Macro Theory B

The Permanent Income Hypothesis

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1 Motivation

1.1 An econometric check

We want to build an empirical model with real data and check which of the two solutions, Autarky or CM, is more likely to be the real one.

The Autarky solution was:

$$c_t^i(s^t) = y_t^i(s^t)$$

We apply *log* on both sides and get:

$$\begin{aligned} log(c_t^i(s^t)) &= log(y_t^i(s^t)) \\ log(c_{t+1}^i(s^t)) &= log(y_{t+1}^i(s^t)) \\ &\downarrow \\ \Delta log(c_t^i) &= \Delta log(y_t^i) \end{aligned}$$

The CM solution was:

$$c_t^i(s^t) = \frac{(\alpha_i)^{\frac{1}{\sigma}}}{\sum_j (\alpha_j)^{\frac{1}{\sigma}}} C_t(s^t)$$

We apply *log* on both sides and get:

$$\begin{aligned} log(c_t^i(s^t)) &= const + log(C_t(s^t)) \\ log(c_{t+1}^i(s^t)) &= const + log(C_{t+1}(s^t)) \\ &\downarrow \\ \Delta log(c_t^i) &= \Delta log(C_t(s^t)) \end{aligned}$$

From here we build the empirical model:

$$\Delta log(c_t^i) = \beta_1 \Delta log(y_t^i) + \beta_2 \Delta log(C_t)$$

We look at the null assumptions:

- Autarky: $H_0 : \beta_1 = 1, \beta_2 = 0$
- CM: $H_0 : \beta_1 = 0, \beta_2 = 1$

Unsurprisingly, *both* assumptions can be rejected! Neither model is a good one.

1.2 Motivations for saving

We are interested in savings, and in modeling heterogenous levels of savings, because we want the model to yield a meaningful distribution of savings in the end, as opposed to a single value described by a representative agent. Since we want to model savings correctly we are interested in the motivations for saving:

1. Intertemporal: $\beta(1+r) \neq 1$.
   In general terms, the HH's problem satisfies: $u'(c_t) = \beta(1+r)u'(c_{t+1})$ (in the case of uncertainty E_t should be added).
   If $\beta(1+r) < 1 \rightarrow$Loan (negative savings): This means that the individuals has a net loss from transferring money to the next period. This is an incentive to consume current income and to loan money from the future to increase consumption now.
   If $\beta(1+r) > 1 \rightarrow$Save: This means that the individuals has a net gain from transferring money to the next period. This is an incentive to forgo consumption in the current period and to save part of this income for the future.
   If $\beta(1+r) = 1 \rightarrow$Indifference: In this case the individual is indifferent bewteen loaning and saving. This is why we derive the $\neq 1$ condition as an incentive for saving (either positive or negative).

2. Borrowing Constraint: In order to smooth consumption the individual must borrow/lend money. If we impose a limit to the amount that it is possible to borrow, the individual has to take this into account (this is a credit market imperfection). He does so by increasing savings at other times to miminize the possibility of reaching the limit when borrowing is needed. We note that such a limit *must be imposed* if we are ever to reach an equilibrium where the markets clear, because if the agents can borrow an infinite amount this will never be possible.

3. Precautionary Savings (Prudence): Here we make an assumption on the properties of $u(c)$: That $u''' > 0$ (i.e. u'' is convex). This means that given a stochastic process with mean $\bar{y}$, if we increase its variance (i.e increase the measure of uncertainty) but maintain the mean, the individual will still want to increase his savings. In other words, inducing a mean preserving transformation on y_t affects the individuals savings.

4. Other: Life cycle, inter generational/bequest etc.

2 PIH - Permanent Income Hypothesis

Assumptions:

1. $\beta(1+r) = 1$
   We want to neutralize this obvious incentive to save/lend, and see if the other facets of the economy can generate savings.

2. $u(c_t) = b_1 c_t - \frac{1}{2} b_2 c_t^2$, $u' > 0 \rightarrow c \leq \frac{b_1}{b_2}$, $u'' < 0 \rightarrow b_2 > 0$
   We assume that the utility function is a parabola with a maximum.

2.1 The HH

This is in essence an extension of the CM model, and so the individual's problem is the same.

The HH's maximize the following:

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

In CM, the sequential formulation for the RC was:

$$a_t^i(s^t) + y_t^i(s^t) = c_t^i(s^t) + \sum_{s^{t+1}} q_{t+1}(s^{t+1}) a_{t+1}^i(s^{t+1})$$

In a bond economy there is by definition only partial insurance, so we have to limit the number of possible assets. We choose to limit it to a single asset which is not dependent on the realization of history: $a_{t+1}(s^t) \equiv a_{t+1}$. Similarly, we give this asset a price that is not dependent on the realization of history: $q_{t+1}(s^{t+1}) \equiv q$.

We get that the RC is:

$$a_t + y_t = c_t + q \cdot a_{t+1}$$

Since q is the same for all assets, it is also the price for the aggregate assets in the economy. It can therefore be thought of the 'price of assets', i.e. the interest rate. Here we formulate the interest rate so it satisfies: $Ra_{t+1} = a_{t+2}$,R:=$\frac{1}{1+r}$. So $q = \frac{1}{1+r}$.

We get:

$$\begin{aligned} a_t + y_t &= c_t + \frac{1}{1+r} a_{t+1} \rightarrow a_{t+1} = (a_t + y_t - c_t)(1+r) \\ &\rightarrow c_t = a_t + y_t - \frac{a_{t+1}}{1+r} \end{aligned} \tag{1}$$

We now formulate fully the HH's problem:

$$\begin{aligned} &max \quad E_0 \textstyle\sum_{t=0}^{\infty} \beta^t u(c_t) \\ &s.t: \\ &\qquad c_t = a_t + y_t - \tfrac{a_{t+1}}{1+r} \end{aligned}$$

Inserting the RC into the problem, we get:

$$max_{a_{t+1}} E_0 \sum_{t=0}^{\infty} \beta^t u(a_t + y_t - \frac{a_{t+1}}{1+r})$$

FOC's w.r.t a_{t+1}:

$$-\beta^t u'(c_t) \frac{1}{1+r} + E_0 \beta^{t+1} u'(c_{t+1}) = 0$$

In period t we get:

$$u'(c_t) = \beta(1+r)E_t u'(c_{t+1})$$

By assumption (1) of PIH we get:

$$u'(c_t) = E_t u'(c_{t+1})$$

From assumption (2) of PIH we get:

$$c_t = E_t(c_{t+1}) \tag{2}$$

This dynamic is called a 'Martingale'.

2.2 Solving for the Consumption

We use the soultion of the HH's problem and plug it back into the RC. But first we formulate some expressions:

- At time t: $c_t = a_t + y_t - \frac{a_{t+1}}{1+r} \rightarrow a_t = c_t - y_t + \frac{a_{t+1}}{1+r}$
  At time $t+1$: $a_{t+1} = c_{t+1} - y_{t+1} + \frac{a_{t+2}}{1+r}$
- At time t: $E_t(a_{t+1}) = E_t(c_{t+1} - y_{t+1} + \frac{a_{t+2}}{1+r})$

Now we plug these expressions into the sequential RC, equation (1), and get:

$$c_t = a_t + y_t - \frac{1}{1+r}[E_t(c_{t+1} - y_{t+1} + \frac{a_{t+2}}{1+r})] = a_t + y_t - E_t(\frac{c_{t+1}}{1+r} - \frac{y_{t+1}}{1+r} + \frac{a_{t+2}}{(1+r)^2})$$

$$\begin{aligned} \underbrace{c_t + \frac{E_t(c_{t+1})}{1+r}}_{\downarrow} &= a_t + \underbrace{y_t + \frac{E_t(y_{t+1})}{1+r}}_{\downarrow} - \underbrace{\frac{E_t(a_{t+2})}{(1+r)^2}}_{\downarrow} \\ \underbrace{\sum_{j=0}^{\infty} \frac{E_t(c_{t+j})}{(1+r)^j}}_{A} &= a_t + \sum_{j=0}^{\infty} \frac{E_t(y_{t+j})}{(1+r)^j} \underbrace{- \lim_{j\to\infty} \frac{E_t(a_{t+j})}{(1+r)^j}}_{B} \end{aligned}$$

For optimality, we impose the transversality condition, which is: $B = 0$. We recall the rationale for this:

- If a_{t+j} is high enough to yield $B > 0$ then this lowers the value of A, meaning it lowers the consumption stream. Since utility is equated with consumption in this model, this is obviously not an optimal path.
- If a_{t+j} is low enough to yield $B < 0$ then this means that the individual has incurred a debt which will never be repayed. We assume that this can't happen since no lender will be willing to loan money which will never be repaid.

So we get:

$$\sum_{j=0}^{\infty} \frac{E_t(c_{t+j})}{(1+r)^j} = a_t + \sum_{j=0}^{\infty} \frac{E_t(y_{t+j})}{(1+r)^j} \tag{3}$$

Note: This is very similar to the CM solution, which was:

$$\sum_{t=0}^{\infty} \sum_{s^t} y_t^i(s^t) P_t^0(s^t) = \sum_{t=0}^{\infty} \sum_{s^t} c_t^i(s^t) P_t^0(s^t)$$

Intuitively, we can see that in our case it means that $P_t^0(s^t) \equiv \frac{1}{(1+r)^j}$. This makes sense because we already defined $q \equiv \frac{1}{1+r}$, and so at time t, the price of an asset which yields value 1 at time $t = t + j$ should be $(\frac{1}{1+r})^j$.

This is still not the final solution, since we can go further with the E_t expressions. We note that from (2) we get:
$E_{t+1}(c_{t+2}) = c_{t+1}$
Taking E_t on both sides:
$E_t[E_{t+1}(c_{t+2})] = E_t(c_{t+1})$
From the law of total expectation:
$E_t(c_{t+2}) = c_t$
From here we see that $E_t(c_{t+j}) = c_t \, \forall j$. Now we plug this into (3) and get:

$$\begin{aligned} \frac{c_t(1+r)}{r} &= a_t + \sum_{j=0}^{\infty} \frac{E_t(y_{t+j})}{(1+r)^j} \\ \rightarrow c_t &= \frac{r}{1+r}[a_t + \sum_{j=0}^{\infty} \frac{E_t(y_{t+j})}{(1+r)^j}] \\ &:= \frac{r}{1+r}[\underbrace{a_t + E_t(\overbrace{W_t}^{Aggregate\,Income})}_{total\,resources}] \end{aligned} \tag{4}$$

We see that the result here is also that c_t is some constant percentage of the aggregate expected resources. This means that even in PIH (no inter-temporal savings motivation) there is a level of insurance (savings) since the solution exhibits consumption smoothing over time.

Certainty Equivalence: An important property of this economy is that if we were to solve the deterministic problem (with y_t *not* being stochastic) we would get an equivalent result. Because consumption is dependent only on the expectancy and not the variance (i.e only on the *first moment*) we could replace the stochastic process $\bar{y}_t$ with a determinisitc one which satisfies:

$$\forall t: \; \bar{y}_t = \sum_{j=0}^{\infty} \frac{E_t(y_{t+j})}{(1+r)^j}$$

If y_t is i.i.d, for example, then this would simply be $\bar{y}_t \equiv const$. This replacement would not affect consumption since the first moment is identical (even if the second one is not).
This is important because we would expect that the addition of uncertainty (variance) would cause the individuals to want to insure themselves beyond the levels of the deterministic case. However, we remain with the *same level* of insurance.

Looking at Assets: We notice that by looking at equation (1) we can get an expression for the change in (or dynamics of) assets between times. From equation (1): $a_{t+1} = (a_t + y_t - c_t)(1+r)$, so:

$$\Delta a_{t+1} = a_{t+1} - a_t = (a_t + y_t - c_t)(1+r) - a_t = (y_t - c_t)(1+r) + ra_t \quad (5)$$

We recall from equation (4) that $c_t = \frac{r}{1+r}[a_t + \sum_{j=0}^{\infty} \frac{E_t(y_{t+j})}{(1+r)^j}]$. Inserting c_t into Δa_{t+1} we get:
$\Delta a_{t+1} = y_t(1+r) - ra_t - rE_t[\sum_{j=0}^{\infty} \frac{y_{t+j}}{(1+r)^j}] + ra_t$
$\Delta a_{t+1} = y_t + ry_t - ry_t - rE_t[\sum_{j=1}^{\infty} \frac{y_{t+j}}{(1+r)^j}]$
$\Delta a_{t+1} = y_t - rE_t[\sum_{j=1}^{\infty} \frac{y_{t+j}}{(1+r)^j}]$
We add and subtract 1 from r and expand:
$\Delta a_{t+1} = y_t - ((r+1)-1)E_t[\sum_{j=1}^{\infty} \frac{y_{t+j}}{(1+r)^j}] = y_t + (1+r)E_t[\sum_{j=1}^{\infty} \frac{y_{t+j}}{(1+r)^j}] - E_t[\sum_{j=1}^{\infty} \frac{y_{t+j}}{(1+r)^j}]$
$\Delta a_{t+1} = y_t + E_t[\sum_{j=1}^{\infty} \frac{y_{t+j}}{(1+r)^{j-1}}] - E_t[\sum_{j=1}^{\infty} \frac{y_{t+j}}{(1+r)^j}]$
We expand both sums and combine the expectancies:
$\Delta a_{t+1} = y_t + E_t[\frac{y_{t+1}}{1+r} + \frac{y_{t+2}}{(1+r)^2} + ... - y_{t+1} - \frac{y_{t+2}}{1+r} - ...] = -E_t[\Delta y_{t+1} + \frac{\Delta y_{t+2}}{1+r} + ...]$
Finally we get:

$$\Delta a_{t+1} = -E_t[\sum_{j=1}^{\infty} \frac{\Delta y_{t+j}}{(1+r)^{j-1}}] \quad (6)$$

We see an inverse relationship between savings and income. This formulation is intuitive since as if we get, for example, a single period positive shock to the income $\Delta y_{t+j} \uparrow$ then we would want to decrease savings and raise consumption. (wealth effect).

2.3 Income (y_t) as a Random Walk:

We now look at an alternative way of formulating the stochastic nature of y_t:

1. $y_t = y_t^p + u_t$
2. $y_t^p = y_{t-1}^p + v_t$

Definitions:

- y_t is the income at period t.

- y_t^p is the permanent component of income at period t.
- u_t is a shock to the transitory componenent of the income at t. $E_{t-j}(u_t) = 0\,\forall j \geq 1$.
- v_t is a shock to the permanent component of the income (called permanent shock). $E_{t-j}(v_t) = 0\,\forall j \geq 1$.

The important distinction between v_t and u_t is that v_t is relevant to every $y_{t+j}\,\forall j \geq 0$, whereas u_t is only relevant to y_t. This become apparent through replacement of one expression into the other:

$$y_{t+1} = y_t^p + v_{t+1} + u_{t+1} = y_{t-1}^p + v_t + v_{t+1} + u_{t+1}$$

u_t does not appear, but all v_{t-j} expression appear. In other words, v_t *persists* completely in y_t^p, since $y_t^p = 1 \cdot y_{t-1}^p + v_t$, where the coefficient 1 means that even though v_t is a temporary shock it becomes a permanent one.

We derive some expressions which we will use later:
$y_{t-1} = y_{t-1}^p + u_{t-1} \rightarrow y_{t-1}^p = y_{t-1} - u_{t-1}$
$y_t = y_t^p + u_t = y_{t-1}^p + v_t + u_t = y_{t-1} - u_{t-1} + v_t + u_t$

At time t we get:
$E_t(y_t) = y_{t-1} - u_{t-1} + v_t + u_t$
$E_t(y_{t+1}) = E_t(y_t - u_t + v_{t+1} + u_{t+1}) = y_t - u_t = y_{t-1} - u_{t-1} + v_t + u_t - u_t = y_{t-1} - u_{t-1} + v_t$
Note that we take the expectation at time t for a value even though it is *known* at time t. We do this for ease of notation.

At time $t-1$ we get:
$E_{t-1}(y_t) = y_{t-1} - u_{t-1} + E_{t-1}(v_t + u_t) = y_{t-1} - u_{t-1}$
$E_{t-1}(y_{t+1}) = E_{t-1}(y_t - u_t + v_{t+1} + u_{t+1})$

Using both of the expectency terms for y_t, we get the operator:
$(E_t - E_{t-1})(y_t) = v_t + u_t$

Using both of the expectency terms for y_{t+1}, we get the operator :

$$\begin{aligned}
(E_t - E_{t-1})(y_{t+1}) &= y_t - u_t - E_{t-1}(y_t - u_t + v_{t+1} + u_{t+1}) \\
&= E_{t-1}[y_t - u_t - (y_t - u_t + v_{t+1} + u_{t+1})] \\
&= E_{t-1}[v_{t+1} + u_{t+1}] \\
&= v_t
\end{aligned}$$

Continuing the relationship, we get finally:

$$(E_t - E_{t-1})(y_{t+j}) = v_t\,\forall j \geq 1 \tag{7}$$

We now look at the expression $\Delta c_t = c_t - c_{t-1}$.
From (2) we get:

$$\Delta c_t = c_t - E_{t-1}(c_t)$$

Replacing c_t according to the second line in (4) we get:

$$\begin{aligned}\Delta c_t &= \frac{r}{1+r}[a_t + \sum_{j=0}^{\infty}\frac{E_t(y_{t+j})}{(1+r)^j}] - \frac{r}{1+r}[a_t + \sum_{j=0}^{\infty}\frac{E_{t-1}(y_{t+j})}{(1+r)^j}] = \\ &= \frac{r}{1+r}\sum_{j=0}^{\infty}\frac{(E_t - E_{t-1})(y_{t+j})}{(1+r)^j}\end{aligned}$$

Using (6), we get:

$$\begin{aligned}\Delta c_t &= \frac{r}{1+r}[v_t + u_t + \frac{v_t}{(1+r)} + \frac{v_t}{(1+r)^2} + ...] \\ &= \frac{r}{1+r}[u_t + \frac{v_t(1+r)}{r}] \\ &= \frac{r}{1+r}u_t + v_t\end{aligned}$$

Intuitively, we get that the change in consumption is caused by the fact that we have learned something about the state of the market (at time t we already *know* v_t and u_t, whereas at time $t = 0$ these are only future values with expectancy zero).
We see that the a shock to the *permanent component* (v_t) causes an equivalent change in the consumption, and that a shock to the non-permanent component causes only a partial change.

Note: We get here the same result as before, meaning that there is *certainty equivalence* despite the fact that we have changed the stochastic nature of y_t.

Looking at assets: Here we assume that $y_{t+1} = y_t + \epsilon_{t+1}$, $E(\epsilon_t) = 0$.
So we get: $\Delta y_{t+1} = y_{t+1} - y_t = y_t + \epsilon_{t+1} - y_t = \epsilon_{t+1} \rightarrow E_t(\Delta y_{t+1}) = E_t(\epsilon_{t+1}) = 0$.
Now, from (7):
$\Delta a_{t+1} = -E_t[\sum_{j=1}^{\infty}\frac{\Delta y_{t+j}}{(1+r)^{j-1}}] = -[\sum_{j=1}^{\infty}\frac{E_t \Delta y_{t+j}}{(1+r)^{j-1}}] = 0$.
The intuition for this is that since y_t is a random walk, the individuals best estimate for the future is that the current shock, ϵ_t, will be *permanent*, since the dynamic of a random walk is that the random error at t persists for every time afterwards (this can easily be seen by recursive replacement using the expression for y_t). The well known result of a permanent shock to income is that it is completely converted into consumption, and so there is no change in savings.

2.4 Income (y_t) as i.i.d variable:

We define $y_t = \bar{y} + \epsilon_t$, where $\bar{y}$ is some constant or average, and $E(\epsilon_t) = 0$. We get:

$\Delta y_{t+1} = y_{t+1} - y_t = \epsilon_{t+1} + \bar{y} - (\epsilon_t + \bar{y}) = \epsilon_{t+1} - \epsilon_t$

$\rightarrow E_t(\Delta y_{t+1}) = E_t(\epsilon_{t+1} - \epsilon_t) = 0 - E_t(\epsilon_t) = -E_t(\epsilon_t) = -\epsilon_t$

Similaly we get:

$\rightarrow E_t(\Delta y_{t+j}) = 0, \forall j \geq 2$

From equation (6) we have:

$\Delta a_{t+1} = -E_t[\sum_{j=1}^{\infty} \frac{\Delta y_{t+j}}{(1+r)^{j-1}}]$

So we get:

$\Delta a_{t+1} = -E_t[-\epsilon_t] = \epsilon_t$

This means that a_{t+1} is also a random walk. The intuition for this is that when y_t is i.i.d a single period shock at time t survives only one period. Since this is happening at time t, i.e 'currently', and at *no other* time, the individual wants to absorb all of the change into the consumption, i.e decrease consumption by the amount of the shock and save it (from consumption smoothing motivations). We note that we are still in PIH, i.e $\beta(1+r) = 1$, but here we *do have* inter-temporal savings, despite the removal of the inter-temporal motivation. Here the savings are generated by the form of the uncertainty (y_t as i.i.d).

2.5 PIH under a borrowing constraint $\bar{a}$:

We have seen PIH in two different cases: y_t as a random walk and as i.i.d.

Random Walk Here we saw that $\Delta a_t = 0$. This means that a shock to the income is completely translated into savings, and not into consumption. We see that any starting value of a will remain constant across time. In this case the imposition of a borrowing constraint $\bar{a}$ will have no effect on savings.

i.i.d Here we saw that $\Delta a_t = \epsilon_t$. This means that a shock to the income is completely translated into savings, and none into consumption. When a variable (in this case a_t) is a random walk there is a positive probability that the variable will reach *any possible value* in its range. This means that under a borrowing constraint $\bar{a}$ on the level of assets there is a positive probability that the individual will reach this level. This means that at some point the individual will have to consume only according to his income, which may be very small. This makes the individual increase savings at all periods in order to account for this possibility.