Macro Theory B

Complete markets (LS 2.8)

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1 Introduction

1.1 Environment and Notation

States and probabilities:

- Let $s_t \in S$ be the current state of the economy.
- Let $s^t = \{s_0, s_1, s_2, ..., s_t\}$ be the history up to time t, with $s^t \in S^t \equiv S_0 \times S_1 \times S_2 \times ... \times S_t$.
- Let $\pi(s^t)$ be the probability of history s^t occurring. $\pi(s^t|s^\tau)$ is the probability of observing s^t conditional on observing s^τ.

Endowment/Income:

- Let $y_t^i(s^t)$ be the realization of individual i's endowment/income upon the realization of history s^t.
- Aggregate endowment: $\sum_i y_t^i(s^t) = Y_t(s^t)$

Individual preferences and utility:

- Let $c_t^i(s^t)$ be the consumption stream of individual i in period t in history s^t
  - $c^i = \{c_t^i(s^t)\}_{t=0}^{\infty}$ is the history dependent consumption plan.

- Consumers order consumption streams by

$$U(c^i) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u\left(c_t^i(s^t)\right) \pi(s^t) = E_0 \sum_{t=0}^{\infty} \beta^t u(c_t^i)$$

  where

  - $0 < \beta \leq 1$ is the discount factor
  - $u'(c) > 0$ (u is strictly increasing).
  - u is twice continuously differentiable.
  - $u''(c) < 0$ (u is strictly concave)
  - Inada Condition: $\lim_{c \to 0} = \infty$

1.2 The Problem

Given an endowment process, constraints, and market arrangements, each consumer has to pick the *optimal* consumption plan.

i.e. the consumption plan that maximizes the value of $U(c^i)$.

In general, once we assume strict concavity of $u(c)$, consumers prefer to smooth consumption over states and time.

Obviously, any allocation must satisfy **feasibility:** $\sum_i c_t^i(s^t) = \sum_i y_t^i(s^t) = Y_t(s^t)$

2 Complete Markets l

2.1 The SP problem

To characterize the efficient (Pareto optimal) allocation, we can use the social planner's solution.

Assume a SP that assigns non-negative **Pareto weights** λ_i to each individual i in the economy.

The planner's objective is to maximize $W = \sum_i \lambda_i U(c^i)$ subject to the feasibility constraint.

The efficient allocation is the allocation that solves this problem for some set of λ_i.

Let $\theta_t(s^t)$ be the Lagrange multiplier on the time t, history s^t constraint

$$L = \sum_{t=0}^{\infty} \sum_{s^t} \left[\sum_i \lambda_i \beta^t u\left(c_t^i(s^t)\right) \pi(s^t) + \theta_t(s^t) \sum_i \left[y_t^i(s^t) - c_t^i(s^t) \right] \right]$$

F.O.C:

$$\lambda_i \beta^t u'\left(c_t^i(s^t)\right) \pi(s^t) = \theta_t(s^t) \qquad \forall \quad i, t, s^t$$

The ratio of FOCs of i and 1 is:

$$\frac{\lambda_i}{\lambda_1} \frac{u'\left(c_t^i(s^t)\right)}{u'\left(c_t^1(s^t)\right)} = 1 \quad \Rightarrow \quad c_t^i(s^t) = u'^{-1}\left(\frac{\lambda_1}{\lambda_i} u'\left(c_t^1(s^t)\right)\right)$$

Substitute in the feasibility constraint:

$$\sum_i u'^{-1}\left(\frac{\lambda_1}{\lambda_i}u'\left(c_t^1(s^t)\right)\right) = \sum_i y_t^i(s^t) = Y_t(s^t)$$

This is *one equation with one unknown* $c_t^1(s^t)$ for each date and state.

2.2 The SP problem: Efficient Allocation

...⇒
An efficient allocation is a function of the realized **aggregate** endowment.

It **does not** depend on

- the specific history leading to s^t
- the realization of individual endowment

A solution method:

- given a set of λ_i, solve for $c_t^1(s^t)$ using the feasibility constraint.
- use the ratio of FOCs to solve for the consumption allocation of all other individuals.

2.3 Arrow-Debreu Securities

Assume:

- Households can trade history contingent claims.
  - i.e. claims on time t consumption, contingent on history s^t.
- Trade occurs at $t = 0$, after the initial state s_0 is realized.
- There exists a complete set of securities.
- The price of a unit of a time t history s^t claim is $q_t^0(s^t)$
  - zero refers to the time of trade.
  - t refers to the time of "settlement" or delivery.

Household i's problem is:

$$\max_{c^i} U(c^i) = \max_{\{c_t^i(s^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty}\sum_{s^t} \beta^t u\left(c_t^i(s^t)\right)\pi(s^t)$$

s.t.:

$$\sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)c_t^i(s^t) \le \sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)y_t^i(s^t)$$

Note that each household has **one budget constraint**.

Let μ_i be the Lagrange multiplier for i's budget constraint.

Form the Lagrangian... the FOC with respect to $c_t^i(s^t)$ is:

$$\beta^t u'\left(c_t^i(s^t)\right)\pi(s^t) = \mu_i q_t^0(s^t) \qquad \forall \quad i, t, s^t$$

Divide by the FOC of $i = 1$:

$$\frac{u'\left(c_t^i(s^t)\right)}{u'\left(c_t^1(s^t)\right)} = \frac{\mu_i}{\mu_1} \quad \Rightarrow \quad c_t^i(s^t) = u'^{-1}\left(\frac{\mu_i}{\mu_1} u'\left(c_t^1(s^t)\right)\right)$$

Substitute in the feasibility constraint:

$$\sum_i u'^{-1}\left(\frac{\mu_i}{\mu_1} u'\left(c_t^1(s^t)\right)\right) = \sum_i y_t^i(s^t) = Y_t(s^t)$$

2.4 Arrow-Debreu Competitive Equilibrium

Definition: A competitive equilibrium is

- An allocation $c^i = \{c_t^i(s^t)\}_{t=0}^{\infty}$ for each i.
- A price system $\{q_t^0(s^t)\}_{t=0}^{\infty}$.

  Such that:

- Given the price system, the allocation solves each household's problem.
- The allocation is feasible.

2.5 Characterization of the CE

Looking at the last equation, observe that:

The CE allocation is a function of the realized **aggregate** endowment.

It **does not** depend on

- the specific history leading to s^t
- the realizations of individual endowment

2.6 CE vs. SP Allocations

Let's look at the equation solving $c_1^t(s^t)$ for the two problems:
SP:

$$\sum_i u'^{-1}\left(\frac{\lambda_1}{\lambda_i}u'\left(c_t^1(s^t)\right)\right) = \sum_i y_t^i(s^t) = Y_t(s^t)$$

CE:

$$\sum_i u'^{-1}\left(\frac{\mu_i}{\mu_1}u'\left(c_t^1(s^t)\right)\right) = \sum_i y_t^i(s^t) = Y_t(s^t)$$

Clearly, if we set the Pareto weights $\lambda_i = \frac{1}{\mu_i}$ then the CE is an efficient allocation. (In other words: the CE is a particular efficient allocation.)

2.7 CE vs. SP Prices

Let's look at the first order conditions for the two problems:
SP:

$$\lambda_i \beta^t u'\left(c_t^i(s^t)\right)\pi(s^t) = \theta_t(s^t) \qquad \forall \quad i, t, s^t$$

CE:

$$\beta^t u'\left(c_t^i(s^t)\right)\pi(s^t) = \mu_i q_t^0(s^t) \qquad \forall \quad i, t, s^t$$

At the efficient allocation, the contingent claims prices $q_t^0(s^t)$ equal the shadow prices $\theta_t(s^t)$ associated with the SP problem.[1]

3 CE solution - Negishi Algorithm

- Fix a positive value for one LM, let's say μ_1.
- Guess some positive values for the remaining $\mu_i's$.
- Use the term for $c_t^i(s^t)$ AND the resource constraint to find a consumption allocation.
- Use the household first order condition to find $q_t^0(s^t)$
- Substitute the price and consumption allocation in the budget constraint of each household
  - If expenditure of household i is greater than income $\rightarrow$ raise μ_i.
  - If expenditure of household i is lower than income $\rightarrow$ lower μ_i.

[1] Up to a scalar multiplication.

- Iterate over the previous steps until convergence.

(see LS p.216 for more details.)

4 Example and a Testable Implication

Assume a CRRA utility function $u\left(c_t^i(s^t)\right) = \frac{c_t^i(s^t)^{1-\sigma}}{1-\sigma}$.

Using the first order conditions for consumers i, j:

$$\frac{\left(c_t^j(s^t)\right)^{-\sigma}}{\left(c_t^i(s^t)\right)^{-\sigma}} = \frac{\mu_j}{\mu_i} \qquad \Leftrightarrow \quad c_t^i(s^t) = \left(\frac{\mu_j}{\mu_i}\right)^{\frac{1}{\sigma}} c_t^j(s^t)$$

sum over all i's to get:

$$C_t(s^t) = \sum_i c_t^i(s^t) = \sum_i \left(\frac{\mu_j}{\mu_i}\right)^{\frac{1}{\sigma}} c_t^j(s^t)$$

Implying that $c_t^j(s^t)$ is a constant fraction of aggregate consumption $C_t(s^t) \Rightarrow$ **individual consumption is perfectly correlated with aggregate consumption for every household.**

5 Remarks...

- Note that full consumption insurance is achieved when the ratio of **marginal utilities** of any two consumers i, j is constant for every date and history.
  i.e. not necessarily a statement about the ratio of consumption levels

- In order to consider the (more realistic) partial insurance, we use exogenous incomplete markets
  - limit the set of assets
  - can have endogenous reasons, which we will not consider