Independent Mistakes in Large Games*

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Abstract

Economic models usually assume that agents play precise best responses to others' actions. It is sometimes argued that this is a good approximation when there are many agents in the game, because if their mistakes are independent, aggregate uncertainty is small. We study a class of games in which players' payoffs depend solely on their individual actions, and on the aggregate of all players' actions. We investigate whether their equilibria are affected by mistakes when the number of players becomes large. Indeed, in generic games with *continuous* payoff functions, independent mistakes wash out in the limit. This may not be the case if payoffs are discontinuous. As a counter-example we present the *n* players Nash bargaining game, as well as a large class of "free-rider games."

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1. Introduction

Certain simplifying assumptions used in modeling economic environments may be justified by the claim that they become good approximations when the number of agents is large. One such assumption is that economic agents always pick the exact optimal actions and never make mistakes. It is sometimes argued that even if players do make mistakes, the effect of the mistakes is negligible when there are many agents in the game: If the mistakes are independent, the law of large numbers will guarantee that they cancel each other in the limit. For instance, in his seminal paper on Rational Expectations, Muth (1961) argues:

> *"Deviations from rationality... their aggregate effect is negligible as long as the deviation from the rational forecast for an individual firm is not strongly correlated with those of the others."*

This paper attempts to verify the above intuition in a formal setting. We argue that, while under some conditions a player may indeed ignore the fact that her opponents make mistakes, there exist cases in which the effect of deviations from optimal play on the outcome may even increase with the number of agents.

A motivating example is the following (generalized) Nash-bargaining game: n dollars are to be split among n people. Each person (simultaneously) submits her demand. If the sum of the demands exceeds n, they all get zero; otherwise each player receives her demand. For every n, the symmetric Nash equilibrium, which is also a reasonable focal point, dictates that each player demand one dollar. For a small number of agents (say n=2), this prediction seems reasonable even in the presence of some uncertainty as to the agents' bids. By contrast, this equilibrium is not very robust if n is, say, the number of people in North America. When the number of agents is large, aggregate uncertainty makes each agent believe that she is not pivotal and thus induces overbidding. Consequently, it stands to reason that for large n, the probability that the prize will eventually be distributed is small. In other words, precisely because there are many agents, random mistakes cannot be ignored.

Formally, we model the agents' imperfect rationality using the concept of "noisy equilibrium". This is defined as the Nash equilibrium of a modified game, in which it is common knowledge among the players that each of them trembles in her play (or in her calculation of the best response) by a random variable, where the trembles are independent across players. In this terminology, our task is to classify games according to whether their Nash and noisy equilibria converge to each other as the number of players increases.

We focus on a class of games in which each player's payoff depends only on her own (real-valued) action and on the average of the actions of all the players. Many economic models that involve a large number of agents fall into this category. Examples include models of markets, competition among firms, public goods, and, in particular, Nash's bargaining game. Games in this class are natural candidates for which the intuitive "law of large numbers" might apply: since the effect of a large number of independent mistakes on the average action is small, one may expect each player to ignore the mistakes made by her opponents.

We find that in generic games with *continuous* payoff functions, Nash and noisy equilibria converge to each other as the number of player goes to infinity. Hence, random deviations from optimal play can be ignored when the number of agents is large. In contrast, Muth's intuition might not apply in games with *discontinuous* payoffs. In such games (including the bargaining game), independent mistakes may change the outcome considerably even when the number of players grows without bound.

Intuitively, this distinction is due to the fact that the introduction of noise "smoothes" the payoff functions. This, of course, has no significant effect on games that already have continuous payoffs. However, games with discontinuous payoffs are altered considerably. When there are many players, each has only a small effect on the aggregate. With discontinuous payoffs, this small effect might still cause large changes in a player's payoff. But when payoffs are smoothed out by noise, each player will ignore her effect on the aggregate and consider only the direct effect of her action on her payoff.

One important implication of this phenomenon pertains to the class of "free rider" games. In such games each player has a direct incentive to take some action, but by doing so the player affects the aggregate and thereby reduces the utility of all the players (herself included). We show that free rider games with many players are doomed to yield inefficient outcomes if players tremble in their play. Even if payoffs are discontinuous, all players act "myopically" and consider only the direct effect of their actions on the outcome. In other words, severe punishments of small deviations, due to discontinuous payoff functions, are not effective when players tremble in their play.

Our solution concept, the noisy equilibrium, is related to Selten's (trembling hand) perfect equilibrium. Both deviate from Nash equilibrium by allowing for mistakes made by the players. However, the two concepts differ considerably. While Selten's perfect equilibrium focuses on infinitesimal mistakes (by analyzing the limit case as the probability of a tremble goes to zero), the noisy equilibrium assumes mistakes of fixed, finite size. To test the robustness of Nash equilibrium to the introduction of mistakes, we take the number of players to infinity rather than the size of mistakes to zero.

The effect of deviations from optimal play has also been studied in the context of more specific economic models. Akerlof and Yellen (1985) consider whether small deviations from rationality can significantly change an economy's competitive equilibrium. They examine whether the effect of mistakes on the equilibrium of a fixed economy is first or second order compared to their magnitude. Wolinsky (1994) considers a dynamic setting and shows that small deviations from maximizing behavior can have a large effect on the outcome. Again, the focus of these models is on infinitesimal trembles, in contrast to our focus on finite size trembles whose average variance becomes small when there are many agents. This approach seems to be closer to the spirit of Muth's quote.[1]

The remainder of the paper is organized as follows: In section 2 we formally define our solution concept, the noisy equilibrium, and the class of games to be studied ("symmetric aggregation games"). In section 3 we study the bargaining game example within that framework and show that independent mistakes cannot be ignored at the limit. Section 4 generalizes the bargaining example to the class of free rider games. We show that while Nash equilibria of such games can be efficient, the outcomes of noisy equilibria must converge to "myopic" outcomes. In section 5 we identify conditions under which the noisy and Nash equilibria converge to each other as the number of players tends to infinity. All the proofs are relegated to the appendix.

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[1] Muth (1961) suggests the question studied in this paper, but pursues a different investigation: he considers a fixed economy and verifies the effect of various forms of stochastic mistakes on the rational expectations equilibrium for different supply and demand functions.

2. The Model

Equilibrium Concept

Let $G=(N,\mathbf{A},\pi)$ be a game, where $N=\{1...n\}$ is the set of players, $\mathbf{A}=\prod_{i=1}^{n}A_i$, where $A_i=A\subset\Re$ are the actions' sets, and $\pi_i:\mathbf{A}\rightarrow\Re$ are the payoff functions. Let $\varepsilon=(\varepsilon_1...\varepsilon_n)$ be a vector of independent, real valued random variables.[2] An *ε-noisy equilibrium* of *G* is a vector of "intended" actions $(a_1^*...a_n^*)\in A^N$ such that for every player *i*, $a_i^*\in\arg\max_{a_i\in A_i}\underset{\varepsilon}{\mathrm{E}}[\pi_i(a_1^*+\varepsilon_1,...,a_n^*+\varepsilon_n)]$.

A noisy equilibrium is simply a Nash equilibrium of the modified game, in which it is common knowledge among the players that they each tremble by a random variable ε_i, where the trembles are independent of each other and of the intended actions a_i. This solution concept is related to Selten's (1975) 'trembling hand' perfect equilibrium, in which players are assumed to make small mistakes and sometimes take actions that are not optimal. But whereas Selten studies the limit of equilibria for vanishing trembles, we keep the errors non-infinitesimal and let the average uncertainty vanish by increasing the number of players. It should also be noted that while Selten's concept is a refinement of Nash equilibrium (i.e., a perfect equilibrium is always a Nash equilibrium), a noisy equilibrium may not be close to any Nash equilibrium of the game. We will later use this concept to test the robustness of Nash equilibria of games that can be replicated in a natural way by comparing the Nash and noisy equilibria when the number of players goes to infinity.

Solution concepts related to the 'noisy equilibrium' introduced here have been investigated in several recent papers. Beja (1990) defines 'Imperfect Performance Equilibrium' for normal form games with finite action space. This is a Nash equilibrium of a game in which players make non-infinitesimal stochastic mistakes that depend on intended actions and on their potential consequences. Beja considers games with a fixed, finite number of players and strategies, but allows payoffs to vary. He shows that for some canonical game forms, the introduction of fixed trembles may alter the equilibrium when payoffs are chosen appropriately. McKelvey and Palfrey (1994) define 'Quantal Response Equilibrium,' in which players receive noisy observations of their payoff entries. This

[2] When no confusion is likely to arise, we will not distinguish between the random variable ε_i and its realization.

approach is related to Harsanyi's (1973) purification of mixed strategies via a Bayesian game, as the deviations from optimal play arise from mistaken observations of the payoffs.[3]

Symmetric Aggregation Games

We wish to test whether noisy and Nash equilibria become close to each other as the number of players in the game grows. Hence, we need to look at games that can be replicated in a natural manner. For $F:\Re^2 \to \Re$ and $n \geq 1$, let the *symmetric aggregation game* $G(F) = G(N, A, F)$ be the game defined by

$$\pi_i(x_1 ... x_n) = F(x_i, \bar{x})$$

where $x_i \in A \subset \Re$ for every *i*, and $\bar{x} \equiv \frac{x_1 + ... + x_n}{n}$.[4] noisy equilbria of this game are Nash equilibria of a noisy game, with payoffs:

$$\pi_i^{\varepsilon}(x_1 ... x_n) = F_n^{\varepsilon}(x_i, \bar{x}) \equiv \underset{\varepsilon_1 ... \varepsilon_n}{\mathrm{E}}\left[F(x_i + \varepsilon_i, \bar{x} + \bar{\varepsilon})\right].$$

As the number of players grows, the aggregate noise $\bar{\varepsilon}$ converges to a degenerate random variable, i.e., to its expected value. In other words, aggregate uncertainty vanishes as $n \to \infty$. We can now inquire whether the effect of the trembles on the equilibrium also vanishes as the number of players tends to infinity. While the class of symmetric aggrega-

[3] It should be noted, however, that the finite action space model is not the appropriate one for the question we are studying. It would only be so if we thought of players as "really" attempting to choose specific mixed strategies and making errors in choosing the mixture. If we don't claim that players actually choose to mix, but rather interpret mixing à la Harsanyi, then the more intuitive definition of errors for discrete games involves a small fraction of the players mistakenly choosing the wrong pure action. But with such an interpretation, there is no reason to expect that a large number of independent errors will cancel out. Assume, for example, that there are two pure actions, L and R, and that players are mixing 1/3L+2/3R (i.e., one third of the types choose left and the rest choose right). Assume that a fraction ε of them mistakenly plays the wrong action. When the number of players tends to infinity, the average play converges to $(1+\varepsilon)/3L + (2-\varepsilon)/3R$. Another example is the stag-hunt game: if all players try to go for the stag, a fraction ε will mistakenly go for the hare, independent of their number. In the continuous model, the 'law of large numbers' intuition is more convincing: players really do attempt to play a (pure) real-valued action. Because of the boundedness of their abilities, they make mistakes and usually choose nearby actions. The average of such independent mistakes should be expected to vanish at the limit.

[4] While the aggregate we treat in this paper is the mean, all our results can be easily extended to other aggregates, such as the median or any other fixed percentile statistic. In fact, the only requirement is that the central limit theorem be applicable to that aggregate of the trembles.

tion games has a simple characterization, it is an appropriate framework to model many interesting economic environments. We mention two examples:

The bargaining game

Recall that a pie of n dollars is to be split between n agents. Each player submits her demand and receives it only if the sum of the demands does not exceed n dollars. For simplicity, assume that the agents' utilities are linear in money.[5] The bargaining game can be expressed within our framework as:

(1) $F(x_i, \overline{x}) \equiv x_i \cdot D(\overline{x})$,

where $D(x) = 1$ for $x \leq 1$ and $D(x) = 0$ for $x > 1$.

Cournot competition

Consider a Cournot game in which n firms compete in a market and have no production costs. The demand side consists of n consumers[6] with identical demand functions $q(p): \Re_+ \to \Re_+$, that are decreasing and smooth. Let the inverse demand function be $p(q)$. It is also positive, decreasing and smooth. Given the quantities supplied, $q_1, ..., q_n$, the market price is $p(\overline{q})$. Assuming that the firms are risk neutral, the Cournot game can be expressed as:

(2) $F(q_i, \overline{q}) \equiv q_i \cdot p(\overline{q})$

3. Motivating Example: The Bargaining Game

We now use the model to formally treat the bargaining example presented above. We assume that trembles ε_i are normally distributed with mean 0 and variance c.

[5] Risk neutrality is a reasonable assumption in this case, since we are dealing with small prizes. To capture other cases we can simply let F equal $u(x_i \cdot D(\overline{x}))$ where u is the utility from money.

[6] To avoid triviality of the problem, we let the demand grow with n. In other words, we think of a replica economy, where the demand per firm is kept constant. If, instead, we were to keep the total demand constant, each firm's production will tend to 0 as n tends to infinity.

Lemma 3.1:

In the bargaining game defined by (1), for large enough *n*, there exists a unique and symmetric ε-noisy equilibrium of the *n*-players game.[7,8]

This result illustrates a typical distinction between the set of Nash equilibria and the set of noisy equilibria. While the former may be the size of the continuum, the second tends to be small. In the bargaining game, every vector of demands that adds up to *n* dollars is a Nash equilibrium. In contrast, the set of noisy equilibria is small and for large *n* it is a singleton. It is interesting to compare this result to Nash's non-cooperative derivation of his solution. Nash perturbs the game by "smoothing" the frontier of the agreement set. The effect is similar to that of our method of adding trembles to the players' actions; the perturbed game has a unique Nash equilibrium. Nash proceeds by showing that when the perturbation becomes small, the equilibrium converges to his cooperative, axiomatic solution. In the symmetric case, where all the players have the same utility functions, the equilibrium of Nash's perturbed game is the symmetric one. This is consistent with our results.

However, while Nash treats bargaining problems with a fixed number of players and studies the limit as trembles vanish, we examine a sequence of games with an increasing number of players. While the number of players increases, the size of the individual trembles stays fixed. In contrast to Nash's solution, which is always efficient (at the limit the players receive the pie with certainty), our model results in asymptotic inefficiency, as stated in the next proposition.

Proposition 3.1:

Let x_n^ε denote the symmetric equilibrium actions of the bargaining game (1). For every $c>0$ there exists $k(c)>1$ such that for all *n*, $x_n^\varepsilon > k(c) > 1$. Hence, $\pi_i^\varepsilon \xrightarrow{n\to\infty} 0$.

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[7] For small *n*, however, there are two equilibrium points.

[8] Note that the unique equilibrium obtains for a *given* error structure, and that the symmetry depends on the errors being identical. By allowing different players to make errors of different relative sizes, one could recapture some of the non-uniqueness of the non perturbed game. However, it is not the case that every Nash equilibrium of the game can be approximated by a noisy equilibrium by changing the ratio of the trembles' sizes. For example, the noisy equilibrium must be almost symmetric if the absolute size of each trembles is small enough; alternatively, in the two-players bargaining game the division will never approach the extreme of 0-2, not even if we set one player's tremble to zero.

Proposition 1 states that the bids in a noisy equilibrium of the bargaining game are bounded below by a constant greater than 1. Since the mean of the bids converges to a degenerate random variable that is greater than 1, the probability that the pie will actually be split between the players tends to 0. Thus, the expected payoff converges to 0. This result contrasts with the (symmetric) Nash equilibria, in which the bids and payoffs are fixed at 1, independent of *n*.

4. Free Rider Games

The bargaining game studied above can be viewed as a free rider problem: Each agent would like to increase her demand if this had no effect on the sum of the demands. On the other hand, when an agent increases her demand there is a negative externality on the other players. We expect free rider environments to become inefficient as the number of agents grow. Surprisingly, the bargaining game has efficient Nash equilibria independent of the number of players. However, this asymptotic efficiency hinges on a very sensitive assumption: that even tiny deviations are met by very severe punishments. Once we introduce trembles into the agents' play, they ignore these discontinuities of the payoff functions. The free rider effect reveals itself again: every player acts as a "price taker" and ignores externalities. As a result, all the agents overbid.

This section focuses on free rider games. We show that agents ignore their effect on the aggregate, regardless of whether payoffs are continuous or not. We start with mild assumptions on the structure of the action sets and the trembles. Let the action space be connected and compact: $A = [a_-, a_+]$, and assume that the trembles $\{\varepsilon_i\}_{i=1}^{\infty}$

(1) have supports uniformly bounded in [-*s*,*s*],

(2) are independent (though not necessarily identically distributed),

(3) satisfy $\sum_{i=1}^{\infty} \text{var}(\varepsilon_i) = \infty$.

(For example, these assumptions are satisfied if $\{\varepsilon_i\}_{i=1}^{\infty}$ are iid on [-*s*,*s*]).

To capture the free ride property, we assume that F satisfies:

Assumption 4.1

$x_1 \geq x_2 \Rightarrow F(x_1, y) \geq F(x_2, y)$

$y_1 \geq y_2 \Rightarrow F(x, y_1) \leq F(x, y_2)$

Note that since F is monotone in both arguments, it is bounded over compact sets. Let it be bounded by M: $|F(x,y)| \leq M$ for any $x, y \in [a_- - s, a_+ + s]$.

Assumption 4.2(a)

There exists $\rho > 0$ such that $\frac{F(\tilde{x}, y) - F(x,y)}{\tilde{x} - x} \geq \rho$ for any $\tilde{x} > x$.

Assumption 4.1 implies that F models a "public bad." For instance, if x is the level of pollution, each agent's payoff decreases as a function of aggregate pollution. On the other hand, each player would prefer to pollute more if aggregate pollution were held fixed. Assumption 4.2(a) says that marginal payoffs are non-negligible. For instance, the extra pleasure from smoking one more cigarette is bounded away from zero. (The setting can be easily changed to account for a "public good." The claims and proofs need only slight modifications.)

Theorem 4.1(a):

Assume 4.1, 4.2(a). There exists N such that for all $n>N$, the noisy equilibrium (in dominant strategies) of the n-players game is for every player i to choose $x_i = a_+$.

In other words, Theorem 4.1(a) says that when there are enough players, they all ignore their effect on the aggregate level of pollution and smoke as much as they can.

We may be interested in cases where Assumption 4.2(a) does not apply. For instance, in the bargaining game there are y's for which $F(x, y) = 0$ for any x. However, when $F(x, y) > 0$, the condition is satisfied. In the Cournot case, the marginal contribution of x can be small if $F(x, y)$ is small. Notice that in both cases, the value 0 for F has economic meaning. Thus, it makes sense to state a condition that is weaker than Assumption 4.2(a), applicable to the case where the (positive) value of F has economic significance:

Assumption 4.2(b)

$F(x,y) \geq 0$, and there exists $\rho > 0$ such that $\frac{F(\tilde{x},y) - F(x,y)}{\tilde{x} - x} \geq \rho \cdot F(x,y)$ for any $\tilde{x} > x$.

Assumption 4.2(b) only requires the increase to be proportional to the value of F, and is implied by 4.2(a).[9] The assumption that F is non-negative entails no loss of generality: we could simply require F to grow at a rate proportional to its excess value above some reference point $m \leq \min_{[a_- - s, a_+ + s]^2} F(x, y)$. Since a constant shift of F does not alter the game, it is clearer to call such a reference point "0".

Theorem 4.1(b):

Assume 4.1, 4.2(b). For any $\eta > 0$ there exists N such that for all $n > N$ and any noisy equilibrium of the n-players game, every player i either (1) receives a payoff smaller than η, or (2) plays $x_i = a_+$ as a dominant strategy.

In other words, when the number of players is large enough, each one smokes as much as she can, unless her equilibrium expected payoff is negligible (in which case the theorem is not able to predict her individual action).

Discussion: Robustness of the Free Rider Effect

There are several examples in the literature of equilibria of games with a finite but large number of agents are very different from the equilibria of their counterparts with a continuum of agents. For example, in the durable good dynamic monopoly problem, a monopolist facing a finite number of buyers can extract all the surplus (Bagnoli, Salant and Swierzbinski (1989)), while in the continuum of buyers case (Fudenberg, Levine and Tirole (1985) and Gul, Sonnenschein and Wilson (1986)), the monopolist sells the good immediately at her reservation price. In the context of corporate takeovers, a potential

[9] Note that our bargaining game (with bounded actions and trembles) satisfies Assumption 4.2(b): when $y>1$, $F(x,y)=0$, so the condition is satisfied regardless of ρ. When $y \leq 1$, $F(x,y)=x$, and the condition is satisfied if ρ is set to the maximal possible value of x.

raider cannot extract any surplus from a continuum of small shareholders (Grossman and Hart (1980)), while an appropriate scheme offered to a large but finite group of shareholders can extract all the efficiency gains due to the takeover (Holmstrom and Nalebuff (1992)).

These examples are driven by discontinuities in the effect of aggregates on each agent's payoff. When a continuum of agents are involved, each has zero effect on the aggregate and hence considers only the direct effect of her action on her own payoff. But when there is a finite number of agents, each has a nonzero effect on the aggregate. This effect, no matter how small, is translated into a large change in her payoff when the payoff function is discontinuous. Hence, the agent is "forced" to take the aggregate into account. In this way, discontinuities in payoffs help "solve" the free rider problem present in each example.

This type of solution to the free rider problem has been criticized in several papers. Chari and Jones (1994) analyze mechanisms attempting to solve a public good problem. They require that the mechanisms satisfy individual rationality and, more importantly, a "uniform continuity" condition: that small changes in some agents' actions have only a small effect on aggregates, uniformly in the number of agents. In other words, severe punishments of small deviations are not permitted. Chari and Jones show that for any such mechanism, provision of the public good goes to zero as the number of agents tends to infinity.

Intuitively, since the agents' effect on the aggregate (the quantity of the public good) is of order $O(1/n)$, agents will care enough about the aggregate only if its affect on their utilities increases with n with order of at least $O(n)$. This could be guaranteed if, for example, the aggregate enters their utility function in a non-continuous way. Chari and Jones' uniform continuity assumption rules out such cases. However, a weaker assumption sufices to rule them out: we only require that the effect not grow too rapidly. For example, if the effect is of order $O(\sqrt{n})$, agents will, at the limit, ignore the aggregate.

There are several ways to justify such weaker versions of the continuity assumption. Mailath and Postlewaite (1990) introduce uncertainty about agents' valuations. In a Bayesian game framework, aggregates of the valuations become almost continuous as the number of agents grows, with a slope that becomes steeper at a sufficiently slow rate.

They show that the probability of provision of the public good goes to zero as the number of agents grows to infinity. Levine and Pesendorfer (1995) and Fudenberg, Levine, and Pesendorfer (1995) introduce noisy observations of the agents' actions. Again, payoff functions are sufficiently smoothed out by the noise and therefore agents act myopically.

The independent stochastic mistakes introduced in this paper could be viewed as another way to make payoffs smooth. Because of others' mistakes, each agent's indirect cost (through her effect on the aggregate) is small and she considers only her direct costs. Thus, our results might be viewed as another critique of the finite models mentioned above. Moreover, in many situations the assumption that agents make mistakes is more reasonable than some of the other assumptions. For instance, consider the corporate raider example. The assumption of uncertainty about agents' private valueations (as in Mailath and Postlewaite) is hard to defend since stock valuations are common. Imperfect observability of the number of small shareholders who gave their consent to sell, is also not compelling in this case. On the other hand, each trader's decision is still subject to error.

5. When Can Mistakes Be Ignored at the Limit?

In sections 3 and 4 we saw that independent trembles can have a large effect on games with discontinuous payoff functions since the noise "smoothed out" the payoffs. One may wonder whether, in games with payoff functions that are already continuous, the effect of the average noise vanishes as the number of players grows. In this section, we focus on the case of a continuous function F and look for conditions under which the noisy and Nash equilibria converge to each other as the number of players grows to infinity. In order to simplify the analysis, we also assume that F is strictly concave. This guarantees single valued best response correspondences.

Assumption 5.1

The function F is continuous and strictly concave.

To test whether the Nash and noisy equilibria converge to each other, we consider an ideal game which has a continuum of players. The ideal game, however, is introduced merely as an analytical tool. We will show later that the Nash and noisy equilibria of finite games approach those of the ideal game as the number of players becomes large. This implies that the Nash and noisy equilibria of the finite games approach each other.

Given the function $F:\Re^2 \to \Re$, we define $G^\infty = (I, \mathbf{A}, \pi)$, the *limit game*, as follows: $I=[0,1]$ is the set of players; $A_i = A \subset \Re$ are the actions sets; $\mathbf{A} \subset A^{[0,1]}$ is the set of measurable actions profiles; and $\pi_i(\{a_\alpha\}_{\alpha\in[0,1]}) = F(a_i, \int_0^1 a_\alpha d\alpha)$ are the payoff functions. For simplicity, we assume that the trembles $\{\varepsilon_i : i \in [0,1]\}$ are i.i.d., with mean 0.[10]

The Nash equilibrium of the limit game is defined as usual. The ε-noisy equilibrium is defined as a vector of "intended" actions $\{a_i^* : i \in [0,1]\} \in \mathbf{A}$ such that for every player $i \in [0,1]$, $a_i^* \in \arg\max_{a_i \in A} \underset{\varepsilon_i}{E}[F(a_i + \varepsilon_i, \int_0^1 a_\alpha^* d\alpha)]$.[11]

There are two differences between the finite case and the limit case. First, the aggregate noise $\bar{\varepsilon}$ appears only in the finite case. Second, the individual effect on the aggregate action appears only in the finite case. We rewrite the payoff functions corresponding to the (Nash and noisy) finite games in a way that captures these two differences. Denoting the aggregate of the other players' actions $\bar{x}_{-i} \equiv \frac{1}{n-1}\sum_{j\neq i} x_j$, the payoff functions are:

$$\pi_i(x_1 \ldots x_n) = F(x_i, \bar{x}) = F\left(x_i, \bar{x}_{-i} + \tfrac{1}{n}(x_i - \bar{x}_{-i})\right), \text{ and}$$

$$\pi_i^\varepsilon(x_1 \ldots x_n) = \underset{\varepsilon}{E}\left[F(x_i + \varepsilon_i, \bar{x} + \bar{\varepsilon})\right] = \underset{\varepsilon_1 \ldots \varepsilon_n}{\mathrm{E}}\left[F\left(x_i + \varepsilon_i, \bar{x}_{-i} + \tfrac{1}{n}(x_i - \bar{x}_{-i}) + \tfrac{1}{n}\sum \varepsilon_i\right)\right].$$

Since F is strictly concave, it is easy to check that the corresponding best response correspondences are single-valued. The Nash and noisy best response functions (to the aggregate of the other players' actions) of the limit and n-players game are, respectively:

$$R(x) = \arg\max_{a_i \in A} F(a_i, x)$$

$$R^\varepsilon(x) = \arg\max_{a_i \in A} \underset{\varepsilon_i}{E}\left[F(a_i + \varepsilon_i, x)\right]$$

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[10] The assumption that the trembles have mean 0 involves no loss of generality. Since players know the structure of their own trembles, changing the mean of the noise is equivalent to relabeling the actions.

[11] The definition implicitly assumes that in the continuum-of-players game, there is no aggregate noise. This assumption conforms to the intuition that the "average" of a continuum of independent random variables is degenerate. As Judd (1985) pointed out, when the cardinality of the set of random variables involved is uncountable, this is an assumption rather than a result.

$R_n(x) = \arg\max_{a_i \in A} F\left(a_i, x + \frac{1}{n}(a_i - x)\right)$

$R_n^{\varepsilon}(x) = \arg\max_{a_i \in A} \underset{\varepsilon_1 \ldots \varepsilon_n}{\mathrm{E}} \left[F\left(a_i + \varepsilon_i, x + \frac{1}{n}(a_i - x) + \frac{1}{n}\sum \varepsilon_i\right)\right].$

Equilibria of the limit games are the (not necessarily unique) symmetric actions profiles $\{a_i = a : i \in [0,1]\}$, satisfying $a = R(a)$ or $a = R^{\varepsilon}(a)$ respectively. Since equilibria are symmetric, we abuse notation and refer to them by the real values representing the (symmetric) actions. Thus, the (Nash and noisy) equilibrium sets are viewed, respectively, as sets of real numbers $E_\infty = \{a : a = R(a)\}$ and $E_\infty^{\varepsilon} = \{a : a = R^{\varepsilon}(a)\}$. Symmetric equilibria of the finite games are similarly characterized by solutions to $a = R_n(a)$ and $a = R_n^{\varepsilon}(a)$. The finite games might also have asymmetric equilibria; however, these must be almost symmetric when the number of players n is large.[12] To simplify the exposition, we focus only on symmetric equilibria. It will be clear that our main result can easily be extended to the general case. We summarize the n-players' (symmetric) equilibrium actions' sets by $E_n = \{a : a = R_n(a)\}$ and $E_n^{\varepsilon} = \{a : a = R_n^{\varepsilon}(a)\}$.

We are not interested in the effect that each player's tremble may have on her own choice of optimal action: such effects should not be expected to vanish when the number of players grows. Only the effect of the aggregate noise can be expected to disappear. To focus on the latter, we assume that the choice of best responses (to the other player's aggregate) is independent of the individual trembles:

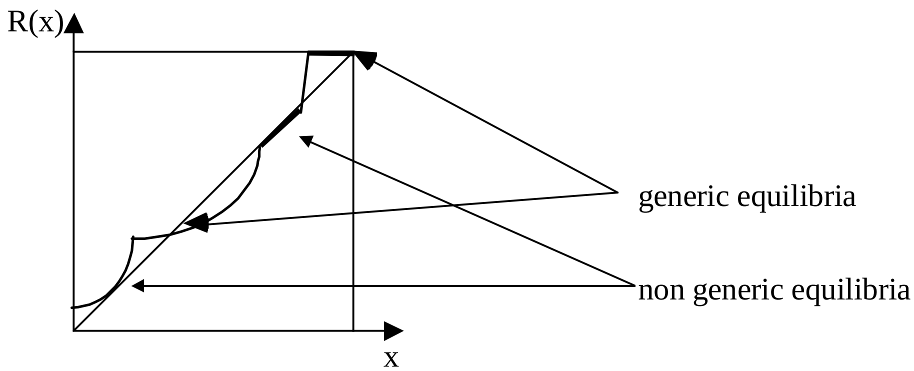
Assumption 5.2:

$R(x) = R^{\varepsilon}(x)$ for all $x \in A$.

Assumption 5.2 implies that the sets of Nash and noisy equilibria of the limit game are identical, i.e., $E_\infty = E_\infty^{\varepsilon}$. Intuitively, since the best response functions for the finite games converge to those of the limit game as the number of players grows, we also expect their equilibrium sets to converge to each other. To gain some intuition for the sequel, we

[12] More formally, for any $\delta>0$ there exists N such that for any $n>N$, and any equilibrium $(a_1 \ldots a_n)$ of the n-players' game, $max\{a_i{:}i=1...n\}-min\{a_i{:}i=1...n\}<\delta$. This result follows directly from the theorem of the maximum, since F is uniformly continuous over the compact set A and since the difference between the $\bar{x}_{-i}$'s tends to 0.

draw a graph of the function $R(x)$ in the square $A \times A$. Equilibria of the limit game are exactly the intersections of $R(x)$ with the 45$^{\mathrm{o}}$ line. Notice that, by the theorem of the maximum, $R(x)$ is a continuous function. Consider the following example:



Consider an equilibrium point in $A = [a_-, a_+]$. There are two possibilities: either it is an interior solution or it is an extreme point of the segment. In the first case, generically, the best response function $R(x)$ crosses the 45$^{\mathrm{o}}$ line. In other words, equilibrium points where $R(x)$ is tangent to the 45$^{\mathrm{o}}$ line are not generic. Notice that in the case where F is differentiable, the slope condition can be characterized using the implicit function theorem: $-\frac{\partial F(x,y)}{\partial y} / \frac{\partial F(x,y)}{\partial x}\Big|_{(x,y)=(a,a)} \neq 1$.

As for the extreme points a_- and a_+, these are equilibrium points if they maximize $F(x, a_-)$ subject to $x \geq a_-$ or $F(x, a_+)$ subject to $x \leq a_+$, respectively. Generically, if that is the case, then the point solves the maximization problem in a neighborhood of an extreme. In other words, generically, if an extreme point is a best response it is so because of the constraint. We summarize this discussion with the following definition:

Generic equilibria are equilibrium points

(a) in the open segment (a_-, a_+), if the best response function crosses the 45$^{\mathrm{o}}$ line.

(b) at the corner a_-, provided $R(x) = a_-$ in a neighborhood a_-.

(c) at the corner a_+, provided $R(x) = a_+$ in a neighborhood a_+.

Our next step is to show that the (Nash and noisy) best response functions of the finite games are close to $R(x)$ when the number of players is large. Recall that, $R_n(x) = \arg\max_{a_i \in A} F\left(a_i, x + \frac{1}{n}(a_i - x)\right)$, and $R_n^{\varepsilon}(x) = \arg\max_{a_i \in A} \underset{\varepsilon_1 \dots \varepsilon_n}{\mathrm{E}} \left[F\left(a_i + \varepsilon_i, x + \frac{1}{n}(a_i - x) + \frac{1}{n}\sum \varepsilon_i\right)\right]$.

We treat $1/n$ as a parameter and consider its effect on the values of x_i, denoted by $R(x, 1/n)$, that maximize these expressions. Intuitively, since F is continuous, a variant of the theorem of the maximum should apply: $R(x, 1/n)$ should not be very sensitive to small perturbations of its second argument. In other words, the best response function changes continuously as $n \to \infty$. We now formally state and prove these claims:

Proposition 5.1

(a) $R_n(\cdot), R_n^{\varepsilon}(\cdot)$ are continuous functions.

(b) $\forall x \in A, R_n(x) \xrightarrow{n\to\infty} R(x)$

(c) $\forall x \in A, R_n^{\varepsilon}(x) \xrightarrow{n\to\infty} R(x)$

Proposition 5.1 implies that the best response functions of the n-players game, both in the noisy and Nash cases, are 'close' to that of the limit game (notice that since A is compact, the convergence of the continuous functions is uniform). Generic equilibria are robust to small shifts of the function $R(x)$. This means that equilibria of finite games with a large number of players are close to those of the respective infinite ones. Since, by Assumption 5.1, the Nash and noisy equilibria of the infinite game are identical, then the Nash and noisy equilibria of the finite game are close to each other. On the other hand, even when F is continuous, non-generic equilibria are not guaranteed to survive. The next theorem concludes the results of this section.

Theorem 5.1

a. (upper hemi-continuity): Let $e_n \in E_n$ ($e_n^{\varepsilon} \in E_n^{\varepsilon}$) be a sequence of Nash (noisy) equilibria of the n-players' games, and assume that $e_n \xrightarrow{n\to\infty} e$ ($e_n^{\varepsilon} \to e$). Then $e \in E_\infty (= E_\infty^{\varepsilon})$, i.e., e is an equilibrium of the limit game.

b. (lower hemi-continuity): Let $e \in E_\infty$ be a generic equilibrium point of the limit game. For any neighborhood of *e* there exists *N* such that for all *n*>*N*, there are Nash (noisy) equilibria $e_n \in E_n$ ($e_n^\varepsilon \in E_n^\varepsilon$) of the *n*-players' game in that neighborhood.

In other words, Theorem 5.1 says that for large *n*, a Nash equilibrium of the *n*-players game is close to a noisy one (and vice versa) if there is a nearby generic equilibrium point of the limit game.

Remark: One should not be tempted to automatically ignore non-generic points. In the context of treating a naturally given economic model, i.e., when it is reasonable to think of the function *F* as somehow being randomly chosen, non generic cases are rare. However, that may not be true in other contexts. Most notably, in the context of mechanism design, where a non-generic case can be selected by the mechanism designer exactly because she is trying to implement a specific outcome, there is no reason to believe that this is a rare case. In other words, the natural metric on the space of functions is not always appropriate.

Appendix: Proofs

Proof of Lemma 3.1:

Given a continuous random variable $\mu : \Re \to \Re_+$, let $PDF_\mu(t)$, $CDF_\mu(t)$ denote its probability and cumulative distribution functions, respectively. Also denote $s_{-i} = \sum_{j \neq i} \varepsilon_j$. Player *i*'s payoff is:

$$\pi_i^\varepsilon = \underset{\varepsilon}{E}[(x_i + \varepsilon_i) \cdot D(\bar{x} + \bar{\varepsilon})] = x_i \cdot \underset{\varepsilon}{E}[D(\bar{x} + \bar{\varepsilon})] + \underset{\varepsilon_i}{E}\left[\varepsilon_i \cdot \underset{\varepsilon_{-i}}{E}[D(\bar{x} + \bar{\varepsilon})]\right] =$$

$$= x_i \cdot CDF_{\bar{\varepsilon}}(1 - \bar{x}) + \int_{-\infty}^{\infty} PDF_{\varepsilon_i}(t) \cdot t \cdot CDF_{s_{-i}}(n - n\bar{x} - t)dt$$

The first order condition is (* denotes equilibrium values):

$$0 = \frac{\partial \pi_i^\varepsilon}{\partial x_i} = CDF_{\bar{\varepsilon}}(1 - \bar{x}^*) - \frac{x_i^*}{n} \cdot PDF_{\bar{\varepsilon}}(1 - \bar{x}^*) - \int_{-\infty}^{\infty} PDF_{\varepsilon_i}(t) \cdot t \cdot PDF_{s_{-i}}(n - n\bar{x}^* - t)dt$$

I.e., at equilibrium (if one exists), every player's choice must satisfy $x_i^* = g(n, \bar{x}^*)$ for the same function g. This implies that an equilibrium must be symmetric.

We next show that the equation:

$$0 = CDF_{\bar{\varepsilon}}(1 - x) - \frac{x}{n} \cdot PDF_{\bar{\varepsilon}}(1 - x) - \int_{-\infty}^{\infty} PDF_{\varepsilon_i}(t) \cdot t \cdot PDF_{s_{-i}}(n - nx - t)dt$$

has a unique solution *x* (for a fixed n). First, we simplify the integral part:

$$I = \int_{-\infty}^{\infty} PDF_{\varepsilon_i}(t) \cdot t \cdot PDF_{s_{-i}}(n - nx - t)dt = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi c}} \cdot \frac{1}{\sqrt{2\pi(n-1)c}} \cdot e^{\frac{-t^2}{2c}} \cdot e^{\frac{-(n-nx-t)^2}{2(n-1)c}} \cdot tdt$$

note that since

$$(n-1)t^2 + (n - nx - t)^2 = (n-1)t^2 + n^2(1-x)^2 - 2n(1-x)t + t^2 =$$
$$= n[n(1-x)^2 - 2(1-x)t + t^2] = n[(1-x-t)^2 + (n-1)(1-x)^2]$$

we have

$$e^{\frac{-(n-1)t^2}{2(n-1)c}} \cdot e^{\frac{-(n-n\bar{x}^*-t)^2}{2(n-1)c}} = e^{\frac{-(1-x-t)^2}{2\frac{n-1}{n}c}} \cdot e^{\frac{-(1-x)^2}{2\frac{c}{n}}}$$

thus,

$$I=\int_{-\infty}^{\infty}\frac{1}{n}\cdot\frac{1}{\sqrt{2\pi\frac{c}{n}}}\cdot\frac{1}{\sqrt{2\pi\frac{n-1}{n}c}}\cdot e^{\frac{-(1-x-t)^2}{2\frac{n-1}{n}c}}\cdot e^{\frac{-(1-x)^2}{2\frac{c}{n}}}\cdot tdt$$

we change the variable of integration to $\tilde{t}=1-x-t$. Thus,

$$I=\frac{1}{n}\cdot\frac{1}{\sqrt{2\pi\frac{c}{n}}}\cdot e^{\frac{-(1-x)^2}{2\frac{c}{n}}}\cdot\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\frac{n-1}{n}c}}\cdot e^{\frac{-\tilde{t}^2}{2\frac{n-1}{n}c}}\cdot(1-x-\tilde{t})\cdot(-dt)$$

the integral can be thought of as taking expectation over a random variable $\mu\sim N\left[0,\frac{n-1}{n}c\right]$. This yields,

$$I=\frac{1}{n}\cdot PDF_{\bar{\varepsilon}}(1-x)\cdot\left(-\underset{\mu}{E}[1-x-\mu]\right)=\frac{-1}{n}\cdot(1-x)\cdot PDF_{\bar{\varepsilon}}(1-x)$$

we can now rewrite the equilibrium equation:

$$0=CDF_{\bar{\varepsilon}}(1-x)-\frac{x}{n}\cdot PDF_{\bar{\varepsilon}}(1-x)+\frac{1}{n}\cdot(1-x)\cdot PDF_{\bar{\varepsilon}}(1-x)$$

or:

$$2x-1=\frac{n\cdot CDF_{\bar{\varepsilon}}(1-x)}{PDF_{\bar{\varepsilon}}(1-x)}$$

We check that the right-hand side is always decreasing:

$$\frac{\partial}{\partial x}\frac{CDF(1-x)}{PDF(1-x)}=\frac{-PDF_{\bar{\varepsilon}}^2(1-x)-\frac{n(1-x)}{c}PDF(1-x)\cdot CDF(1-x)}{PDF^2(1-x)}=$$

$$=\frac{-\frac{n(1-x)}{c}\cdot CDF(1-x)-PDF(1-x)}{PDF(1-x)}$$

For $x\le 1$ the expression is negative. To show that it is negative also for $x>1$, check that the numerator (1) is negative when $x=1$, (2) goes to 0 as $x\to\infty$, and (3) its derivative is positive. (1)-(3) imply that the numerator is negative also for $x>1$.

The left-hand side is increasing and has range $(-\infty,\infty)$, and both right and left hand sides are continuous. Thus, there is exactly one solution. I.e., an equilibrium, if one exists, is unique. Now note that the solution is clearly a maximum - just verify that choosing extreme values for x_i yields a lower payoff than, say, choosing $x_i=1$. This guaranties existence.

Proof of Proposition 3.1:

Consider again the (Simplified) FOC for equilibrium:

$$2x-1=\frac{n\cdot CDF_{\bar{\varepsilon}}(1-x)}{PDF_{\bar{\varepsilon}}(1-x)}$$

We do not have a convenient algebraic expression for the right-hand side. Rather, the following lemma (stated without proof) helps us find bounds on its value:

<u>Lemma:</u> Let $\mu \sim N[0,\sigma^2]$ and assume that $z<-\sigma$. Then $\frac{\sigma^2}{2}<\frac{(-z)CDF_{\mu}(z)}{PDF_{\mu}(z)}<2\cdot\sigma^2$.

Plugging into the equilibrium equation, we get:

$$\frac{3+\sqrt{1+16c}}{4}>x>\frac{3+\sqrt{1+4c}}{4}.$$

And since x_n^{ε} is bounded from below by a constant greater than 1, the payoffs $\pi_i^{\varepsilon}=\underset{\varepsilon}{E}\left[\left(x_n^{\varepsilon}+\varepsilon_i\right)\cdot D\left(x_n^{\varepsilon}+\bar{\varepsilon}\right)\right]$ converge to 0 when n grows to infinity.

Proof of Theorem 4.(a):

In the n-players game, if player i chooses $x_i<a_+$, then an increase of x_i by $\delta=a_+-x_i>0$ is feasible. We will show that such a deviation is profitable for any x_{-i}. Hence, a_+ is the dominant action for player *i*.

To estimate the difference in utility resulting from the deviation, we separate it into two parts: the direct effect, *D*, which is the gain from the deviation keeping the aggregate constant, and the indirect effect, *I*, which is the loss in utility due to the increase in the aggregate:

$$\Delta u_i=\underset{\varepsilon}{E}\left[F\left(x_i+\varepsilon_i+\delta,\bar{x}+\bar{\varepsilon}+\delta/n\right)-F\left(x_i+\varepsilon_i,\bar{x}+\bar{\varepsilon}\right)\right]=D+I$$

where $D=\underset{\varepsilon_i}{E}\left[\underset{\varepsilon_{-i}}{E}\left[F\left(x_i+\varepsilon_i+\delta,\bar{x}+\bar{\varepsilon}\right)-F\left(x_i+\varepsilon_i,\bar{x}+\bar{\varepsilon}\right)\right]\right]$

and $I=\underset{\varepsilon_i}{E}\left[\underset{\varepsilon_{-i}}{E}\left[F\left(x_i+\varepsilon_i+\delta,\bar{x}+\bar{\varepsilon}+\delta/n\right)-F\left(x_i+\varepsilon_i+\delta,\bar{x}+\bar{\varepsilon}\right)\right]\right]$

By Assumption 4.2(a), the (positive) direct effect is independent of the number of players n: $D \ge \rho \cdot \delta > 0$. We next show that the (negative) indirect effect vanishes when n grows to infinity. To find a bound on I, we first write $\bar{\varepsilon}$ as a weighted average of ε_i and $\bar{\varepsilon}_{-i} = \frac{1}{n-1}\sum_{j \ne i} \varepsilon_j$, and break the integral over ε_{-i} into strips $J_k = [k\frac{\delta}{n}, (k+1)\frac{\delta}{n})$, of width $\frac{\delta}{n}$:

$$I = \underset{\varepsilon_i}{E}\left[\underset{\varepsilon_{-i}}{E}\left[\left(\sum_{\{k \in Z: J_k \cap [-s,s] \ne \phi\}} \chi_{\bar{\varepsilon}_{-i} \in J_k}\right) \cdot \begin{bmatrix} F\left(x_i + \varepsilon_i + \delta, \bar{x} + (\frac{1}{n}\varepsilon_i + \frac{n-1}{n}\bar{\varepsilon}_{-i}) + \frac{\delta}{n}\right) - \\ F\left(x_i + \varepsilon_i + \delta, \bar{x} + (\frac{1}{n}\varepsilon_i + \frac{n-1}{n}\bar{\varepsilon}_{-i})\right) \end{bmatrix}\right]\right]$$

Since F is monotone in its second argument, we can bind its value over each strip by its values at the end-points of the strip. We also change the order of the summation and expectation operators. Thus,

$$I \ge \underset{\varepsilon_i}{E}\left[\sum_{\{k \in Z: J_k \cap [-s,s] \ne \phi\}} \underset{\varepsilon_{-i}}{E}\left[\chi_{\bar{\varepsilon}_{-i} \in J_k} \cdot \begin{bmatrix} F\left(x_i + \varepsilon_i + \delta, \bar{x} + \frac{1}{n}\varepsilon_i + \frac{n-1}{n}(k+1+\frac{n}{n-1})\frac{\delta}{n}\right) - \\ F\left(x_i + \varepsilon_i + \delta, \bar{x} + \frac{1}{n}\varepsilon_i + \frac{n-1}{n}k\frac{\delta}{n}\right) \end{bmatrix}\right]\right]$$

Since the difference is constant over each strip, it can be taken out of the expectation operator This yields:

$$I \ge \underset{\varepsilon_i}{E}\left[\sum_{\{k \in Z: J_k \cap [-s,s] \ne \phi\}} \begin{bmatrix} F\left(x_i + \varepsilon_i + \delta, \bar{x} + \frac{1}{n}\varepsilon_i + \frac{n-1}{n}(k+3)\frac{\delta}{n}\right) - \\ F\left(x_i + \varepsilon_i + \delta, \bar{x} + \frac{1}{n}\varepsilon_i + \frac{n-1}{n}k\frac{\delta}{n}\right) \end{bmatrix} \cdot \underset{\varepsilon_{-i}}{E}\left[\chi_{\bar{\varepsilon}_{-i} \in J_k}\right]\right]$$

We now estimate the maximum weight over each strip. Denote:

$$W_n = \max_k \underset{\varepsilon_{-i}}{E}[\chi_{\bar{\varepsilon}_{-i} \in J_k}] = \max_k \text{prob}[\bar{\varepsilon}_{-i} \in J_k]$$

By the central limit theorem (see Shirayayev, pp. 326,332),

$$\mu_n = \sqrt{\sum_{\substack{j \ne i \\ 1 \le j \le n}} \text{var}(\varepsilon_j)} \cdot \left(\bar{\varepsilon}_{-i} - E[\bar{\varepsilon}_{-i}]\right) \xrightarrow{d} N(0,1)$$

and since the c.d.f. of the limit is continuous, the convergence of the c.d.f.'s is uniform. Also, since the ε_j's have bounded support, the sum of their variances is, at most, of order $O(n)$. Hence,

$$\sqrt{\sum_{\substack{j\neq i\\1\leq j\leq n}}\text{var}(\varepsilon_j)}\cdot\frac{\delta}{n}\xrightarrow[n\to\infty]{}0$$, implying

$$\sup_t \text{prob}\left[\mu_n\in[t,t+\sqrt{\sum_{\substack{j\neq i\\1\leq j\leq n}}\text{var}(\varepsilon_j)}\cdot\frac{\delta}{n}\right]\xrightarrow[n\to\infty]{}0$$

or equivalently, $W_n\xrightarrow[n\to\infty]{}0$. Hence, we have:

$$I\geq \underset{\varepsilon_i}{E}\left[W_n\cdot\sum_{\{k\in Z:J_k\cap[-s,s]\neq\phi\}}\left[\begin{matrix}F\left(x_i+\varepsilon_i+\delta,\bar{x}+\frac{1}{n}\varepsilon_i+\frac{n-1}{n}(k+3)\frac{\delta}{n}\right)-\\ F\left(x_i+\varepsilon_i+\delta,\bar{x}+\frac{1}{n}\varepsilon_i+\frac{n-1}{n}k\frac{\delta}{n}\right)\end{matrix}\right]\right]$$

W.l.o.g., assume that F is bounded by M over a square a little larger than $[a_- - s, a_+ + s]^2$, i.e., the square $[a_- - s-\delta/n, a_+ + s+\delta/n]^2$, such that the value of F is bounded over all the strips. All the terms in the telescopic sum cancel out, except for the three first and three last, which are bounded by +M and -M. Therefore,

$$I\geq \underset{\varepsilon_i}{E}[-W_n\cdot 6M]=-W_n\cdot 6M\xrightarrow[n\to\infty]{}0$$

Thus, for a large n, $\Delta u_i=D+I>0$. I.e., a deviation is profitable.

Proof of Theorem 4.1(b):

The proof is very similar to that of 4.1(a). As above we have $I\xrightarrow[n\to\infty]{}0$. We also have:

$$D=\underset{\varepsilon_i}{E}\left[\underset{\varepsilon_{-i}}{E}\left[F(x_i+\varepsilon_i+\delta,\bar{x}+\bar{\varepsilon})-F(x_i+\varepsilon_i,\bar{x}+\bar{\varepsilon})\right]\right]\geq \underset{\varepsilon_i}{E}\left[\underset{\varepsilon_{-i}}{E}\left[\rho\cdot F(x_i+\varepsilon_i,\bar{x}+\bar{\varepsilon})\cdot\delta\right]\right]=$$
$$=\rho\cdot\delta\cdot\underset{\varepsilon}{E}\left[F(x_i+\varepsilon_i,\bar{x}+\bar{\varepsilon})\right]=\rho\cdot\delta\cdot u_i$$

Therefore, for n large enough such that $|I<\rho\cdot\delta\cdot\eta|$, we either have $u_i<\eta$, or a deviation from $x_i(<a_+)$ would be profitable.

Proof of Proposition 5.1

Parts (a): Continuity follows directly from the theorem of the maximum since the functions $F(\ )$ and $E[F()]$ are continuous, while single-valuedness follows from the concavity of F. Part (b) also follows from the theorem of the maximum. Unfortunately, that theorem cannot be applied to (c), because for each n the 'expectation' operation is taken over a different set of random variables. However, we can employ an argument resembling the proof of the theorem of the maximum.

Let $r_n = R_n^{\varepsilon}(x)$ and $r = R(x)$, but assume that r_n does not converge to *r*. Hence, there exists $s \neq r$ such that a subsequence $\{r_{n_k}\}_{k=1}^{\infty} \subseteq \{r_n\}_{n=1}^{\infty}$ converges to *s*. To save on notation, assume that r_n is already the converging subsequence.

Intuitively, the continuity of F implies that, for *n* large enough,

$$\underset{\varepsilon_1 \dots \varepsilon_n}{\mathrm{E}}\left[F\left(r+\varepsilon_i, x+\frac{1}{n}(r-x)+\frac{1}{n}\sum\varepsilon_i\right)\right] \cong \underset{\varepsilon_i}{\mathrm{E}}\left[F(r+\varepsilon_i, x)\right] >$$
$$> \underset{\varepsilon_i}{\mathrm{E}}\left[F(s+\varepsilon_i, x)\right] \cong \underset{\varepsilon_i}{\mathrm{E}}\left[F(r_n+\varepsilon_i, x)\right] \cong \underset{\varepsilon_1 \dots \varepsilon_n}{\mathrm{E}}\left[F\left(r_n+\varepsilon_i, x+\frac{1}{n}(r_n-x)+\frac{1}{n}\sum\varepsilon_i\right)\right],$$

in contradiction to the assumption that r_n is a maximizer. We now prove the “inequality” formally. Let

$$B_n = \left\{(\varepsilon_1 \dots \varepsilon_n) : \bar{\varepsilon} \in [E[\varepsilon] - 1/\sqrt[3]{n}, E[\varepsilon] + 1/\sqrt[3]{n}\right\}.$$

By the central limit theorem, $\mathrm{Prob}(B_n) \xrightarrow{n\to\infty} 1$. Consider the difference,

$$d_1(n) = \underset{\varepsilon_1 \dots \varepsilon_n}{\mathrm{E}}\left[F\left(r+\varepsilon_i, x+\frac{1}{n}(r-x)+\frac{1}{n}\sum\varepsilon_i\right)\right] - \underset{\varepsilon_i}{\mathrm{E}}\left[F(r+\varepsilon_i, x)\right] =$$
$$= \left(\int_{B_n} + \int_{B_n^c}\right)\left(F\left(r+\varepsilon_i, x+\frac{1}{n}(r-x)+\frac{1}{n}\sum\varepsilon_i\right) - F(r+\varepsilon_i, x)\right)$$

Since the trembles have uniformly bounded support, and since F is continuous, the integrand converges to 0, uniformly over B_n. More precisely, for any $\delta > 0$ there exists *N* such that for all *n*>*N* and all $(\varepsilon_1 \dots \varepsilon_n) \in B_n$,

$$\left|F\left(r+\varepsilon_i, x+\frac{1}{n}(r-x)+\frac{1}{n}\sum\varepsilon_i\right) - F(r+\varepsilon_i, x)\right| < \delta \quad .$$

Since the probability of B_n^c tends to 0, we have $|d_1(n)| \xrightarrow{n\to\infty} 0$.

In a similar way, one can verify that also

$$|d_2(n)| = \left|\underset{\varepsilon_i}{\mathrm{E}}\left[F(r_n+\varepsilon_i, x)\right] - \underset{\varepsilon_1 \dots \varepsilon_n}{\mathrm{E}}\left[F\left(r_n+\varepsilon_i, x+\frac{1}{n}(r_n-x)+\frac{1}{n}\sum\varepsilon_i\right)\right]\right| \xrightarrow{n\to\infty} 0$$

Since we assumed that $s \in \lim_{n\to\infty} r_n$ and since F in continuous, we also have,

$$|d_3(n)| = \left| \underset{\varepsilon_i}{\mathrm{E}}[F(s+\varepsilon_i, x)] - \underset{\varepsilon_i}{\mathrm{E}}[F(r_n+\varepsilon_i, x)] \right| \xrightarrow{n\to\infty} 0$$

Finally, since $r = R(x) = R^{\varepsilon}(x)$, we have

$\underset{\varepsilon_i}{\mathrm{E}}[F(r+\varepsilon_i, x)] - \underset{\varepsilon_i}{\mathrm{E}}[F(s+\varepsilon_i, x)] = \eta > 0$, independent of *n*. Thus, for *n* large enough,

$$\underset{\varepsilon_1\ldots\varepsilon_n}{\mathrm{E}}\left[F\left(r+\varepsilon_i, x+\frac{1}{n}(r-x)+\frac{1}{n}\sum\varepsilon_i\right)\right] - \underset{\varepsilon_1\ldots\varepsilon_n}{\mathrm{E}}\left[F\left(r_n+\varepsilon_i, x+\frac{1}{n}(r_n-x)+\frac{1}{n}\sum\varepsilon_i\right)\right] >$$
$$> \eta - |d_1(n)| - |d_2(n)| - |d_3(n)| > 0$$

In contradiction to the assumption that r_n is the maximizer.

Proof of Theorem 5.1

Part (a) follows directly from Proposition 5.1, since $R_n(x)$ and $R_n^{\varepsilon}(x)$ are uniformly continuous in *x* and continuous in *n* at $n \to \infty$.

Part (b): Assume first that *e* is in the open segment (a_-, a_+). Let the neighborhood be $[e-\delta, e+\delta]$ for $\delta > 0$. Since *e* is a generic equilibrium point, there exist $x_1, x_2 \in [e-\delta, e+\delta]$ and $\rho > 0$ such that $R(x_1) - x_1 > \rho > 0$ and $R(x_2) - x_2 < -\rho < 0$. By Proposition 5.1(b), $R_n(x) \xrightarrow{n\to\infty} R(x)$. Furthermore, since the neighborhood $[e-\delta, e+\delta]$ is closed, the convergence is uniform over it. Hence, for *n* large enough, $|R_n(x) - R(x)| < \rho$ for any $x \in [e-\delta, e+\delta]$. Thus, $R_n(x_1) - x_1 > 0$ and $R_n(x_2) - x_2 < 0$. Since, by Proposition 5.1(a), $R_n(\cdot)$ is continuous, there exist $\tilde{x}$ between x_1 and x_2 such that $R_n(\tilde{x}) - \tilde{x} = 0$. This means that $\tilde{x}$ is a Nash equilibrium of the *n*-players' game.

In a similar manner, using $R_n^{\varepsilon}(\cdot)$ instead of $R_n(\cdot)$, we show the existence of a noisy equilibrium in the neighborhood. In the case *e* is one of the extreme points of the segment $[a_-, a_+]$, we extend the definition of *F* a little bit beyond its original range, in a manner consistent with Assumption 5.1. Because of the constraint, the values of $R(x)$, $R_n(x)$, and $R_n^{\varepsilon}(x)$ lie, for *x*'s that are a little bit beyond the segment $[a_-, a_+]$, on the other side of the 45° line. The rest of the proof is the same as in the case where *e* is in the open segment (a_-, a_+).

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