

# 1 Appendix B

In problem (4) we look for the vectors  $\mu^b, \mu^s$  that maximize an  $a$ -weighted sum of the ex-ante utilities subject to the specified constraints. In this appendix we derive the necessary conditions for an optimal solution using Lagrange multipliers and provide an alternative proof for lemma 4.

Recall that since  $\mu^j = (\mu_{-1}^j, \dots, \mu_{N-1}^j)$  for  $j \in \{s, b\}$ , the maximization problem involves  $2N + 2$  variables and  $4N + 3$  constraints. We rewrite the problem as follows:

$$\begin{aligned} & \text{MAX}_{\mu^s, \mu^b} \quad \alpha^s T^s(\mu^s) + \alpha^b T^b(\mu^b) \\ & \text{subject to: } 1. \quad T^s(\mu^s) + T^b(\mu^b) - S(\mu^s, \mu^b) = 0 \\ & \quad \quad \quad 2. \quad \mu_m^j \geq 0, \quad m \in \{-1, \dots, N-1\}, j \in \{s, b\} \\ & \quad \quad \quad 3. \quad \mu_m^j \leq 1, \quad m \in \{0, \dots, N-1\}, j \in \{s, b\} \end{aligned}$$

Denote by  $\lambda, \{\delta_m^b\}_{m=-1}^{N-1}, \{\delta_m^s\}_{m=-1}^{N-1}, \{\gamma_m^b\}_{m=0}^{N-1}, \{\gamma_m^s\}_{m=0}^{N-1}$  the respective Lagrange multipliers. The Lagrangean function is then:

$$\begin{aligned} \mathcal{L}(\mu^s, \mu^b) &= \alpha^s T^s(\mu^s) + \alpha^b T^b(\mu^b) \\ &\quad - \lambda [T^s(\mu^s) + T^b(\mu^b) - S(\mu^s, \mu^b)] \\ &\quad + \sum_{m=-1}^{N-1} \delta_m^b \mu_m^b + \sum_{m=-1}^{N-1} \delta_m^s \mu_m^s \\ &\quad - \sum_{m=0}^{N-1} \gamma_m^b (\mu_m^b - 1) - \sum_{m=0}^{N-1} \gamma_m^s (\mu_m^s - 1) \end{aligned}$$

The necessary conditions for an optimal solution (Kuhn-Tucker conditions) are given by:

1. Equating all partial derivatives of  $\mathcal{L}(\mu^s, \mu^b)$  w.r.t  $\mu_m^j$  to 0:

$$(1a) \quad \frac{\partial \mathcal{L}(\mu^s, \mu^b)}{\partial \mu_m^j} = \alpha^j \frac{\partial T^j(\mu^j)}{\partial \mu_m^j} - \lambda \left[ \frac{\partial T^j(\mu^j)}{\partial \mu_m^j} - \frac{\partial S(\mu^s, \mu^b)}{\partial \mu_m^j} \right] + \delta_m^j - \gamma_m^j = 0, \text{ for } m \in \{0, \dots, N-1\}, j \in \{s, b\}$$

$$(1b) \quad \frac{\partial \mathcal{L}(\mu^s, \mu^b)}{\partial \mu_{-1}^j} = (\alpha^j - \lambda) + \delta_{-1}^j = 0, \text{ for } j \in \{s, b\}.$$

2. Complementary slackness conditions:

$$(2a) \quad \delta_m^j \geq 0 \quad (\delta_m^j = 0 \text{ if } \mu_m^j > 0) \quad \text{for } m \in \{-1, \dots, N-1\}, j \in \{s, b\}$$

$$(2b) \quad \gamma_m^j \geq 0 \quad (\gamma_m^j = 0 \text{ if } \mu_m^j < 1) \quad \text{for } m \in \{0, \dots, N-1\}, j \in \{s, b\}$$

3. The original constraints:

$$(3a) \quad T^s(\mu^s) + T^b(\mu^b) - S(\mu^s, \mu^b) = 0$$

$$(3b) \quad \mu_m^j \geq 0 \quad \text{for } m \in \{-1, \dots, N-1\}, j \in \{s, b\}$$

$$(3c) \quad \mu_m^j \leq 1 \quad \text{for } m \in \{0, \dots, N-1\}, j \in \{s, b\}$$

The slackness conditions (2a),(2b) and constraints (3b),(3c) imply that for every  $j \in \{s, b\}$  and  $m \in \{0, \dots, N-1\}$ , either  $\delta_m^j = 0$ , or  $\gamma_m^j = 0$ , or both. Also, both are non-negative. Thus,

$$\begin{aligned} \text{if } (\gamma_m^j - \delta_m^j) < 0 & \text{ then } (\delta_m^j > 0, \gamma_m^j = 0) \\ \text{if } (\gamma_m^j - \delta_m^j) = 0 & \text{ then } (\delta_m^j = 0, \gamma_m^j = 0) \\ \text{if } (\gamma_m^j - \delta_m^j) > 0 & \text{ then } (\delta_m^j = 0, \gamma_m^j > 0) \end{aligned}$$

Recall that  $r_m^j$  is defined as follows:

$$r_m^j = \frac{1}{\alpha^j} \frac{\partial [T^j(\mu^j) - S(\mu^s, \mu^b)] / \partial \mu_m^j}{\partial T^j(\mu^j) / \partial \mu_m^j} \in [-\infty, +\infty]$$

and note that  $r_{-1}^j = 1/\alpha^j$  and that (1a) and (1b) reduce to:

$$\begin{aligned} (1a') \quad \alpha^j \frac{\partial T^j(\mu^j)}{\partial \mu_m^j} (1 - \lambda r_m^j) &= \gamma_m^j - \delta_m^j \\ (1b') \quad \alpha^j (1 - \lambda r_{-1}^j) &= -\delta_{-1}^j \end{aligned}$$

Note also that since (1b) holds for both  $j = b$  and  $j = s$ , and since  $\delta_{-1}^j \geq 0$  then in every optimal solution the budget constraint is binding, that is  $\lambda > 0$ .

**Claim 1** *In the optimum,  $\text{sign}[1 - \lambda r_m^j] = \text{sign}[\gamma_m^j - \delta_m^j]$  for every  $j \in \{s, b\}$  and  $m \in \{0, \dots, N-1\}$*

**Proof.** If  $\alpha^j > 0$  the claim follows directly from (1a') and the fact that  $\frac{\partial T^j(\mu^j)}{\partial \mu_m^j} > 0$ .

If  $\alpha^j = 0$  then the definition of  $r_m^j$  and (1a) imply:

$$\text{sign}[r_m^j] = \text{sign}\left[\frac{\partial T^j(\mu^j)}{\partial \mu_m^j} - \frac{\partial S(\mu^s, \mu^b)}{\partial \mu_m^j}\right] = \text{sign}[\delta_m^j - \gamma_m^j]$$

since  $\alpha^j = 0$  implies that  $r_m^j$  is either  $\infty$  or  $-\infty$ , we obtain:

$$\text{sign}[1 - \lambda r_m^j] = \text{sign}[\gamma_m^j - \delta_m^j]$$

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Claim 1, (2a) and (2a) imply that for every  $j \in \{s, b\}$  and  $m \in \{0, \dots, N-1\}$ :

$$\begin{aligned} r_m^j < 1/\lambda & \implies (\gamma_m^j - \delta_m^j) > 0 \implies \mu_m^j = 1 \\ r_m^j = 1/\lambda & \implies (\gamma_m^j - \delta_m^j) = 0 \implies 0 < \mu_m^j < 1 \\ r_m^j > 1/\lambda & \implies (\gamma_m^j - \delta_m^j) < 0 \implies \mu_m^j = 0 \end{aligned}$$

For  $m = -1$ , (1b') and (2a) imply that in every optimal solution  $r_{-1}^j \geq \frac{1}{\lambda}$ , and if  $r_{-1}^j > \frac{1}{\lambda}$  then  $\mu_{-1}^j = 0$ .

Thus, in an optimal solution, for every  $j, j' \in \{b, s\}$  and  $l, k \in \{-1, \dots, N-1\}$ : if  $r_k^j < r_l^{j'}$  then either  $r_k^j < \frac{1}{\lambda}$  or  $r_l^{j'} > \frac{1}{\lambda}$  (or both). And therefore, either  $\mu_k^j$  attains its upper bound, or  $\mu_l^{j'} = 0$ .