Macro Theory B

An Island Model (LS 3.28.2)

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1 Introduction

Before we saw the optimization problem of a single unemployed agent who searched for a job by drawing from an exogenous wage offer distribution. In this model there is a continuum of agents who interact across a large number of spatially separated labor markets.

1.1 Setup

- There is a large number of islands, and each one has a labor market with an aggregate production function of $\theta f(n)$. n is the island employment level and $\theta > 0$ is an idiosyncractic (specific to each island) production shock.
- The production funciton satisfies:

$$f' > 0,\ f'' < 0,\ lim_{n \to 0} f'(n) = \infty \tag{28.2.1}$$

- The productivity shock takes on m possible values, $\theta_1 < ... < \theta_m$, and is goverened by strictly positive transition probabilities: $\pi(\theta, \theta') > 0$, and the CDF satisfies: $Pr(\theta' \leq \theta_k | \theta) = \sum_{i=1}^{k} \pi(\theta, \theta_i)$ and is decreasing in θ. This means 1. That a shock *will* happen (a move from θ to θ') and that the higher the original θ is, the higher the probability to reach an even higher θ' in the next period.

1.2 Dynamic

- At the beggining of each period the agents are distributed in some way over the islands.
- After observing the productivity shock (in each island - complete information) the agents decide whether or not to move to another island - meaning $A_i = \{move, work\}$.
- A mover *gives up* his labor earnings in the period of the move. His choice maximizes the expected present value of his earnings stream.
- Wages are determined competitively so that each island's labor market clears with a wage rate equal to the marginal product of labor: $F_n = w$. The labor force which enters into production is the one minus those who left, but without those who are on the way.

1.3 A Single Market

$Market\,State = (x, \theta)$, where x is the beggining period labor force. In equilibrium there will be two mappings from the market state to the employment level $n(\theta, x)$ and wage rate $w(\theta, x)$. These equation must satisfy the market clearing conditions:

The wage is equal to the marginal product of labor, given the current amount of employment n: $w(\theta, x) = \frac{\partial}{\partial n}(\theta f(n)) = \theta f'(n(\theta, x))$.
The amount of employment must not exceed the labor force: $n(\theta, x) \leq x$. There is an inequality since at this period some agents may choose to leave the market. If none do, then $n = x$.

- $v(\theta, x)$ is the value of the optimization problem for an agent in market (θ, x) at the beggning of a period.
- v_u is the expected value obtained in next period by an agent leaving the market (this is the present discounted value of being out of work this period and beggining the next period at the island the agent is moving to. We assume for now that such a value exists and is exogenous, and that at equilibrium this value is the same for all agents in all islands looking to move to any other island). Because this value relates to the *beggining of next period*, in this period it is valued at βv_u. The agent's bellman equation is given by:

$$v(\theta, x) = max\{move, work\} = max\{\beta v_u,\ w(\theta, x) + \beta E[v(\theta', x')|\theta, x]\} \tag{28.2.2}$$

Where θ', x' are the market state of the island in the next period.

1.4 No Agents Leave

If no agents choose to leave the market, then we get $n(\theta, x) = x$, and then $w(\theta, x) = \theta f'(x)$. We look at two cases:

1.5 Case 1: Some additional agents arrive next period

This means that the expected value of working at this island in the next period must satisfy $\beta E[v(\theta', x')|\theta, x] = \beta v_u$. If it was less than this, then no new agent's would choose to move to this island, and if it was more than this, more agents would choose to move to this island until the value of working there was driven down to βv_u. Therefore the resulting bellman equation (from choosing to stay) is:

$$v(\theta, x) = \theta f'(x) + \beta v_u$$

1.6 Case 2: No additional agents arrive next period

If no agents arrive and all stay then $x' = x$. The lack of arrivals implies that $E[v(\theta', x'|\theta, x)] \leq \beta v_u$. So we get that the current bellman equation (from choosing to stay) is:

$$v(\theta, x) = \theta f'(x) + \beta E[v(\theta', x')|\theta, x] \leq \theta f'(x) + \beta v_u$$

1.7 General Case:

The combination of both cases give the bellman equation in general:

$$v(\theta, x) = max\{\beta v_u,\, \theta f'(x) + min\{\beta v_u,\, \beta E[v(\theta', x')|\theta, x]\}\} \tag{28.2.3}$$

In evaluating the stay option, the agent checks if $E[v(\theta', x')|\theta, x]$ is greater than βv_u or not. After checking he picks the worst case, since he knows thats what will govern the payoff: If it is less than βv_u then noone will arrive and that will be the payoff, and if it is greater than βv_u then enough agents will arrive to equal the next period payoff to βv_u exactly. He then maximizes, given that worst case payoff, is it better to move or work. Given a value for v_u this is a well behaved function with a *unique* solution $v(\theta, x)$ which is nondecreasing in θ (having a better production shock in the current island can't hurt the payoff the agent) of and nonincreasing in x (having a larger labor force means that there are possibly *more* workers, meaning a lower wage, which can either hurt or not affect the agent's payoff).

2 General evolution of island's labor force

2.1 Case 1: Some agents leave

If agents leave then $E[v(\theta', x')|\theta, x] \leq \beta v_u$ and so no additional agents will arive next period, and therefore there will be no new wokers next period: $x' = n$. In equilibrium we get that whoever chooses to work (meaning to stay) receives the same payoff as those who chose to leave:

$$\theta f'(x') + \beta E[v(\theta', x')|\theta, x] = \beta v_u \tag{28.2.4}$$

This means that given βv_u and the fact that agents choose to leave, we can derive a bound for the labor force, $x^+(\theta)$. Agents will leave this period, reducing x (which will actually be x') until the expected payoff from staying to work equals βv_u, which can be restated as until x equals some bound. We denote this bound by $x^+(\theta)$. If $x \geq x^+(\theta)$ then at the next period $x' = x^+(\theta)$, since agents will be leaving and lowering the labor force until it reaches the bound.

2.2 Case 2: All agents remain, and some workers arrive next period

The arriving workers expect to get the payoff v_u next period. This has to hold, therefore, for anyone expecting to work next period at this island. Therefore x' must satisfy

$$E[v(\theta', x')|\theta, x] = v_u \tag{28.2.5}$$

Like before, we can derive from this equation another bound on the labor force, $x^-(\theta)$. Agents will be arriving this period and increasing the labor force

x (which will actually be x') until the payoff reaches v_u, or until $x = x^-(\theta)$. So, if $x \leq x^-(\theta)$ then $x' = x^-(\theta)$.

It can be seen that $x^-(\theta) < x^+(\theta)$.

2.3 Case 3: All agents remain, no additional workers arrive next period

This actually means that $x^-(\theta) < x < x^+(\theta)$: Noone wants to leave so $E[v(\theta', x')|\theta, x] > \beta v_u$ and so $x < x^+(\theta)$. Noone wants to move to this island, so $E[v(\theta', x')|\theta, x] < v_u$ and so $x^-(\theta) < x$. Obviously, in this case $x = x'$.

3 Solving the model for 2 islands (equilibrium)

We assume that there are two islands, and we look at two different shocks $\theta_1 < \theta_2$:

In general, we know that $x^-(\theta_1) < x^+(\theta_1)$, $x^-(\theta_2) < x^+(\theta_2)$. We also notice that if $\theta \uparrow$ then $v \uparrow$, and so $x^+ \uparrow$, since it is derived by lowering x until the expected value reaches βv_u (so since the epected value is higher, the x that satisfies it is also higher), and so we get $x^+(\theta_2) \geq x^+(\theta_1)$. Similarly, we get that $x^-(\theta_2) \geq x^-(\theta_1)$. From this we can also trivially derive that $x^+(\theta_2) \geq x^-(\theta_1)$. The unknown is what is the relationship between $x^-(\theta_2)$ and $x^+(\theta_1)$. We look for an equilibrium solution, meaning a range R which satisfies $x \in R \rightarrow x' = x$. We know that given a value for θ, if $x \in [x^-(\theta), x^+(\theta)]$ then $x' = x$ if θ were to stay the same ($\theta' = \theta$). So we extrapolate from this to the case that $\theta' \neq \theta$. Again, we look at the case $\theta_1 < \theta_2$.

3.1 Case 1 $x^+(\theta_1) > x^-(\theta_2)$:

In this case we have look at the nonempty interval $I := [x^-(\theta_1), x^+(\theta_1)] \cap [x^-(\theta_2), x^+(\theta_2)] = [x^-(\theta_2), x^+(\theta_1)]$. If $x \in I$ then $x' = x$. Otherwise the result depends on the direction of the move in θ, i.e. it won't always be the case that $x' = x$ and so these intervals don't work:

Assume $x < x_2^-$: If $\theta_1 \rightarrow \theta_2$ then $x \uparrow x_2^-$ (so $x' \neq x$).

Assume $x > x_1^+$: If $\theta_2 \rightarrow \theta_1$ then $x \downarrow x_1^+$ (so $x' \neq x$).

3.2 Case 2 $x^+(\theta_1) < x^-(\theta_2)$:

In this case $I = \emptyset$. However, we look at the nonempty interval $J = [x^+(\theta_1), x^-(\theta_2)]$. For an $x \in J$, if $\theta_1 \rightarrow \theta_2$ then $\forall x' \in J : x' = x^-(\theta_2)$, and if $\theta_2 \rightarrow \theta_1$then $\forall x' \in J : x' = x^+(\theta_1)$, meaning that all x values in this interval result in one of these two points. Therefore, only if $x \in \{x^+(\theta_1), x^-(\theta_2)\}$ we get that $x' = x$.

4 Solving the model for general θ:

We can summarize the equilibrium modes for general θ. Assume like before that the θvalues are ordered $\{\theta_1, ..., \theta_m\}$. So we get:

$$x = \begin{cases} \{x \in \{x^-(\theta_i),\, x^+(\theta_i) | x^+(\theta_1) \leq x \leq x^-(\theta_m)\}\}\, if & x^+(\theta_1) \leq x^-(\theta_m) \\ \{x \in [x^-(\theta_m), x^+(\theta_1)]\} & otherwise \end{cases}$$