB' condition and of the objective function, both w.r.t. μ_m^j. Since both functions are linear in μ_m^j, r_m^j is a constant:

$$r_m^j = \frac{\partial\left[T^s(\mu^s) + T^b\left(\mu^b\right) - S\left(\mu^s, \mu^b\right)\right] / \partial\mu_m^j}{\partial\left[\alpha^s T^s(\mu^s) + \alpha^b T^b\left(\mu^b\right)\right] / \partial\mu_m^j} = \frac{1}{\alpha^j} \frac{\partial\left[T^j\left(\mu^j\right) - S\left(\mu^s, \mu^b\right)\right] / \partial\mu_m^j}{\partial T^j\left(\mu^j\right) / \partial\mu_m^j} \in [-\infty, +\infty]. \tag{5}$$

(If the denominator is 0 we define r_m^j to be $-\infty$ or $+\infty$, according to the sign of the nominator.) In an optimal solution of the linear program (4), μ_m^j's with lower r_m^j are set to their upper bound, and those with higher r_m^j are set to 0:

Lemma 4 *In an optimal solution of (4), for $j, j' \in \{b, s\}$ and $l, k \in \{-1, ..., N-1\}$, if $r_k^j < r_l^{j'}$ then either μ_k^j attains its upper bound, or $\mu_l^{j'} = 0$.*

Crucially, for each agent j, r_m^j is monotonically decreasing in m:

Lemma 5 *For each agent j, if $-1 \le l < k \le N$ then $r_k^j < r_l^j$.*

By Lemma 4 (and since the objective function is increasing in all μ_m^j), there is a unique cutoff $r^* \in \{r_m^j\}_{j\in\{b,s\},\ m\in\{-1,...,N-1\}}$, determined by BB', such that, in an optimal solution of (4), $\mu_m^j = 0$ if $r_m^j > r^*$, μ_m^j attains its upper bound if $r_m^j < r^*$, and μ_m^j is strictly above 0 and weakly below its upper bound (1 if $m \ge 0$ or ∞ if $m = -1$) if $r_m^j = r^*$.[10] By Lemma 5, there can be at most two μ's for which $r_m^j = r^*$, belonging each to a different agent. In case there is only one, its value is uniquely determined by BB'. If there are two, then BB' uniquely determines their sum $\mu_m^j + \mu_{m'}^{-j}$.

Note that Lemma 4 provides a necessary but not sufficient condition for an optimal solution of (4). Nevertheless, since the only degree of freedom in specifying the solution is how to split the sum $\mu_m^j + \mu_{m'}^{-j}$ into μ_m^j and $\mu_{m'}^{-j}$, and since any way of doing that yields the same value for the objective function, then any such pair defines an optimal solution. By changing μ_m^j and $\mu_{m'}^{-j}$ while keeping their sum constant, utility is transferred between the agents at a constant rate. Thus, the set of all such pairs spans an α-facet of the Pareto frontier.

This simple structure of optimal solutions to (4) implies a simple structure to the mechanisms that induce them. Fix an optimal solution (μ^s, μ^b). If $\mu_m^j > 0$ for some $m \ge 0$, let M^j denote the smallest $m \in \{0, ..., N-1\}$ for which $\mu_m^j > 0$ and let $\sigma^j = \mu_{M^j}^j$; otherwise, set $M^j = N$. Let $\hat{\Gamma} = \langle \hat{p}, \hat{\tau}^b, \hat{\tau}^s \rangle$ with $\hat{p}$ as defined in Theorem 1 with parameters M^j, σ^j, and with $\hat{\tau}^j$ satisfying (3). Then $\hat{\Gamma}$ induces (μ^s, μ^b) and is TSLDIC-optimal. To complete the proof we show that this mechanism is monotone and hence fully incentive compatible.

To define monotonicity, we introduce the following notation. Given a symmetric mechanism Γ, denote the probability that type $w^{j,m}$ trades an object with valuation $v^j \in \{v_e^j, v_r^j\}$, by $p^j(v^j, m)$. That is, $p^j(v^j, m) = E_{w^{-j}}\left[p\left(i; w^j, w^{-j}\right)\right]$ where $w^j \in G_m^j$ and $w_i^j = v^j$ (this is well defined by symmetry). We say that Γ is monotone if the probability to trade objects with eager valuations is always weakly higher the that of trading objects with a reluctant valuation, and if the probability to trade an object with a given valuation is weakly increasing in m:

Definition 4 *A symmetric mechanism is monotone if (1) $p^j(v_e^j, k) \ge p^j(v_r^j, m)$ for all m, k and (2) $p^j(v^j, m+1) \ge p^j(v^j, m)$ for all m and $v^j \in \{v_e^j, v_r^j\}$.*[11]

[10] Existence of r^* follows from the continuity of BB' in the vector of μ's, and since it is positive (negative) if all the μ's are set to 0 (their upper bound).

[11] Note that $p^j(v_e^j, m)$ is defined only for $m \in \{1...N\}$ and $p^j(v_r^j, k)$ is defined for $k \in \{0...N-1\}$. The conditions

$\hat{\Gamma}$ is monotone since $p^j(v_r^j, m) = q^{-j} \cdot \mu_m^j + \left(1 - q^{-j}\right) \cdot 0$ where $\mu_m^j \leq 1$ is weakly increasing in m, and since $p^j(v_e^j, k) = q^{-j} \cdot 1 + \left(1 - q^{-j}\right) \cdot \rho^j$ where $\rho^j \geq 0$ is constant (the probability of trade depends only on the type of agent $-j$ who is reluctant to trade the object).

Lemma 6 *A monotone and TSLDIC mechanism is incentive compatible.*

Thus, the TSLDIC-optimal mechanism $\hat{\Gamma}$ satisfies IC and is therefore α-optimal.

Computation of the payments to the agents

Given the trade probabilities described above, any payment scheme that provides type $w^{j,m}$ with the expected rent given in (3) completes the specification of the mechanism. The simplest scheme fixes the payment to $w^{j,m}$ independently of $-j$'s type; plugging in M^j and σ^j, we obtain:

$$\tau^j\left(w^{j,m}, w^{-j}\right) = Max\left\{0, c^j + \left(m - M^j(\alpha) - 1 + \sigma^j(\alpha)\right)\Delta^j q^{-j}\right\}$$

That is, the agent is paid $\sigma_j \Delta^j q_{-j}$ for the first "eager" report above the threshold M_j, and $\Delta^j q_{-j}$ for each of the others. Recall that the rents τ are translated back into payments t using (2).

Note that this simple specification satisfies only the ex ante budget balance constraint. To re-obtain ex post budget balance, simply redefine the payments as follows:

$$\hat{t}^j\left(w^j, w^{-j}\right) = t^j\left(w^j, w^{-j}\right) + \frac{1}{2}\left[R\left(w^j, w^{-j}\right) - E_{w^{-j}}\left[R\left(w^j, w^{-j}\right)\right] + E_{w^j}\left[R\left(w^j, w^{-j}\right)\right]\right],$$

where $R\left(w^j, w^{-j}\right)$ is the revenue of the mechanism, as defined in (1).

Appendix : Proofs

Proof of the second part of Theorem 1: $M^j(\alpha)$ is decreasing in α^j. Consider an α-optimal mechanism with thresholds $\left(M^s(\alpha), M^b(\alpha)\right)$ and an $\hat{\alpha}$-optimal mechanism with thresholds $\left(M^s(\hat{\alpha}), M^b(\hat{\alpha})\right)$. Denote the respective derivatives' ratios by r_m^j and $\hat{r}_m^j$ and the cutoffs by r^* and $\hat{r}^*$. Assume to the contrary that $\alpha^j < \hat{\alpha}^j$ but $M^j(\alpha) < M^j(\hat{\alpha})$. By definition of M^j, and since r_m^j decreases in α^j for all m, $r^* \geq r^j_{M^j(\alpha)} > \hat{r}^j_{M^j(\alpha)} \geq \hat{r}^*$. Since the r_m^{-j}'s decrease in $\alpha^{-j} = 1 - \alpha^j$ and since $r^* > \hat{r}^*$, $r_m^{-j} \geq r^*$ implies $\hat{r}_m^{-j} > \hat{r}^*$ for all m. Thus, also for $-j$ fewer types trade. But this implies that both $T^s(\mu^s)$ and $T^b\left(\mu^b\right)$ are lower, contradicting optimality.

Lemma 1. Eager-Eager Encounters: Assume to the contrary that $\Gamma = \left\langle p, \tau^s, \tau^b\right\rangle$ is a LDIC-optimal mechanism with $p\left(k; \hat{w}^s, \hat{w}^b\right) < 1$ for some $\left(\hat{w}^s, \hat{w}^b\right)$ such that $\hat{w}_k^s = v_e^s$ and $\hat{w}_k^b = v_e^b$. Modify the mechanism by setting $p\left(k; \hat{w}^s, \hat{w}^b\right) = 1$. The modified mechanism is also LDIC since

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are required to hold only when $p^j(\cdot, \cdot)$ is defined.

for each agent j (i) The utility from reporting any type other than $\hat{w}^j$ (either truthfully or not) is unchanged; (ii) For type $\hat{w}^j$, the utility from reporting truthfully is unchanged; (iii) For any type $\tilde{w}^j \neq \hat{w}^j$, the expected utility from pretending to be $\hat{w}^j$ is weakly lower (same if $\tilde{w}^j_k = v^j_e$ and strictly lower if $\tilde{w}^j_k = v^j_r$). The ex ante surplus of the mechanism is, however, increased by $\left[1 - p(k, \hat{w}^s, \hat{w}^b)\right] \cdot prob(\hat{w}^b) \cdot prob(\hat{w}^s) \cdot \left(v^b_e - v^s_e\right)$, where $prob(\hat{w}^j)$ is the prior probability that agent j is of type $\hat{w}^j$. This additional surplus can be transferred to one of the agents (as a lump sum, so that incentives are unaffected), contradicting the optimality of Γ.

Reluctant-Reluctant Encounters: Assume to the contrary that $\Gamma = \langle p; \tau^s, \tau^b \rangle$ is LDIC-optimal where $p\left(k; \hat{w}^s, \hat{w}^b\right) > 0$ for some $\left(\hat{w}^s, \hat{w}^b\right)$ such that $\hat{w}^s_k = v^s_r$ and $\hat{w}^b_k = v^b_r$. Modify Γ by setting $p\left(k; \hat{w}^s, \hat{w}^b\right) = 0$. The same arguments as above (with the slight change that for type $\tilde{w}^j$, pretending to be $\hat{w}^j$ yields the same expected utility if $\tilde{w}^j_k = v^j_r$ and strictly lower if $\tilde{w}^j_k = v^j_e$) imply that the modified mechanism is also LDIC. The ex ante surplus of the mechanism is increased by $p(k; \tilde{w}^s, \tilde{w}^b) \cdot prob(\tilde{w}^b) \cdot prob(\tilde{w}^s) \cdot (v^s_r - v^b_r)$, again contradicting the optimality of Γ.

Lemma 3. Since Γ is symmetric and LDIC, then $\bar{\tau}\left(w^{j,m+1}\right) \geq \bar{\tau}\left(w^{j,m}\right) + q^{-j}\mu^j_m\Delta^j$ for $j \in \{b, s\}$, $m \in \{0, ..., N-1\}$. Suppose now that Γ is not tight. Then, there is an agent j for which at least one of these N inequalities is strict. Denote the index of the first one by k:

$$
\begin{aligned}
&q^{-j}\mu^j_m\Delta^j = \bar{\tau}^j\left(w^{j,m+1}\right) - \bar{\tau}^j\left(w^{j,m}\right) \quad \text{for } m \in \{0, ..., k-1\} \\
&q^{-j}\mu^j_k\Delta^j < \bar{\tau}^j\left(w^{j,k+1}\right) - \bar{\tau}^j\left(w^{j,k}\right) \\
&q^{-j}\mu^j_m\Delta^j \leq \bar{\tau}^j\left(w^{j,m+1}\right) - \bar{\tau}^j\left(w^{j,m}\right) \quad \text{for } m \in \{k+1, ..., N-1\}
\end{aligned}
$$

Let $\hat{\Gamma} = \left(p, \hat{\tau}^s, \hat{\tau}^b\right)$ denote a mechanism with p as in Γ, and $\hat{\tau}^j$ defined as follows:

$$
\hat{\tau}^j\left(w^{j,m}, w^{-j}\right) = \begin{cases} \tau\left(w^{j,m}, w^{-j}\right) + d\left[1 - F_{BD}\left(k, N, q^j\right)\right] & \text{for } m \in \{0, ..., k\} \\ \tau\left(w^{j,m}, w^{-j}\right) + d\left[1 - F_{BD}\left(k, N, q^j\right)\right] - d & \text{for } m \in \{k+1, ..., N-1\} \end{cases}
$$

where $d = \bar{\tau}\left(w^{j,k+1}\right) - \bar{\tau}\left(w^{j,k}\right) - q^{-j}\mu^j_k\Delta^j$ and F_{BD} is the c.d.f. of the Binomial distribution.

Since the prior probability of having at least $k+1$ eager valuations is $1 - F_{BD}\left(k, N, q^j\right)$, the mechanism's ex ante revenue and the agents' ex ante utilities are the same as under Γ. Moreover,

$$
\begin{aligned}
&q^{-j}\mu^j_m\Delta^j = \bar{\hat{\tau}}^j\left(w^{j,m+1}\right) - \bar{\hat{\tau}}^j\left(w^{j,m}\right) \quad \text{for } m \in \{0, ..., k\} \\
&q^{-j}\mu^j_m\Delta^j \leq \bar{\hat{\tau}}^j\left(w^{j,m+1}\right) - \bar{\hat{\tau}}^j\left(w^{j,m}\right) \quad \text{for } m \in \{k+1, ..., N-1\}
\end{aligned}
$$

If $\hat{\Gamma}$ is TSLDIC - we are done, otherwise - repeat the process of eliminating a non-binding constraint (finitely many times) until the mechanism is tight.

Lemma 4.[12] Suppose to the contrary that $r_k^j < r_l^{j'}$ but there exists an optimal solution $\left(\mu^s, \mu^b\right)$ for which μ_k^j does not attain its upper bound and $\mu_l^{j'} > 0$. Define the following constants:

$$a_k^j = \partial\left[\alpha^s T^s\left(\mu^s\right) + \alpha^b T^b\left(\mu^b\right)\right]/\partial\mu_k^j \qquad a_l^{j'} = \partial\left[\alpha^s T^s\left(\mu^s\right) + \alpha^b T^b\left(\mu^b\right)\right]/\partial\mu_l^{j'}$$

$$b_k^j = \partial\left[T^s\left(\mu^s\right) + T^b\left(\mu^b\right) - S\left(\mu^b, \mu^s\right)\right]/\partial\mu_k^j \qquad b_l^{j'} = \partial\left[T^s\left(\mu^s\right) + T^b\left(\mu^b\right) - S\left(\mu^b, \mu^s\right)\right]/\partial\mu_l^{j'}$$

and let $r_k = b_k^j/a_k^j$, $r_l = b_l^{j'}/a_l^{j'}$. Consider now an alternative solution $\left(\tilde{\mu}^s, \tilde{\mu}^b\right)$ which is identical to μ^s, μ^b, except that $\tilde{\mu}_k^j = \mu_k^j + \varepsilon$ and $\tilde{\mu}_l^{j'}$ is such that the budget constraint is satisfied, that is, $b_k^j(\tilde{\mu}_k^j - \mu_k^j) = b_l^{j'}(\mu_l^{j'} - \tilde{\mu}_l^{j'})$. The value of the objective function for $\left(\tilde{\mu}^s, \tilde{\mu}^b\right)$ is:

$$\begin{aligned}
\alpha^s T^s(\tilde{\mu}^s) + \alpha^b T^b(\tilde{\mu}^b) &= \alpha^s T^s(\mu^s) + \alpha^b T^b(\mu^b) + a_k^j(\tilde{\mu}_k^j - \mu_k^j) + a_l^{j'}(\tilde{\mu}_l^{j'} - \mu_l^{j'}) \\
&= \alpha^s T^s(\mu^s) + \alpha^b T^b(\mu^b) + (\tilde{\mu}_k^j - \mu_k^j)b_k^j/r_k - (\mu_l^{j'} - \tilde{\mu}_l^{j'})b_l^{j'}/r_l \\
&> \alpha^s T^s(\mu^s) + \alpha^b T^b(\mu^b) + [(\tilde{\mu}_k^j - \mu_k^j)b_k^j - (\mu_l^{j'} - \tilde{\mu}_l^{j'})b_l^{j'}]/r_k \\
&= \alpha^s T^s(\mu^s) + \alpha^b T^b(\mu^b),
\end{aligned}$$

which is a contradiction to the optimality of $\left(\mu^s, \mu^b\right)$.

Lemma 5. For $m \geq 0$, the numerator of r_m^j is $\partial\alpha^j T^j\left(\mu^j\right)/\partial\mu_m^j = \alpha^j q^{-j}\Delta^j\left(1 - F_{BD}\left(m, N, q^j\right)\right)$. The denominator is $\partial\left[T^j\left(\mu^j\right) - S\left(\mu^b, \mu^s\right)\right]/\partial\mu_m^j = \partial T^j\left(\mu^j\right)/\partial\mu_m^j - q^{-j}(v_e^b - v_e^s - \Delta^j)\left(N - m\right) f_{BD}(m, N, q^j)$. Thus,

$$r_m^j = \frac{1}{\alpha^j}\left[1 - \frac{(v_e^b - v_e^s - \Delta^j)}{\Delta^j} \cdot \frac{(N-m) \times f_{BD}(m, N, q^j)}{(1 - F_{BD}\left(m, N, q^j\right))}\right] \tag{6}$$

For $m = -1$, that is for c^j, $r_{-1}^j = 1/\alpha^j > r_m^j$ for all $m \geq 0$. To show that r_m^j decreases in m, it remains to show that $(N-m)\frac{[f_{BD}(m,N,q)]}{[1-F_{BD}(m,N,q)]}$ is strictly increasing in m for every $q \in [0, 1]$.

Let $p_m = f_{BD}(m, N, q) = \binom{N}{m}\left(q^j\right)^m\left(1 - q^j\right)^{N-m}$. Following Chechile (2003), define $W = \frac{p_{k+j}p_{\hat{k}}}{p_{\hat{k}+j}p_k}$ for every $\hat{k}, k, j \in \{0, ..., N\}$. Observe that $W = \frac{\left[(\hat{k}+j)\cdot\ldots\cdot(\hat{k}+1)\right]\left[(n-k)\cdot\ldots\cdot(n-k-j+1)\right]}{\left[(k+j)\cdot\ldots\cdot(k+1)\right]\left[(n-\hat{k})\cdot\ldots\cdot(n-\hat{k}-j+1)\right]}$, and multiply both sides by $\frac{(n-\hat{k})}{(n-k)}$ to get: $\frac{p_{k+j}p_{\hat{k}}}{p_{\hat{k}+j}p_k}\frac{(n-\hat{k})}{(n-k)} = \frac{\left[(\hat{k}+j)\cdot\ldots\cdot(\hat{k}+1)\right]\left[(n-k-1)\cdot\ldots\cdot(n-k-j+1)\right]}{\left[(k+j)\cdot\ldots\cdot(k+1)\right]\left[(n-\hat{k}-1)\cdot\ldots\cdot(n-\hat{k}-j+1)\right]}$.

Let $\hat{k} > k$. Since each of the terms in the numerator of the RHS is larger than the corresponding term in the denominator, then $\frac{p_{k+j}p_{\hat{k}}}{p_{\hat{k}+j}p_k}\frac{n-\hat{k}}{n-k} > 1$ or, equivalently, $\frac{1}{n-k}\frac{p_{k+j}}{p_k} > \frac{1}{n-\hat{k}}\frac{p_{\hat{k}+j}}{p_{\hat{k}}}$. Thus, $\frac{1}{n-k}\frac{\sum_{i=k+1}^{i=N-(\hat{k}-k)}p_i}{p_k} > \frac{1}{n-\hat{k}}\frac{\sum_{i=\hat{k}+1}^{i=N}p_i}{p_{\hat{k}}}$, and therefore also $\frac{1}{n-k}\frac{\sum_{i=k+1}^{i=N}p_i}{p_k} > \frac{1}{n-\hat{k}}\frac{\sum_{i=\hat{k}+1}^{i=N}p_i}{p_{\hat{k}}}$. Since $\sum_{i=0}^{i=N}p_i = 1$ then $\frac{1}{n-k}\frac{1-\sum_{i=0}^{i=k}p_i}{p_k} > \frac{1}{n-\hat{k}}\frac{\sum_{i=0}^{i=\hat{k}}p_i}{p_{\hat{k}}}$. Recall that $F_{BD}(k, N, q) = 1 - \sum_{i=0}^{i=k}p_i$, and therefore: $(n-k)\frac{f_{BD}(k,N,q)}{1-F_{BD(k,N,q)}} < (n-\hat{k})\frac{f_{BD}(\hat{k},N,q)}{1-F_{BD(k,N,q)}}$.

Lemma 6. Consider two different types $w^{j,m}$ and $\hat{w}^{j,m+k}$ where $k \geq 0$. In a TSLDIC mechanism,

[12] An alternative proof of this lemma, using Lagrange-multipliers, is available upon request from the authors.

$\bar{\tau}\left(\hat{w}^{j,m+k}\right)-\bar{\tau}\left(w^{j,m}\right)=\sum_{l=0}^{k-1}p^j(v_r^j,m+l)\Delta^j$. By monotonicity:

$$k\Delta^j p^j(v_r^j,m)\leq\bar{\tau}\left(\hat{w}^{j,m+k}\right)-\bar{\tau}\left(w^{j,m}\right)\leq k\Delta^j p^j(v_e^j,m+k). \tag{7}$$

Define the *distance* $\left\|w^{j,m},\hat{w}^{j,m+k}\right\|$ between $w^{j,m}$ and $\hat{w}^{j,m+k}$ as the number of elements by which the two vectors differ. By monotonicity, and since $\left\|w^{j,m},\hat{w}^{j,m+k}\right\|\geq k$, we have:

$$\begin{aligned}&\frac{1}{2}\left(\left\|w^{j,m},\hat{w}^{j,m+k}\right\|-k\right)\Delta^j\left(p^j(v_r^j,m)-p^j(v_e^j,m)\right)\leq 0\leq\\&\frac{1}{2}\left(\left\|w^{j,m},\hat{w}^{j,m+k}\right\|-k\right)\Delta^j\left(p^j(v_e^j,m+k)-p^j(v_r^j,m+k)\right)\end{aligned} \tag{8}$$

Adding (7) and (8) and applying simple algebraic manipulations, we get:

$$\bar{\tau}\left(\hat{w}^{j,m+k}\right)\geq\bar{\tau}\left(w^{j,m}\right)+\frac{1}{2}\left(\left\|w^{j,m},\hat{w}^{j,m+k}\right\|+k\right)\Delta^j p^j(v_r^j,m)-\frac{1}{2}\left(\left\|w^{j,m},\hat{w}^{j,m+k}\right\|-k\right)\Delta^j p^j(v_e^j,m) \tag{9}$$

$$\bar{\tau}\left(w^{j,m}\right)\geq\bar{\tau}\left(\hat{w}^{j,m+k}\right)-\frac{1}{2}\left(\left\|w^{j,m},\hat{w}^{j,m+k}\right\|+k\right)\Delta^j p^j(v_e^j,m+k)+\frac{1}{2}\left(\left\|w^{j,m},\hat{w}^{j,m+k}\right\|-k\right)\Delta^j p^j(v_r^j,m+k) \tag{10}$$

Note that there are exactly $\frac{1}{2}\left(\left\|w^{j,m},\hat{w}^{j,m+k}\right\|+k\right)$ objects that $\hat{w}^{j,m+k}$ is eager to trade and $w^{j,m}$ is reluctant to trade, and $\frac{1}{2}\left(\left\|w^{j,m},\hat{w}^{j,m+k}\right\|-k\right)$ objects that $\hat{w}^{j,m+k}$ is reluctant to trade $w^{j,m}$ is eager to trade. Thus, for every k, (9) implies that $\hat{w}^{j,m+k}$ does not gain from pretending to be $w^{j,m}$, and (10) implies that $w^{j,m}$ does not gain from pretending to be $\hat{w}^{j,m+k}$. This means that the mechanism is incentive compatible.

Proposition 1. The ex ante utility of agent j is determined by $\bar{\tau}^j$, which is, in turn, determined by $\{\mu_m^j\}_{m=-1}^{N-1}$ according to (3). Lemma 5 implies that in an α-optimal mechanism μ_m^j is (weakly) increasing in m and thus $\bar{\tau}^j$ is positive iff $\mu_{N-1}^j>0$. By (5), for $\alpha^j=0$ the value of r_{N-1}^j is either $+\infty$ or $-\infty$, implying $\mu_{N-1}^j=0$ or $\mu_{N-1}^j=1$, respectively. By (6), for $\alpha^j=0$, $r_{N-1}^j=-\infty$ iff:

$$\frac{\Delta^j\cdot\left(1-F_{BD}\left(N-1,N,q^j\right)\right)-\left(v_e^b-v_e^s-\Delta^j\right)\cdot f_{BD}(N-1,N,q^j)}{\Delta^j\cdot\left(1-F_{BD}\left(N-1,N,q^j\right)\right)}<0$$

Note that the denominator is always positive, substitute $f_{BD}(N-1,N,q^j)=N\left(q^j\right)^{N-1}\left(1-q^j\right)$ and $F_{BD}\left(N-1,N,q^j\right)=1-\left(q^j\right)^N$ to get that $r_{N-1}^j=-\infty$ iff $N\geq\frac{q^j}{1-q^j}\cdot\frac{\Delta^j}{v_e^b-v_e^s-\Delta^j}$.

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